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Interval-valued fuzzy permutation method and experimental analysis on cardinal and ordinal evaluations

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ABSTRACT

This paper presents interval-valued fuzzy permutation (IVFP) methods for solving multi-attribute decision making problems based on interval-valued fuzzy sets. First, we evaluate alternatives according to the achievement levels of attributes, which admits cardinal or ordinal representation. The relative importance of each attribute can also be measured by interval or scalar data. Next, we identify the concordance, midrange concordance, weak concordance, discordance, midrange discordance and weak discordance sets for each ordering. The proposed method consists of testing each possible ranking of the alternatives against all others. The evaluation value of each permutation can be computed either by cardinal weights or by solving programming problems. Then, we choose the permutation with the maximum evaluation value, and the optimal ranking order of alternatives can be obtained. An experimental analysis of IVFP rankings given cardinal and ordinal evaluations is conducted with discussions on consistency rates, contradiction rates, inversion rates, and average Spearman correlation coefficients.

1. Introduction

Fuzzy multiattribute decision making problems have become an important research field in multiple criteria decision analysis (MCDA). The key for solving MCDA is how to obtain the decision maker's preference information through the form of attributes or alternatives. There are necessary steps in utilizing MCDA involving numerical measures of the relative importance of attributes and the performance of each alternative on these attributes. In real-world cases, exact values may be difficult to be precisely determined since decision makers' judgments are often vague. The imprecision may result from unquantifiable information, incomplete information, nonobtainable information, or partial ignorance [5]. Therefore, an extension to the fuzzy environment is a natural generalization of MCDA models.

Nevertheless, it is not always certain that the evaluation of membership values in real applications. There may be some hesitation degree between belongingness and nonbelongingness. In view that there are many real life situations where due to inadequacy in information availability, interval-valued fuzzy sets (IVFSs) with ill-known membership grades are appropriate to cope with such problems. IVFS is defined by an interval-valued membership function [22,28]. That is, the degree of membership of an element to a set is characterized by a closed subinterval of $[0,1]$. In view of the fact that the membership degrees are considered as intervals, the aim of this paper is to develop a new outranking method for solving MCDA problems with interval-valued fuzzy data.

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In the present paper we suggest how to determine the optimal ranking order of the decision alternatives whose performance evaluations are not necessarily unambiguous and admit some hesitance. We propose a simple and flexible outranking model for such imprecise, vague, and hesitant decision environment based on IVFSs. The proposed model can be used for cardinal or ordinal data, even for missing information or noncomparable outcomes. Then, the level of concordance of the complete preference order can be measured to determine the best ranking order of the alternatives. Next, we show some numerical examples to illustrate the proposed method. Finally, enormous random MCDA problems are generated and computational studies are undertaken to compare preference orders determined by different methods.

2. Multiattribute decision environment based on interval-valued fuzzy data

A multiattribute decision making problem can be concisely expressed in a decision matrix, whose element indicates the evaluation or value of the ith alternative, A_i, with respect to the jth attribute, x_j. In the present paper, we extend the canonical matrix format to interval-valued fuzzy decision matrix D; that is, decision makers are expected to assign an extent of membership grades that captures the degree of the alternative A_i satisfies the attribute x_j according to their opinions. Let X be the discussion universe containing decision attributes in the multiattribute decision problem setting. Denote the set of all attributes $X = \{x_1, x_2, \ldots, x_n\}$. Let $\mathrm{Int}([0,1])$ stand for the set of all closed subintervals of $[0,1]$. An IVFS A_i of the ith alternative on X is given by:

$$A_i = \{\langle x_j, M_{A_i}(x_j)\rangle \mid x_j \in X\}, \tag{1}$$

where $M_{A_i} : X \to \mathrm{Int}([0,1])$, such that $x_j \to M_{A_i}(x_j) = [M^-_{A_i}(x_j), M^+_{A_i}(x_j)]$. M_{A_i} indicates the possible degree to which the alternative A_i satisfies attribute x_j. $M^-_{A_i}(x_j)$ and $M^+_{A_i}(x_j)$ are the lower bound and the upper bound, respectively, of the interval $M_{A_i}(x_j)$.

It is worthwhile to mention that IVFS theory is mathematically equivalent to Atanassov's intuitionistic fuzzy set (A-IFS) theory [6,7,19]. The concept of A-IFSs, introduced by [1], is a generalization of ordinary fuzzy sets [2,19]. A-IFSs assign to each element of the universe not only a membership degree but also a nonmembership degree, and furthermore the sum of these two degrees is less than or equal to 1. In this paper, let $\mu_{A_i}(x_j)$ be the degree to which the alternative A_i satisfies attribute x_j, where $\mu_{A_i}(x_j) : X \to [0,1]$. Similarly, let $\nu_{A_i}(x_j)$ be the degree to which the alternative A_i does not satisfy attribute x_j with $\nu_{A_i}(x_j) : X \to [0,1]$. In addition, $0 \leqslant \mu_{A_i}(x_j) + \nu_{A_i}(x_j) \leqslant 1$ for all $x_j \in X$.

The A-IFS theory has been applied to many different fields, such as decision making, logic programming, topology, medical diagnosis, pattern recognition, machine learning and market prediction [26]. Especially, there exist many useful methods for MCDA on a basis of A-IFSs [15,17,18,21,26,27]. Although A-IFS and IVFS constitute an isomorphism [25], A-IFS and IVFS are based on different semantics, such as weighing/modeling preferences versus imprecise membership [7,10]. Furthermore, the semantic is crucial for real applications [3,25]. From the practical viewpoint, the membership degree and nonmembership degree in A-IFSs are exact without any assumption on indeterminacy, except for $\mu_{A_i}(x_j) + \nu_{A_i}(x_j) \leqslant 1$, and more or less independent, while IVFSs assign an interval for approximating the correct membership degree [18].

As mentioned above, the decision maker's evaluation lies in the closed interval $[M^-_{A_i}(x_j), M^+_{A_i}(x_j)]$. Let $M^-_{A_i}(x_j) = \mu_{A_i}(x_j)$ and $M^+_{A_i}(x_j) = 1 - \nu_{A_i}(x_j)$, and thus $[M^-_{A_i}(x_j), M^+_{A_i}(x_j)] = [\mu_{A_i}(x_j), 1 - \nu_{A_i}(x_j)]$. An interval $[M^-_{A_i}(x_j), M^+_{A_i}(x_j)]$ can be mapped bijectively onto a couple $(\mu_{A_i}(x_j), 1 - \nu_{A_i}(x_j))$ [18]. Since IVFS and A-IFS are equipollent generalizations of ordinary fuzzy set [4], we can also express the decision matrix using A-IFS notation. An A-IFS A_i of the ith alternative on X is given by:

$$A_i = \{\langle x_j, \mu_{A_i}(x_j), \nu_{A_i}(x_j)\rangle \mid x_j \in X\}. \tag{2}$$

For each element $x_j \in X$, the intuitionistic index of x_j in A_i is defined as follows [2,23]:

$$\pi_{A_i}(x_j) = 1 - \mu_{A_i}(x_j) - \nu_{A_i}(x_j), \tag{3}$$

where $\pi_{A_i}(x_j) \in [0,1]$ $\forall x_j \in X$. $\pi_{A_i}(x_j)$ reflects the fact that the decision maker may not always be certain of membership grades. In other words, an interval $[M^-_{A_i}(x_j), M^+_{A_i}(x_j)]$ shows all possible degrees of membership and the decision maker is hesitated to the extent $\pi_{A_i}(x_j)$. This hesitation margin plays an important role for A-IFSs, such as measurement of distances [8,23], similarity [13,16], entropy [4,12,24], etc.

Let A and B denote two IVFSs of the universe of discourse X. [4] defined the following expressions:

$$A \leqslant B \quad \text{if and only if} \quad M^-_A(x) \leqslant M^-_B(x) \quad \text{and} \quad M^+_A(x) \leqslant M^+_B(x) \quad \text{for all } x \in X;$$

$$A \preccurlyeq B \quad \text{if and only if} \quad M^-_A(x) \leqslant M^-_B(x) \quad \text{and} \quad M^+_A(x) \geqslant M^+_B(x) \quad \text{for all } x \in X.$$

In addition, $A \geqslant B$ if and only if $B \leqslant A$; $A \succcurlyeq B$ if and only if $B \preccurlyeq A$. The above definitions can be extended to the A-IFSs as follows:

$$A \leqslant B \quad \text{if and only if} \quad \mu_A(x) \leqslant \mu_B(x) \quad \text{and} \quad \nu_A(x) \geqslant \nu_B(x) \quad \text{for all } x \in X;$$

$$A \preccurlyeq B \quad \text{if and only if} \quad \mu_A(x) \leqslant \mu_B(x) \quad \text{and} \quad \nu_A(x) \leqslant \nu_B(x) \quad \text{for all } x \in X.$$

$A = B$ if and only if $M_A^-(x) = M_B^-(x)$ and $M_A^+(x) = M_B^+(x)$ for all $x \in X$ (or $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ by A-IFS notation).

Consider $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; and $[M_{A_i}^-(x_j), M_{A_i}^+(x_j)]$ (or equivalently, $(\mu_{A_i}(x_j), \nu_{A_i}(x_j))$ by A-IFS notation) representing the performance measure of the ith alternative in terms of the jth attribute. The interval-valued fuzzy decision matrix D is defined as the following form:

$$D = \begin{bmatrix} [M_{A_1}^-(x_1), M_{A_1}^+(x_1)] & [M_{A_1}^-(x_2), M_{A_1}^+(x_2)] & \cdots & [M_{A_1}^-(x_n), M_{A_1}^+(x_n)] \\ [M_{A_2}^-(x_1), M_{A_2}^+(x_1)] & [M_{A_2}^-(x_2), M_{A_2}^+(x_2)] & \cdots & [M_{A_2}^-(x_n), M_{A_2}^+(x_n)] \\ \vdots & \vdots & \ddots & \vdots \\ [M_{A_m}^-(x_1), M_{A_m}^+(x_1)] & [M_{A_m}^-(x_2), M_{A_m}^+(x_2)] & \cdots & [M_{A_m}^-(x_n), M_{A_m}^+(x_n)] \end{bmatrix}$$
$$= \begin{bmatrix} (\mu_{A_1}(x_1), \nu_{A_1}(x_1)) & (\mu_{A_1}(x_2), \nu_{A_1}(x_2)) & \cdots & (\mu_{A_1}(x_n), \nu_{A_1}(x_n)) \\ (\mu_{A_2}(x_1), \nu_{A_2}(x_1)) & (\mu_{A_2}(x_2), \nu_{A_2}(x_2)) & \cdots & (\mu_{A_2}(x_n), \nu_{A_2}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{A_m}(x_1), \nu_{A_m}(x_1)) & (\mu_{A_m}(x_2), \nu_{A_m}(x_2)) & \cdots & (\mu_{A_m}(x_n), \nu_{A_m}(x_n)) \end{bmatrix}. \tag{4}$$

Since all attributes cannot be assumed to be of equal importance, we must receive a set of grades of importance, denoted as W, from the decision maker. The IVFS can also be expressed as the subjective importance of decision attributes during the decision maker's evaluation process. An IVFS W in X is an object having the form:

$$W = \{\langle x_j, M_W(x_j)\rangle \mid x_j \in X\}$$
$$= \{\langle x_j, \mu_W(x_j), \nu_W(X_j)\rangle \mid x_j \in X\}, \tag{5}$$

where $M_W : X \to \text{Int}([0, 1])$, such that $x_j \to M_W(x_j) = [M_W^-(x_j), M_W^+(x_j)]$. In addition, $\mu_W(x_j) : X \to [0, 1]$ and $\nu_W(x_j) : X \to [0, 1]$ define the degree of importance and the degree of unimportance for an attribute, respectively, where $0 \leqslant \mu_W(x_j) + \nu_W(x_j) \leqslant 1$. For each $x_j \in X$, the intuitionistic index toward the importance of an attribute is as follows:

$$\pi_W(x_j) = 1 - \mu_W(x_j) - \nu_W(x_j). \tag{6}$$

The intuitionistic index $\pi_W(x_j)$ allows decision makers to change their evaluating the relative importance of an attribute between the highest weight and the lowest one. The grades of attribute importance are usually given by a set of weights, w_j's, which is normalized to sum to 1. Hence, a set of weights lying in the closed interval $[M_W^-(x_j), M_W^+(x_j)] = [\mu_W(x_j), \mu_W(x_j) + \pi_W(x_j)]$ must satisfies the following conditions:

$$M_W^-(x_j) \leqslant w_j \leqslant M_W^+(x_j), \quad j = 1, 2, \ldots, n; \tag{7}$$
$$\sum_{j=1}^{n} w_j = 1. \tag{8}$$

For the sake of obtaining a set of feasible weights, we assume that $\sum_{j=1}^{n} M_W^-(x_j) \leqslant 1$ and $\sum_{j=1}^{n} M_W^+(x_j) \geqslant 1$ in this paper.

3. Interval-valued fuzzy permutation (IVFP) method

Similar to the permutation method [14,20], the proposed IVFP method measures the level of concordance of the complete preference order. According to interval-valued fuzzy decision matrix, we test each possible ranking of the alternatives against all others. Then, the best order of the alternatives can be chosen by the evaluation criteria consisting of the levels of concordance and of discordance. The IVFP method is a useful approach owing to its simplicity and flexibility with regard to cardinal and ordinal rankings.

3.1. IVFP method with cardinal evaluations of alternatives given

Consider the interval-valued fuzzy decision matrix D that refers to m alternatives on n attributes. Then, $m!$ permutations of the ranking of the alternatives exist. Let P_i denote the ith permutation:

$$P_i = (\ldots, A_k, \ldots, A_l, \ldots), \quad \text{for } i = 1, 2, \ldots, m!, \tag{9}$$

where A_k is ranked higher than A_l. Next, we define six subsets of all attributes according to the inequality relations of IVFSs and the accuracy function. [11] discussed MCDA problems based on the vague set theory. They proposed an accuracy function defined by the sum of the degrees of membership and nonmembership, i.e., one minus the intuitionistic index. [17] defined the same accuracy function for an A-IFS. The accuracy function can be used to validate the evaluation precision and help the decision maker to make decisions more credibly. For the real decision making problems, as [18] indicated, we need to reduce the level of uncertainty as much as possible, especially to a conservative decision maker. Thus, in addition to inequality relations $\geqslant$, $\succcurlyeq$, $\leqslant$, and $\preccurlyeq$, we consider two conditions of $\pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j)$ and $\pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j)$ to order A_k

and A_l. We define the concordance set C_{kl}, midrange concordance set C'_{kl}, weak concordance set C''_{kl}, discordance set D_{kl}, midrange discordance set D'_{kl}, and weak discordance set D''_{kl} as follows, where they are expressed equivalently using either IVFS or A-IFS notation:

$$\begin{aligned} C_{kl} &= \{ j \mid A_k(x_j) \geqslant A_l(x_j) \text{ and } \pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid M^-_{A_k}(x_j) \geqslant M^-_{A_l}(x_j), M^+_{A_k}(x_j) \geqslant M^+_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid \mu_{A_k}(x_j) \geqslant \mu_{A_l}(x_j), \nu_{A_k}(x_j) \leqslant \nu_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j) \}, \quad k, l = 1, 2, \ldots, m, k \neq l, \end{aligned} \tag{10}$$

$$\begin{aligned} C'_{kl} &= \{ j \mid A_k(x_j) \geqslant A_l(x_j) \text{ and } \pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid M^-_{A_k}(x_j) \geqslant M^-_{A_l}(x_j), M^+_{A_k}(x_j) \geqslant M^+_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid \mu_{A_k}(x_j) \geqslant \mu_{A_l}(x_j), \nu_{A_k}(x_j) \leqslant \nu_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j) \}, \quad k, l = 1, 2, \ldots, m, k \neq l, \end{aligned} \tag{11}$$

$$\begin{aligned} C''_{kl} &= \{ j \mid A_k(x_j) \succcurlyeq A_l(x_j) \} \\ &= \{ j \mid M^-_{A_k}(x_j) \geqslant M^-_{A_l}(x_j) \text{ and } M^+_{A_k}(x_j) \leqslant M^+_{A_l}(x_j) \} \\ &= \{ j \mid \mu_{A_k}(x_j) \geqslant \mu_{A_l}(x_j) \text{ and } \nu_{A_k}(x_j) \geqslant \nu_{A_l}(x_j) \}, \quad k, l = 1, 2, \ldots, m, k \neq l, \end{aligned} \tag{12}$$

$$\begin{aligned} D_{kl} &= \{ j \mid A_k(x_j) \leqslant A_l(x_j) \text{ and } \pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid M^-_{A_k}(x_j) \leqslant M^-_{A_l}(x_j), M^+_{A_k}(x_j) \leqslant M^+_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid \mu_{A_k}(x_j) \leqslant \mu_{A_l}(x_j), \nu_{A_k}(x_j) \geqslant \nu_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j) \}, \quad k, l = 1, 2, \ldots, m, k \neq l, \end{aligned} \tag{13}$$

$$\begin{aligned} D'_{kl} &= \{ j \mid A_k(x_j) \leqslant A_l(x_j) \text{ and } \pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid M^-_{A_k}(x_j) \leqslant M^-_{A_l}(x_j), M^+_{A_k}(x_j) \leqslant M^+_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j) \} \\ &= \{ j \mid \mu_{A_k}(x_j) \leqslant \mu_{A_l}(x_j), \nu_{A_k}(x_j) \geqslant \nu_{A_l}(x_j) \text{ and } \pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j) \}, \quad k, l = 1, 2, \ldots, m, k \neq l, \end{aligned} \tag{14}$$

$$\begin{aligned} D''_{kl} &= \{ j \mid A_k(x_j) \preccurlyeq A_l(x_j) \} \\ &= \{ j \mid M^-_{A_k}(x_j) \leqslant M^-_{A_l}(x_j) \text{ and } M^+_{A_k}(x_j) \geqslant M^+_{A_l}(x_j) \} \\ &= \{ j | \mu_{A_k}(x_j) \leqslant \mu_{A_l}(x_j) \text{ and } \nu_{A_k}(x_j) \leqslant \nu_{A_l}(x_j) \}, \quad k, l = 1, 2, \ldots, m, k \neq l. \end{aligned} \tag{15}$$

In a particular ranking, if the partial ranking $A_k \geqslant A_l$ appears, the fact that $A_k(x_j) \geqslant A_l(x_j)$ and $\pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j)$ will be rated w_j, $A_k(x_j) \geqslant A_l(x_j)$ and $\pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j)$ being rated $\frac{2}{3} w_j$, $A_k(x_j) \succcurlyeq A_l(x_j)$ being rated $\frac{1}{3} w_j$, $A_k(x_j) \leqslant A_l(x_j)$ and $\pi_{A_k}(x_j) \geqslant \pi_{A_l}(x_j)$ being rated $-w_j$, $A_k(x_j) \leqslant A_l(x_j)$ and $\pi_{A_k}(x_j) \leqslant \pi_{A_l}(x_j)$ being rated $-\frac{2}{3} w_j$, and $A_k(x_j) \preccurlyeq A_j(x_j)$ being rated $-\frac{1}{3} w_j$. The evaluation criterion of the chosen hypothesis for ranking of the alternatives is the algebraic sum of w_j's corresponding to the element by element consistency.

In the proposed IVFP method, the evaluation value $E(P_i)$ of the ith permutation P_i is defined by

$$E(P_i) = \sum_{j \in C_{kl}} w_j + \frac{2}{3} \sum_{j \in C'_{kl}} w_j + \frac{1}{3} \sum_{j \in C''_{kl}} w_j - \sum_{j \in D_{kl}} w_j - \frac{2}{3} \sum_{j \in D'_{kl}} w_j - \frac{1}{3} \sum_{j \in D''_{kl}} w_j, \quad i = 1, 2, \ldots, m!. \tag{16}$$

For each permutation P_i, its optimal weight values can be computed via the following linear programming (LP):

$$\begin{aligned} \max \quad & E(P_i) = \sum_{j \in C_{kl}} w_j + \frac{2}{3} \sum_{j \in C'_{kl}} w_j + \frac{1}{3} \sum_{j \in C''_{kl}} w_j - \sum_{j \in D_{kl}} w_j - \frac{2}{3} \sum_{j \in D'_{kl}} w_j - \frac{1}{3} \sum_{j \in D''_{kl}} w_j, \\ \text{subject to} \quad & M^-_W(x_j) \leqslant w_j \leqslant M^+_W(x_j) \quad (j = 1, 2, \ldots, n), \\ & \sum_{j=1}^{n} w_j = 1 \end{aligned} \tag{17}$$

for each $i = 1, 2, \ldots, m!$.

Solving Eq. (17) by Simplex method, we can obtain its optimal solution of attribute weights $\overline{\mathbf{w}}^i = (\overline{w}^i_1, \overline{w}^i_2, \ldots, \overline{w}^i_n)^T$ and the optimal evaluation value $\overline{E(P_i)}$ of the ith permutation. In total, $m!$ LP problems need to be solved since there are $m!$ permutations in the alternative set. Then, we choose the maximum value among $\overline{E(P_i)}$'s, and the optimal ranking order of the alternatives can be found correspondingly.

What has to be noticed is that the proposed IVFP method can be used for the attribute information to be in a scalar form, not interval-valued fuzzy data. Assume that a set of cardinal weights w_j, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$ be given to the set of decision attributes. Applying Eq. (16), the evaluation value $E(P_i)$ of each permutation can be computed as an evaluation criterion. The permutation with the maximum $E(P_i)$ value represents the best order of the alternatives.

3.2. IVFP method with ordinal evaluations of alternatives given

Besides the cardinal evaluations of alternatives given, the IVFP method can be used for the ordinal evaluations given. Assume that the decision maker only give ordering or ranking information of the alternative on each attribute. Moreover, the relative importance among attributes can be determined either by interval-valued fuzzy data or by cardinal weights. The proposed method in this subsection features limited information requirements because the decision maker has no need to scale the qualitative attributes in the decision matrix.

There is one simple way to transform the attributewise ranks into the interval-valued fuzzy data. The method, similar to Grzegorzewski's method [9], is to calculate the number of alternatives surely worse than (e.g., the inequality relation $\leqslant$ in the interval-valued fuzzy decision matrix D, exclusive of $=$) and surely better than (e.g., the inequality relation $\geqslant$ in D, exclusive of $=$) a particular alternative. The point we wish to emphasize is that the method admits incomplete ordinal data since not all alternatives can been ranked with respect to an attribute. Considering the situation with missing information or noncomparable outcomes, we define two functions, $\alpha_j(A_i)$ and $\beta_j(A_i)$ for each A_i with respect to x_j. Let $\alpha_j(A_i)$ denote the number of alternatives $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, A_{i+2}, \ldots, A_m$ surely worse than A_i, while $\beta_j(A_i)$ denotes the number of alternatives $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, A_{i+2}, \ldots, A_m$ surely better than A_i. The degrees of membership and nonmembership are given as follows, respectively.

$$\mu_{A_i}(x_j) = \frac{\alpha_j(A_i)}{m-1}, \tag{18}$$

$$\nu_{A_i}(x_j) = \frac{\beta_j(A_i)}{m-1}. \tag{19}$$

Correspondingly, the lower bound and upper bound of the interval $M_{A_i}(x_j)$ are as follows:

$$M^-_{A_i}(x_j) = \frac{\alpha_j(A_i)}{m-1}, \tag{20}$$

$$M^+_{A_i}(x_j) = \frac{m-1-\beta_j(A_i)}{m-1}. \tag{21}$$

The situation that $\alpha_j(A_i) + \beta_j(A_i) < m - 1$ (i.e., $\pi_{A_i}(x_j) > 0$) occurs when the decision maker assigns the same rank to more than one alternative or some alternatives are not comparable with the others. Taking the interval-valued fuzzy decision matrix D for example, the attributewise preference in the weak concordance C''_{kl} set or the weak discordance set D''_{kl} belongs to the noncomparable relations.

In such a way, the ordinal evaluations of alternatives on each attribute can be easily converted into interval-valued fuzzy data. Then, the IVFP method can be also applied the situation with ties, missing information or noncomparable evaluation data.

3.3. The presented algorithm

The IVFP method for solving a MCDA problem can be summed up as a series of successive steps:

Step 1. Generating relevant attributes for the MCDA problem setting.
Step 2. Developing a limited (and countably small) number of predetermined noninferior alternatives.
Step 3. Evaluating alternatives in terms of attributes. The alternatives have associated with them a level of the achievement of the attributes, which admits cardinal or ordinal information.
- Step 3-1. For cardinal information, construct the interval-valued fuzzy decision matrix D to concisely express the MCDA problem of concern.
- Step 3-2. For ordinal information, Eqs. (20) and (21) (or Eqs. (18) and (19) based on A-IFSs) are utilized to construct the interval-valued fuzzy decision matrix D.

Step 4. Receiving a set of grades of importance for decision attributes. The relative importance of each attribute can be given by interval or scalar data. The premises of interval-valued fuzzy weights are both $\sum M^-_W(x_j) \leqslant 1$ and $\sum M^+_W(x_j) \geqslant 1$; while the premise of cardinal weights is $\sum w_j = 1$.
Step 5. Identifying concordance and discordance sets for each ordering. There are $m!$ permutations of the alternatives which have to be tested. Using Eqs. (10)–(15), we can find C_{kl}, C'_{kl}, C''_{kl}, D_{kl}, D'_{kl} and D''_{kl} for pairwise partial rankings.
Step 6. Computing the evaluation value $E(P_i)$ of the permutation P_i.
- Step 6-1. In the case of cardinal weights, compute $E(P_i)$ by using Eq. (16).
- Step 6-2. In the case of interval-valued fuzzy weights, solve Eq. (17) to acquire the optimal solution of attribute weights $\overline{\mathbf{w}}^i$ and the evaluation value $\overline{E(P_i)}$.

Step 7. Selecting the largest value among the evaluation values. The permutation with the maximum evaluation value is the optimal ranking order of the alternatives.

In the following, we present numerical examples connected with a decision making problem. The case of cardinal evaluations given will be discussed first, then the ordinal ones.

4. Numerical examples and discussions

4.1. Case of cardinal evaluations of alternatives on each attribute

In this subsection, we work out a numerical example to illustrate the IVFP method for MCDA problems with cardinal cases. Consider a sneakers-choice problem. Suppose that five attributes x_1 (styling), x_2 (color), x_3 (price), x_4 (air-sole), and x_5 (brand image) are taken into consideration in the selection problem. Denote the set of all attributes by $X = \{x_1, x_2, x_3, x_4, x_5\}$. (Note that Step 1 has been done.) Suppose that there exist four nondominated brands A_1, A_2, A_3, and A_4. Denote the alternative set by $A = \{A_1, A_2, A_3, A_4\}$. (Note that Step 2 has been done.) Assume that a decision maker has indicated an extent of membership grades that captures the degree of the brand A_i satisfies the attribute x_j. (Note that Step 3 has been done.) The interval-valued fuzzy decision matrix D in Step 3-1 is given below:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} [0.3379, 0.6639] & [0.5119, 0.6238] & [0.1640, 0.3414] & [0.2556, 0.8424] & [0.2766, 0.7013] \\ [0.0059, 0.7713] & [0.2561, 0.3808] & [0.3010, 0.4029] & [0.5434, 0.8702] & [0.5522, 0.8509] \\ [0.2585, 0.4713] & [0.2132, 0.8856] & [0.1243, 0.4259] & [0.4664, 0.6260] & [0.8205, 0.9602] \\ [0.6388, 0.9890] & [0.3997, 0.4897] & [0.0492, 0.6926] & [0.3202, 0.5387] & [0.9065, 0.9433] \end{bmatrix} \end{array}$$

$$= \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{bmatrix} (0.3379, 0.3361) & (0.5119, 0.3762) & (0.1640, 0.6586) & (0.2556, 0.1576) & (0.2766, 0.2987) \\ (0.0059, 0.2287) & (0.2561, 0.6192) & (0.3010, 0.5971) & (0.5434, 0.1298) & (0.5522, 0.1491) \\ (0.2585, 0.5287) & (0.2132, 0.1144) & (0.1243, 0.5741) & (0.4664, 0.3740) & (0.8205, 0.0398) \\ (0.6388, 0.0110) & (0.3997, 0.5103) & (0.0492, 0.3074) & (0.3202, 0.4613) & (0.9065, 0.0567) \end{bmatrix}.$$

Assume that the subjective importance of attributes, W, in Step 4 is given by the decision maker as:

$$W = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \big[\, [0.1228, 0.8670] \quad [0.0030, 0.6884] \quad [0.0879, 0.5912] \quad [0.5527, 0.8887] \quad [0.1387, 0.6718] \,\big] \end{array}$$

$$= \big[\, (0.1228, 0.1330) \quad (0.0030, 0.3116) \quad (0.0879, 0.4088) \quad (0.5527, 0.1113) \quad (0.1387, 0.3282) \,\big].$$

It should be noted that $\sum M_W^-(x_j) = 0.9051 \leqslant 1$ and $\sum M_W^+(x_j) = 3.7071 \geqslant 1$.

There are $24 (= 4!)$ permutations of the ranking for all alternatives that have to be tested in Step 5. They are:

$$\begin{array}{lll}
P_1 = (A_1, A_2, A_3, A_4), & P_9 = (A_2, A_3, A_1, A_4), & P_{17} = (A_3, A_4, A_1, A_2), \\
P_2 = (A_1, A_2, A_4, A_3), & P_{10} = (A_2, A_3, A_4, A_1), & P_{18} = (A_3, A_4, A_2, A_1), \\
P_3 = (A_1, A_3, A_2, A_4), & P_{11} = (A_2, A_4, A_1, A_3), & P_{19} = (A_4, A_1, A_2, A_3), \\
P_4 = (A_1, A_3, A_4, A_2), & P_{12} = (A_2, A_4, A_3, A_1), & P_{20} = (A_4, A_1, A_3, A_2), \\
P_5 = (A_1, A_4, A_2, A_3), & P_{13} = (A_3, A_1, A_2, A_4), & P_{21} = (A_4, A_2, A_1, A_3), \\
P_6 = (A_1, A_4, A_3, A_2), & P_{14} = (A_3, A_1, A_4, A_2), & P_{22} = (A_4, A_2, A_3, A_1), \\
P_7 = (A_2, A_1, A_3, A_4), & P_{15} = (A_3, A_2, A_1, A_4), & P_{23} = (A_4, A_3, A_1, A_2), \\
P_8 = (A_2, A_1, A_4, A_3), & P_{16} = (A_3, A_2, A_4, A_1), & P_{24} = (A_4, A_3, A_2, A_1).
\end{array}$$

Let us, for example, compute the testing results of the ordering $P_{16} = (A_3, A_2, A_4, A_1)$ derived from the interval-valued fuzzy decision matrix, D. Applying Step 5, we draw the procedure for determining concordance and discordance sets on account of alternatives A_4 and A_1. Observe that $M_{A_4}^-(x_5)(= 0.9065) \geqslant M_{A_1}^-(x_5)(= 0.2766)$ (or equivalently, $\mu_{A_4}(x_5)(= 0.9065) \geqslant \mu_{A_1}(x_5)(= 0.2766)$), $M_{A_4}^+(x_5)(= 0.9433) \geqslant M_{A_1}^+(x_5)(= 0.7013)$ (or $\nu_{A_4}(x_5)(= 0.0567) \leqslant \nu_{A_1}(x_5)(= 0.2987)$), and $\pi_{A_4}(x_5)(= 0.0368) \leqslant \pi_{A_1}(x_5)(= 0.4247)$, the concordance set C_{41} is:

$$C_{41} = \big\{ j \bigm| A_4(x_j) \geqslant A_1(x_j) \text{ and } \pi_{A_4}(x_j) \leqslant \pi_{A_1}(x_j) \big\} = \{5\}.$$

Because $M_{A_4}^-(x_1)(= 0.6388) \geqslant M_{A_1}^-(x_1)(= 0.3379)$, $M_{A_4}^+(x_1)(= 0.9890) \geqslant M_{A_1}^+(x_1)(= 0.6639)$, and $\pi_{A_4}(x_1)(= 0.3502) \geqslant \pi_{A_1}(x_1)(= 0.3260)$, the midrange concordance set C'_{41} is:

$$C'_{41} = \big\{ j \bigm| A_4(x_j) \geqslant A_1(x_j) \text{ and } \pi_{A_4}(x_j) \geqslant \pi_{A_1}(x_j) \big\} = \{1\}.$$

Since $M^-_{A_4}(x_4)(=0.3202) \geqslant M^-_{A_1}(x_4)(=0.2556)$ and $M^+_{A_4}(x_4)(=0.5387) \leqslant M^+_{A_1}(x_4)(=0.8424)$, we know that the weak concordance set C''_{41} is:

$$C''_{41} = \{ j \mid A_4(x_j) \succcurlyeq A_1(x_j) \} = \{4\}.$$

Therefore, the concordance testing result concerning alternatives A_4 and A_1 is $\frac{2}{3}w_1 + \frac{1}{3}w_4 + w_5$.

On the part of discordance sets, $M^-_{A_4}(x_2)(=0.3997) \leqslant M^-_{A_1}(x_2)(=0.5119)$, $M^+_{A_4}(x_2)(=0.4897) \leqslant M^+_{A_1}(x_2)(=0.6238)$, and $\pi_{A_4}(x_2)(=0.0900) \leqslant \pi_{A_1}(x_2)(=0.1119)$. Thus, the midrange discordance set D'_{41} is:

$$D'_{41} = \{ j \mid A_4(x_j) \leqslant A_1(x_j) \text{ and } \pi_{A_4}(x_j) \leqslant \pi_{A_1}(x_j) \} = \{2\}.$$

Moreover, on account of $M^-_{A_4}(x_3)(=0.0492) \leqslant M^-_{A_1}(x_3)(=0.1640)$ and $M^+_{A_4}(x_3)(=0.6926) \geqslant M^+_{A_1}(x_3)(=0.3414)$, we have the weak discordance set D''_{41} is:

$$D''_{41} = \{ j \mid A_4(x_j) \preccurlyeq A_1(x_j) \} = \{3\}.$$

The discordance testing result concerning alternatives A_4 and A_1 is $\frac{2}{3}w_2 + \frac{1}{3}w_3$.

The rest of P_{16} is conducted in a similar manner. The complete testing results of P_{16} are presented in the following matrix:

$$\begin{array}{c} \\ 3 \\ 2 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 2 & 4 & 1 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3}w_1 + w_5 & \frac{1}{3}w_3 + w_4 & \frac{1}{3}w_4 + w_5 \\ \frac{1}{3}w_2 + \frac{1}{3}w_3 + \frac{2}{3}w_4 & 0 & \frac{1}{3}w_3 + \frac{2}{3}w_4 & w_3 + w_4 + w_5 \\ \frac{2}{3}w_1 + \frac{1}{3}w_2 + \frac{1}{3}w_5 & w_1 + w_2 + w_5 & 0 & \frac{2}{3}w_1 + \frac{1}{3}w_4 + w_5 \\ \frac{2}{3}w_1 + \frac{1}{3}w_2 + \frac{1}{3}w_3 & \frac{1}{3}w_1 + w_2 & \frac{2}{3}w_2 + \frac{1}{3}w_3 & 0 \end{bmatrix} \end{array}.$$

Applying Step 6, the evaluation value of P_{16}, $E(P_{16})$, is:

$$\begin{aligned} E(P_{16}) &= \sum_{j\in C_{kl}} w_j + \frac{2}{3}\sum_{j\in C'_{kl}} w_j + \frac{1}{3}\sum_{j\in C''_{kl}} w_j - \left(\sum_{j\in D_{kl}} w_j + \frac{2}{3}\sum_{j\in D'_{kl}} w_j + \frac{1}{3}\sum_{j\in D''_{kl}} \right) \\ &= -\frac{5}{3}w_1 - \frac{11}{3}w_2 + \frac{2}{3}w_3 + \frac{8}{3}w_4 + \frac{8}{3}w_5, \end{aligned}$$

where $\sum_{j\in C_{kl}} w_j + \frac{2}{3}\sum_{j\in C'_{kl}} w_j + \frac{1}{3}\sum_{j\in C''_{kl}} w_j$ is the sum of the upper-triangular elements of the above matrix in accordance with the hypothesis: $A_3 \geqslant A_2 \geqslant A_4 \geqslant A_1$, and $\sum_{j\in D_{kl}} w_j + \frac{2}{3}\sum_{j\in D'_{kl}} w_j + \frac{1}{3}\sum_{j\in D''_{kl}} w_j$ is the sum of the lower-triangular elements in conflict with the hypothesis. Since the grades of attribute importance here are given by a set of interval-valued fuzzy weights, we solve the following LP problem by Simplex method according to Step 6-2:

$$\begin{aligned} \max \quad & E(P_{16}) = -\frac{5}{3}w_1 - \frac{11}{3}w_2 + \frac{2}{3}w_3 + \frac{8}{3}w_4 + \frac{8}{3}w_5, \\ \text{subject to} \quad & 0.1228 \leqslant w_1 \leqslant 0.8670, \\ & 0.0030 \leqslant w_2 \leqslant 0.0684, \\ & 0.0879 \leqslant w_3 \leqslant 0.5912, \\ & 0.5527 \leqslant w_4 \leqslant 0.8887, \\ & 0.1387 \leqslant w_5 \leqslant 0.6718, \\ & w_1 + w_2 + w_3 + w_4 + w_5 = 1. \end{aligned}$$

The optimal solution can be obtained as follows:

$$\bar{\mathbf{w}}^{16} = \left(\bar{w}_1^{16}, \bar{w}_2^{16}, \bar{w}_3^{16}, \bar{w}_4^{16}, \bar{w}_5^{16}\right)^T = (0.1228, 0.0030, 0.0879, 0.5527, 0.2336)^T.$$

Correspondingly, the optimal evaluation value of P_{16}, $\overline{E(P_{16})}$, can be computed as follows:

$$\overline{E(P_{16})} = -\frac{5}{3}\bar{w}_1^{16} - \frac{11}{3}\bar{w}_2^{16} + \frac{2}{3}\bar{w}_3^{16} + \frac{8}{3}\bar{w}_4^{16} + \frac{8}{3}\bar{w}_5^{16} = 1.9397.$$

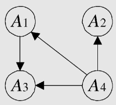
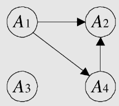
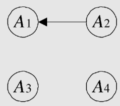
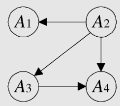
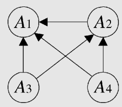
The optimal evaluation values $\overline{E(P_i)}$'s of 24 permutations can be derived in a similar way. The results are:

$$\begin{array}{llll}
\overline{E(P_1)} = -0.4194, & \overline{E(P_2)} = -1.2619, & \overline{E(P_3)} = -0.9209, & \overline{E(P_4)} = -0.9977, \\
\overline{E(P_5)} = -1.3387, & \overline{E(P_6)} = -1.8402, & \overline{E(P_7)} = 1.2095, & \overline{E(P_8)} = 0.2405, \\
\overline{E(P_9)} = 1.6943, & \overline{E(P_{10})} = 2.5045, & \overline{E(P_{11})} = 0.9242, & \overline{E(P_{12})} = 1.4089, \\
\overline{E(P_{13})} = -0.5311, & \overline{E(P_{14})} = -0.6395, & \overline{E(P_{15})} = 1.1295, & \overline{E(P_{16})} = 1.9397, \\
\overline{E(P_{17})} = 0.2024, & \overline{E(P_{18})} = 1.8629, & \overline{E(P_{19})} = -0.6549, & \overline{E(P_{20})} = -1.0300, \\
\overline{E(P_{21})} = 0.7525, & \overline{E(P_{22})} = 1.2689, & \overline{E(P_{23})} = -0.6401, & \overline{E(P_{24})} = 1.0204.
\end{array}$$

In applying Step 7 to this example, $\overline{E(P_{10})} = 2.5045$ gives the maximum value. Thus, the best order of the alternatives is $P_{10} = (A_2, A_3, A_4, A_1)$ (i.e., $A_2 \geqslant A_3 \geqslant A_4 \geqslant A_1$) and the best choice is Brand A_2.

4.2. Case of ordinal evaluations of alternatives on each attribute

Consider the same sneakers-choice problem, but the input data are replaced with ordinal evaluations. We adopt a simple method to produce attributewise ranks based on the original interval-valued data, i.e., the inequality relation $\geqslant$ is designated as "better than," $\leqslant$ as "worse than," and $\not\geqslant$ and $\not\leqslant$ as "noncomparable." The original interval-valued fuzzy decision matrix roughly renders the following overranking relationships, and these relationships are well illustrated by the graphical representation.

Attribute	x_1	x_2	x_3	x_4	x_5
Outranking relationship	$A_1 > A_3$ $A_4 > A_1$ $A_4 > A_2$ $A_4 > A_3$	$A_1 > A_2$ $A_1 > A_4$ $A_4 > A_2$	$A_2 > A_1$	$A_2 > A_1$ $A_2 > A_3$ $A_2 > A_4$ $A_3 > A_4$	$A_2 > A_1$ $A_3 > A_1$ $A_3 > A_2$ $A_4 > A_1$ $A_4 > A_2$
Graphical representation					

Take attribute x_2 for example. Since $M^-_{A_1}(x_2)(= 0.5119) > M^-_{A_2}(x_2)(= 0.2561)$ and $M^+_{A_1}(x_2)(= 0.6238) > M^+_{A_2}(x_2)(= 0.3808)$, alternative A_1 is better than A_2 with respect to x_2 and denoted as $A_1 > A_2$. In a similar way, we get $A_1 > A_4$ and $A_4 > A_2$. As mentioned before, the attributewise preference in the weak concordance set C''_{kl} or the weak discordance set D''_{kl} belongs to the noncomparable relations; thus, attribute x_2 has tied attributewise rankings between A_1 and A_3, A_2 and A_3, A_3 and A_4, respectively.

From the ordinal information of attributewise outranking relationships, we apply Eqs. (20) and (21) in Step 3-2 to reconstruct a new interval-valued fuzzy decision matrix D'.

$$D' = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} [0.3333, 0.6667] & [0.6667, 1.0000] & [0.0000, 0.6667] & [0.0000, 0.6667] & [0.0000, 0.0000] \\ [0.0000, 0.6667] & [0.0000, 0.3333] & [0.3333, 1.0000] & [1.0000, 1.0000] & [0.3333, 0.3333] \\ [0.0000, 0.3333] & [0.0000, 1.0000] & [0.0000, 1.0000] & [0.3333, 0.6667] & [0.6667, 1.0000] \\ [1.0000, 1.0000] & [0.3333, 0.6667] & [0.0000, 1.0000] & [0.0000, 0.3333] & [0.6667, 1.0000] \end{bmatrix} \end{array}$$

$$= \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{bmatrix} (0.3333, 0.3333) & (0.6667, 0.0000) & (0.0000, 0.3333) & (0.0000, 0.3333) & (0.0000, 1.0000) \\ (0.0000, 0.3333) & (0.0000, 0.6667) & (0.3333, 0.0000) & (1.0000, 0.0000) & (0.3333, 0.6667) \\ (0.0000, 0.6667) & (0.0000, 0.0000) & (0.0000, 0.0000) & (0.3333, 0.3333) & (0.6667, 0.0000) \\ (1.0000, 0.0000) & (0.3333, 0.3333) & (0.0000, 0.0000) & (0.0000, 0.6667) & (0.6667, 0.0000) \end{bmatrix}.$$

Given D' and W, we implement the IVFP method again. Taking $P_{16}(= (A_3, A_2, A_4, A_1))$ for example, the testing results are presented in the following matrix:

$$\begin{array}{c} \\ 3 \\ 2 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 2 & 4 & 1 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3}w_1 + \frac{2}{3}w_2 + \frac{2}{3}w_5 & \frac{1}{3}w_3 + \frac{2}{3}w_4 + \frac{1}{3}w_5 & \frac{2}{3}w_3 + \frac{1}{3}w_4 + \frac{2}{3}w_5 \\ w_3 + w_4 & 0 & \frac{1}{3}w_3 + w_4 & \frac{2}{3}w_3 + w_4 + \frac{2}{3}w_5 \\ w_1 + \frac{1}{3}w_2 & w_1 + \frac{2}{3}w_2 + \frac{2}{3}w_5 & 0 & w_1 + \frac{2}{3}w_3 + \frac{1}{3}w_4 + \frac{2}{3}w_5 \\ \frac{2}{3}w_1 + w_2 & w_1 + \frac{2}{3}w_2 & \frac{2}{3}w_2 & 0 \end{bmatrix} \end{array}.$$

The evaluation value $E(P_{16})$ is equal to $-\frac{7}{3}w_1 - \frac{8}{3}w_2 + \frac{5}{3}w_3 + \frac{7}{3}w_4 + \frac{7}{3}w_5$. Next, the following LP problem can be obtained:

$$\begin{aligned}
\max \quad & E(P_{16}) = -\frac{7}{3}w_1 - \frac{8}{3}w_2 + \frac{5}{3}w_3 + \frac{7}{3}w_4 + \frac{7}{3}w_5,\\
\text{subject to} \quad & 0.1228 \leqslant w_1 \leqslant 0.8670,\\
& 0.0030 \leqslant w_2 \leqslant 0.6884,\\
& 0.0879 \leqslant w_3 \leqslant 0.5912,\\
& 0.5527 \leqslant w_4 \leqslant 0.8887,\\
& 0.1387 \leqslant w_5 \leqslant 0.6718,\\
& w_1 + w_2 + w_3 + w_4 + w_5 = 1.
\end{aligned}$$

Solving the above LP, its optimal solution can be acquired as follows:

$$\bar{\mathbf{w}}^{16} = \left(\bar{w}_1^{16}, \bar{w}_2^{16}, \bar{w}_3^{16}, \bar{w}_4^{16}, \bar{w}_5^{16}\right)^T = (0.1228, 0.0030, 0.0879, 0.5527, 0.2336)^T.$$

The optimal evaluation value of P_{16}, $\overline{E(P_{16})}$, can be calculated as follows:

$$\overline{E(P_{16})} = -\frac{7}{3}\bar{w}_1^{16} - \frac{8}{3}\bar{w}_2^{16} + \frac{5}{3}\bar{w}_3^{16} + \frac{7}{3}\bar{w}_4^{16} + \frac{7}{3}\bar{w}_5^{16} = 1.6867.$$

The optimal evaluation values $\overline{E(P_i)}$'s of 24 permutations can be derived in a similar way. The results are:

$$\begin{array}{llll}
\overline{E(P_1)} = 0.3745, & \overline{E(P_2)} = -0.0832, & \overline{E(P_3)} = -0.7031, & \overline{E(P_4)} = -1.3646,\\
\overline{E(P_5)} = -0.7447, & \overline{E(P_6)} = -1.7907, & \overline{E(P_7)} = 1.8042, & \overline{E(P_8)} = 1.2199,\\
\overline{E(P_9)} = 2.7141, & \overline{E(P_{10})} = 2.9540, & \overline{E(P_{11})} = 1.5231, & \overline{E(P_{12})} = 2.3381,\\
\overline{E(P_{13})} = 0.0171, & \overline{E(P_{14})} = -0.7077, & \overline{E(P_{15})} = 1.4467, & \overline{E(P_{16})} = 1.6867,\\
\overline{E(P_{17})} = -0.3412, & \overline{E(P_{18})} = 1.0251, & \overline{E(P_{19})} = -0.3150, & \overline{E(P_{20})} = -1.4242,\\
\overline{E(P_{21})} = 0.7983, & \overline{E(P_{22})} = 1.4868, & \overline{E(P_{23})} = -0.7673, & \overline{E(P_{24})} = 0.5358,
\end{array}$$

where $\overline{E(P_{10})} = 2.9540$ gives the maximum value. Therefore, the best order of the alternatives is $P_{10} = (A_2, A_3, A_4, A_1)$ and the best alternative is A_2. The point to observe is that this result matches the solution of the cardinal evaluations of alternatives given.

In the case of interval-valued fuzzy weights, no matter what type of evaluation values is given, the numerical examples show the same results of best order for the alternatives. What is true for the illustrative examples could be to a considerable extent true for general cases as well. If there are no significantly different results between the methods with cardinal evaluations and with ordinal evaluations, this phenomenon implies that the data requirement in the proposed method can be simplified. That is, the cardinal evaluations of alternatives can be replaced with ordinal ones to construct the decision matrix required in the IVFP method. For the above reasons, test problems for the cases of cardinal and ordinal evaluations will be generated, and a simulation validation of different given cases will be investigated.

5. Design of computational experiments

The computational experiments will be conducted in a similar manner to the analysis of the illustrative examples. Fig. 1 depicts four classes of experimental scenarios based on the solution approach and the data type of alternative evaluations. The input data of Scenario I (upper left-hand side) include a set of cardinal weights and cardinal evaluations of alternatives based on IVFSs. The required data in Scenario II (upper right-hand side) consists of cardinal weights and ordinal evaluations based on IVFSs. Scenario III (lower left-hand side) refers to interval-valued fuzzy weights and cardinal evaluations, while Scenario IV (lower right-hand side) indicates interval-valued fuzzy weights and ordinal evaluations. When interval-valued fuzzy weights are given, the exact value of attribute importance must be derived by using a LP approach for solving Eq. (17). Nevertheless, it has no need to use the LP approach when given weights are just scalars. The experimental analysis is intended to compare the IVFP results according to cardinal and ordinal evaluations when the non-LP and LP approaches are separately used.

A MATLAB computer program is written to generate random data and to solve MCDA problems with all possible combinations of $3, 4, 5, \ldots, 10$ alternatives and $3, 4, 5, \ldots, 10$ attributes. Hence, $64(= 8 \times 8)$ different instances will be examined in this study. Let us consider an IVFS $A_i \in X$. First, two real numbers, η_1 and η_2, are uniformly distributed over the interval [0, 1]. Next, let $M_{A_i}^-(x_j) = \min\{\eta_1, \eta_2\}$ (or equivalently, $\mu_{A_i}(x_j) = \min\{\eta_1, \eta_2\}$) and $M_{A_i}^+(x_j) = \max\{\eta_1, \eta_2\}$ (or equivalently, $\nu_{A_i}(x_j) = 1 - \max\{\eta_1, \eta_2\}$). Then, $\pi_{A_i}(x_j) = 1 - \mu_{A_i}(x_j) - \nu_{A_i}(x_j)$. In a similar manner, the simulation data of $[M_W^-(x_j), M_W^+(x_j)]$'s can be randomly generated, but more noteworthy are the conditions of $\sum M_W^-(x_j) \leqslant 1$ and

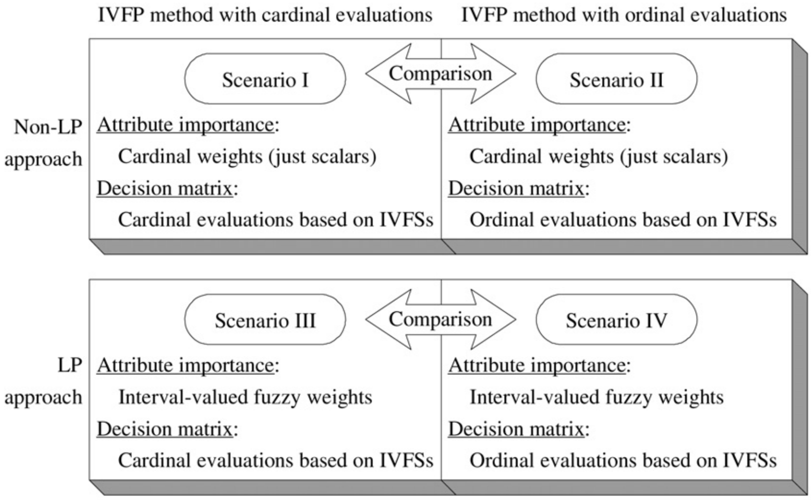


Fig. 1. Four types of experimental scenarios.

$\sum M_W^+(x_j) \geqslant 1$. As for cardinal weights, let w_j be uniformly distributed over [0, 1] and all w_j's must satisfy $\sum w_j = 1$. For each instance in Scenarios I and III, 1000 different interval-valued fuzzy decision matrices D's and subjective importance of attributes w_j's and W's, respectively, are randomly produced.

On the other hand, in both Scenarios II and IV, their decision matrices need only ordering or ranking information. Thus, we specify the attributewise ranks of alternatives according to the interval-valued fuzzy decision matrix D in Scenarios I and III. Then, transform these ordinal data into new interval-valued fuzzy data that probably differ from the original IVFSs in D. For each A_i with respect to x_j, determine $\alpha_j(A_i)$ which is equal to the number of alternatives surely worse than A_i (i.e., $A_i(x_j) \geqslant A_k(x_j)$ and $A_i(x_j) \neq A_k(x_j)$ for $k \in \{1, 2, \ldots, i-1, i+1, \ldots, m\}$). Similarly, $\beta_j(A_i)$ is equal to the number of alternatives surely better than A_i (i.e., $A_i(x_j) \leqslant A_k(x_j)$ and $A_i(x_j) \neq A_k(x_j)$ for $k \in \{1, 2, \ldots, i-1, i+1, \ldots, m\}$). Applying Eqs. (20) and (21) (or Eqs. (18) and (19)) we can get new interval-valued fuzzy data. In such a way, 1000 interval-valued fuzzy decision matrices for each instance in Scenarios II and IV are generated according to random data in Scenarios I and III, respectively. In addition, the subjective importance of attributes in Scenarios II and IV coincides with attribute importance in Scenarios I and III, respectively. Therefore, it follows from what has been mentioned that a total of $256{,}000 (= 64 \times 1000 \times 4)$ sets of experimental cases are generated for four scenarios.

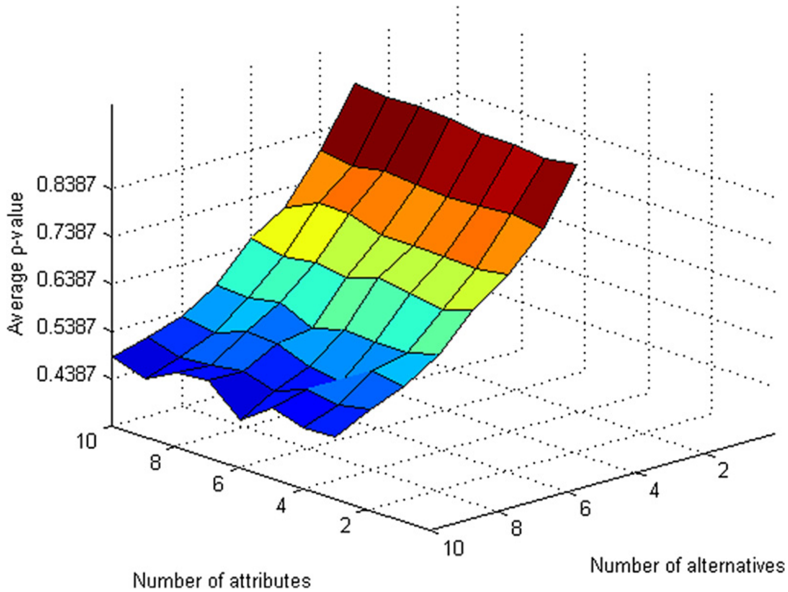
In order to compare the IVFP results yielded by cardinal and ordinal evaluations, several approaches are applied to determine whether the similar results or not. We conduct a comprehensive comparative study of preference rank orders, consisting of average Spearman correlation coefficients, consistency rates, contradiction rates of the best alternative, and inversion rates between the better alternatives and the worse ones. Finally, a second-order regression will be further implemented to realize the influence of the number of alternatives, the number of attributes, and non-LP and LP approaches on the mean of Spearman correlation coefficients. In the following, we present the major computational results and comparison analysis.

6. Analysis of computational results

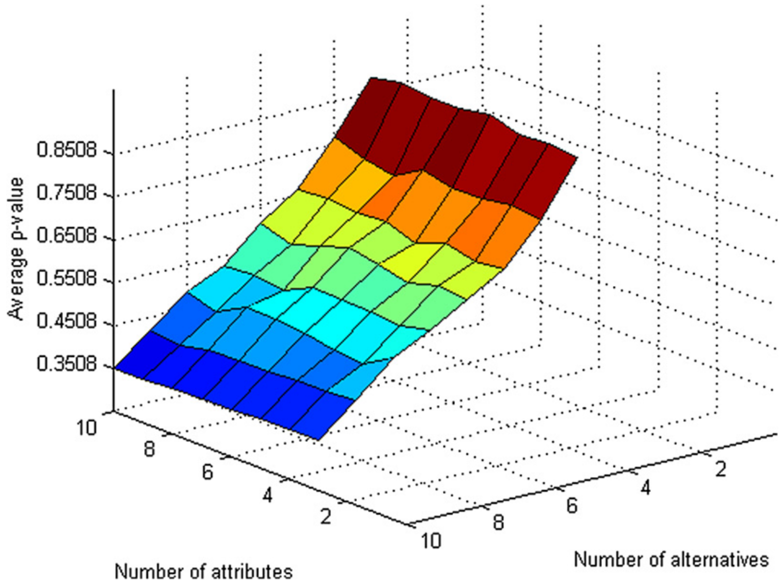
6.1. Spearman correlation coefficients

The first examination approach is comparison of best order derived from cardinal and ordinal evaluations by using average Spearman correlation coefficients. We compute the mean of Spearman correlation coefficients of 1000 experiment observations for non-LP and LP approaches, respectively. The results are presented in Fig. 2 and the detailed figures are revealed in Tables 1 and 2. As the plots in Fig. 2 illustrate, the preference orders between cardinal and ordinal evaluations have very high Spearman correlation coefficients in small m values. The highest coefficients are 0.9164 in pair (Scenario I, Scenario II) and 0.9023 in pair (Scenario III, Scenario IV). For both non-LP and LP approaches, the average Spearman correlation coefficients are around 0.6 to 0.9 when $m < 7$. Moreover, the fewer alternatives are involved, the more likely it is that the ranking orders between cardinal and ordinal evaluations will be highly related.

For each pair of cardinal and ordinal evaluations, there exists a consistent trend that the mean of Spearman correlation coefficients decreases with the number of alternatives. Besides, the standard deviations of Spearman correlation coefficients are almost around 0.2 or 0.3. Hence, the discrepancy of average Spearman correlation coefficients is moderately unobvious as a whole. On the other hand, the number of attributes produces no significant effects upon Spearman correlation coefficients. The mean of Spearman correlation coefficients undergoes little change as the number of attributes increases. In addition, as n increases, no special trend was found regarding the standard deviation of Spearman correlation coefficients.



(a) Average ρ-values of (Scenario I, Scenario II).



(b) Average ρ-values of (Scenario III, Scenario IV).

Fig. 2. Experimental results: average Spearman correlation coefficients (ρ).

6.2. Ranking consistency

The consistency rate measures the level of concordance between two complete preference orders yielded by different types of evaluations for each $m \times n$ combination. The results in Fig. 3 correspond to 1000 experiment observations for each pairwise comparison. The computational results indicate that the consistency rates are rather high (i.e., around 80% to 50% or 40%) when the number of alternatives in a decision problem is rather small (i.e., equal to 3, 4 or 5). Therefore, when m is small, there has high percentage that the overall preference ranking of alternatives based on ordinal evaluations completely matches the solution based on cardinal evaluations. Nevertheless, as m increases, the consistency rates gradually decrease. The consistency rate approaches below 10% when the value of m is greater than 8 for non-LP and LP situations. As sketched here, it seems reasonable to recognize that the preference orders using cardinal and ordinal data will be unlikely identical when m is greater than 10.

Also in the same figure, the influence of the number of attributes on consistency rates does not seem to be important. This is indicated by the closeness of the curves that correspond to different numbers of attributes. No matter how

Table 1
Average Spearman correlation coefficients of pair (Scenario I, Scenario II).

Number of alternatives	Number of attributes							
	3	4	5	6	7	8	9	10
3	0.9020 (0.1986)[a]	0.8916 (0.2061)	0.8998 (0.2003)	0.8995 (0.2005)	0.9114 (0.1910)	0.9138 (0.1890)	0.9113 (0.1911)	0.9164 (0.1867)
4	0.7866 (0.2998)	0.7988 (0.2954)	0.7922 (0.3030)	0.7888 (0.2979)	0.7918 (0.3041)	0.8002 (0.2965)	0.7874 (0.3042)	0.7976 (0.2972)
5	0.7099 (0.3159)	0.7005 (0.3196)	0.7046 (0.3287)	0.7066 (0.3289)	0.7062 (0.3285)	0.7275 (0.3249)	0.7288 (0.3154)	0.6958 (0.3268)
6	0.6562 (0.3193)	0.6354 (0.3266)	0.6493 (0.3212)	0.6589 (0.3175)	0.6367 (0.3267)	0.6470 (0.3280)	0.6364 (0.3284)	0.6530 (0.3086)
7	0.5828 (0.3158)	0.5639 (0.3175)	0.5639 (0.3141)	0.5672 (0.3227)	0.5480 (0.3187)	0.5814 (0.3189)	0.5709 (0.3114)	0.5748 (0.3183)
8	0.5376 (0.2968)	0.5447 (0.3002)	0.5012 (0.2910)	0.5105 (0.2978)	0.5389 (0.2943)	0.5406 (0.3030)	0.5147 (0.2914)	0.5304 (0.2982)
9	0.5075 (0.2814)	0.4948 (0.2812)	0.5054 (0.2832)	0.4817 (0.2855)	0.5010 (0.2854)	0.4914 (0.2930)	0.4834 (0.2821)	0.5040 (0.2893)
10	0.4708 (0.2724)	0.4689 (0.2682)	0.4871 (0.2841)	0.4387 (0.2578)	0.5012 (0.2734)	0.4960 (0.2946)	0.4592 (0.2703)	0.4812 (0.2956)

[a] Standard deviations are in parentheses.

Table 2
Average Spearman correlation coefficients of pair (Scenario III, Scenario IV).

Number of alternatives	Number of attributes							
	3	4	5	6	7	8	9	10
3	0.8821 (0.2124)[a]	0.8878 (0.2087)	0.8840 (0.2112)	0.9023 (0.1984)	0.8901 (0.2072)	0.8906 (0.2068)	0.9012 (0.1992)	0.8858 (0.2101)
4	0.7581 (0.3269)	0.7645 (0.3144)	0.7529 (0.3285)	0.7510 (0.3257)	0.7677 (0.3148)	0.7296 (0.3317)	0.7423 (0.3311)	0.7666 (0.3174)
5	0.6637 (0.3402)	0.6530 (0.3474)	0.6724 (0.3377)	0.6463 (0.3507)	0.6672 (0.3392)	0.6657 (0.3392)	0.6688 (0.3386)	0.6661 (0.3434)
6	0.6140 (0.3263)	0.6184 (0.3263)	0.5991 (0.3271)	0.6158 (0.3290)	0.6276 (0.3249)	0.6071 (0.3237)	0.5855 (0.3307)	0.6068 (0.3331)
7	0.5622 (0.3105)	0.5452 (0.3143)	0.5516 (0.3169)	0.5529 (0.3070)	0.5573 (0.3228)	0.5227 (0.3238)	0.5345 (0.3157)	0.5284 (0.3138)
8	0.5171 (0.3044)	0.4777 (0.3013)	0.4824 (0.3022)	0.4977 (0.2982)	0.5035 (0.2993)	0.5055 (0.3103)	0.4683 (0.2974)	0.4876 (0.2960)
9	0.4392 (0.2883)	0.4343 (0.2840)	0.4318 (0.2835)	0.4293 (0.2830)	0.4268 (0.2930)	0.4242 (0.2819)	0.4108 (0.2400)	0.4192 (0.2808)
10	0.3684 (0.2694)	0.3659 (0.2688)	0.3634 (0.2683)	0.3609 (0.2678)	0.3584 (0.2672)	0.3558 (0.2667)	0.3533 (0.2661)	0.3508 (0.2656)

[a] Standard deviations are in parentheses.

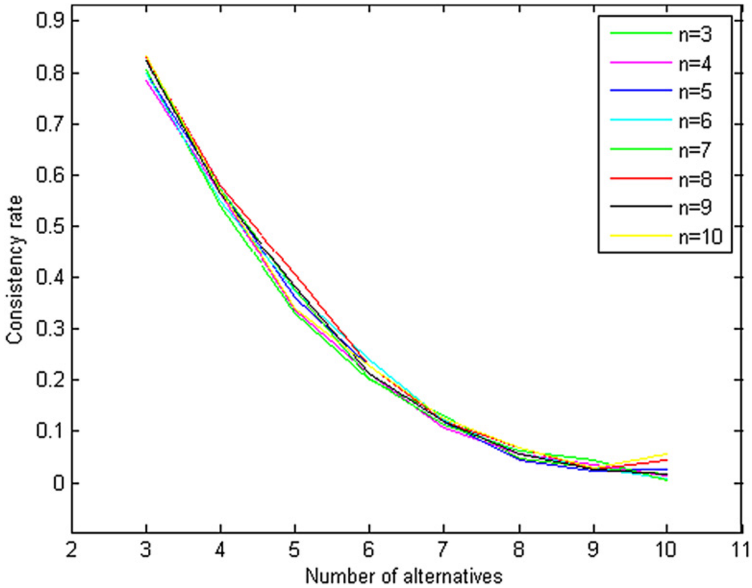
many attributes are considered, the results of consistency rates yield similar patterns. Therefore, changes in the number of alternatives are more meaningful toward consistency rates than changes in the number of attributes.

6.3. Ranking contradiction

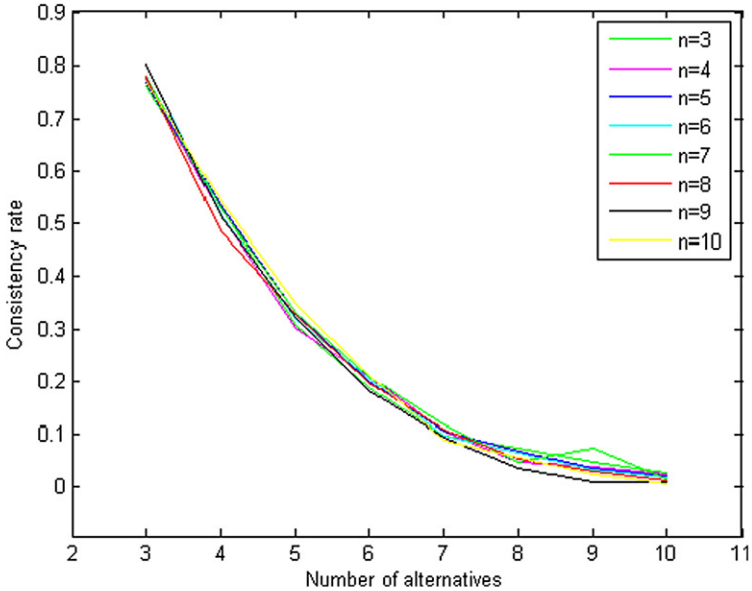
Two kinds of ranking inconsistency merit to be examined. The first kind is the contradiction rate of the best alternative in the chosen permutation. Since decision makers are always concerned about the best alternative, the frequency of matching the top rank seems to be more important than matching all ranks. Thus, we further observe the contradiction rate of the top rank between two results using cardinal and ordinal evaluations. For example, if the ranking of a set of six alternatives is equal to $(\underline{4}, 1, 5, 2, 6, 3)$ (i.e., $A_4 \geqslant A_1 \geqslant A_5 \geqslant A_2 \geqslant A_6 \geqslant A_3$) based on cardinal evaluations and the other method using ordinal evaluations yields $(\underline{1}, 4, 5, 2, 6, 3)$, then a case of a ranking contradiction of the best alternative has occurred.

Fig. 4 shows the contradiction rate for the best alternative. The contradiction rate gently increases with the number of alternatives, but it seems to have a little irregular pattern with the number of attributes. Among $m \times n$ combinations for pair (Scenario I, Scenario II), the case of $m = 3$ and $n = 10$ has the lowest contradiction rate (0.0796) and thus has largest concordance proportion of the best choice; whereas the case of $m = 10$ and $n = 6$ receives the highest contradiction rate (0.3144) and become the less common top choices. The contradiction rate (0.0855) of the best alternative in the cases of $m = 3$ and $n = 9$ is the smallest among the rest of $m \times n$ combinations for pair (Scenario III, Scenario IV). On the contrary, the contradiction rate (0.4966) in the cases of $m = n = 10$ is relatively higher than the rest.

Fig. 4 depicts a phenomenon that most of the contradiction rates lies in 0.1 to 0.3 for pair (Scenario I, Scenario II) and 0.1 to 0.5 for pair (Scenario III, Scenario IV). In regard to non-LP case, the probability that the most preferred alternative using cardinal and ordinal evaluations are contradictory is estimated to be 10 to 30 percent. It implies that the top choice yielded



(a) The consistency rates of (Scenario I, Scenario II).



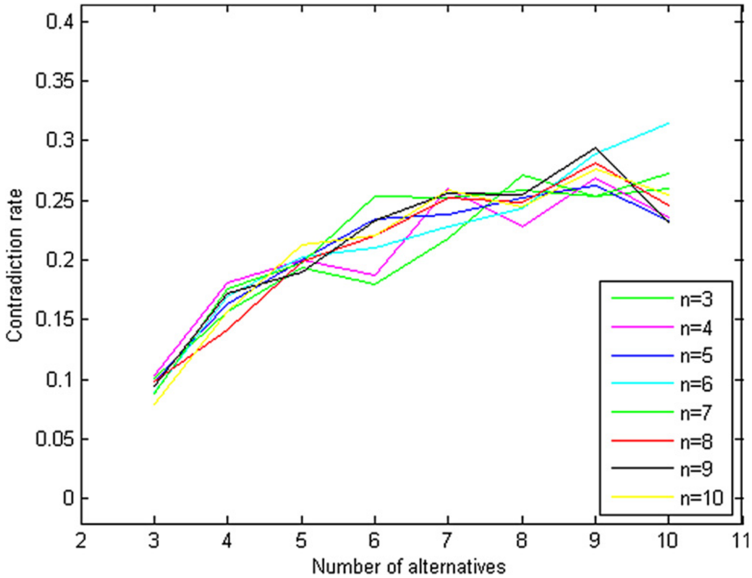
(b) The consistency rates of (Scenario III, Scenario IV).

Fig. 3. Experimental results: the consistency rates. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

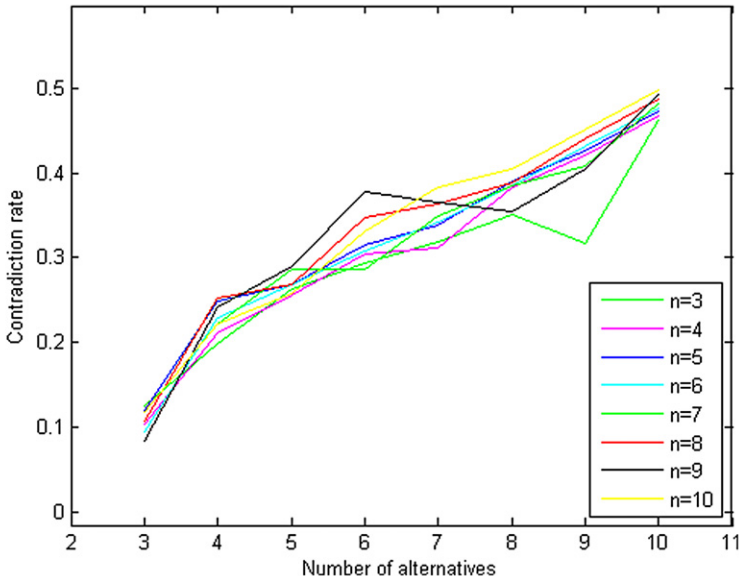
by using ordinal evaluations in the IVFP method is often in common with the selected alternative obtained by cardinal data. The concordance proportion of the top choice is, on average, up to the range of 70–90%.

6.4. Ranking inversion

The second kind of ranking inconsistency is the inversion rate between the better alternatives and the worse alternatives. Let better alternatives denote the first half alternatives in the final ranking; similarly, worse alternatives for the last half. The event that one of the better alternatives by using cardinal evaluations becomes the worse one by using ordinal evaluations will cause decision makers quite a confusion, and vice versa. The higher the degree of ranking inversion, the more difficult the final decision. As an example, if a ranking based on ordinal evaluations of a set of six alternatives is equal to $(\underbrace{6, 4, 1}, \underline{3, 5, 2})$ but cardinal data yielded $(\underbrace{6, 5, 1}, \underline{3, 4, 2})$, then a case of a ranking inversion between the better alternatives and the worse ones has occurred. Notice that the determination approach of better and worse alternatives is rounding to the nearest integer. The actual numbers of better and worse alternatives are not equal proportion when m is odd.



(a) The contradiction rates of (Scenario I, Scenario II).



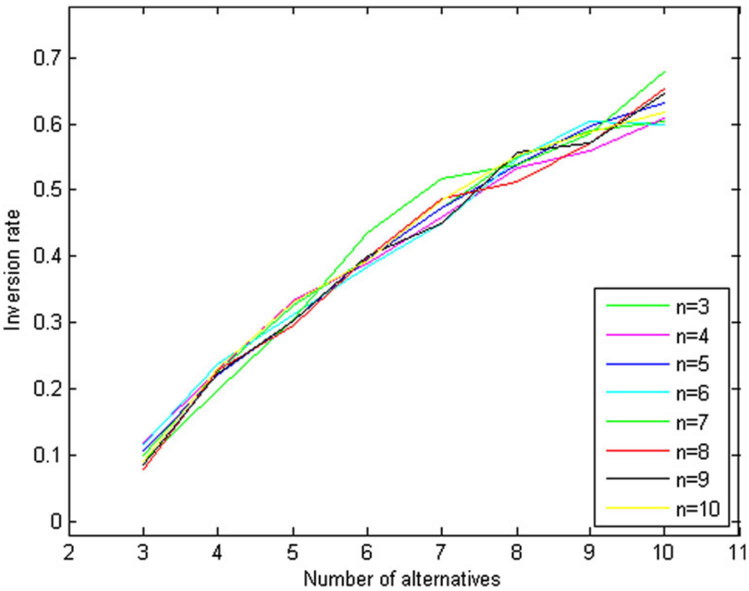
(b) The contradiction rates of (Scenario III, Scenario IV).

Fig. 4. Experimental results: the contradiction rates. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

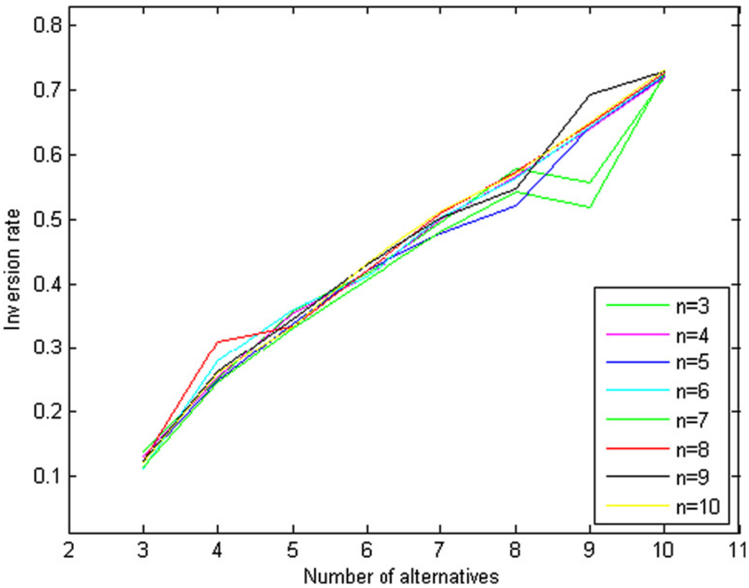
Fig. 5 illustrates the inversion rate between the better alternatives and the worse ones. For each pair of scenarios, it should be noted that the inversion rates for all experimental instances constitute increasing curves. Most of the inversion rates rise from around 0.1 (when $m = 3$) to 0.6 (when $m = 10$). For pair (Scenario I, Scenario II), the minimal inversion rate (0.0793) occurs in the case of $m = 3$ and $n = 8$, while the maximum (0.6777) in the case of $m = 10$ and $n = 7$. For pair (Scenario III, Scenario IV), the minimal inversion rate (0.1137) can be found in the case of $m = 3$ and $n = 6$, while the maximum (0.7295) in the case of $m = n = 10$. When m is small (e.g., 3, 4, 5), the better and worse alternatives yielded by ordinal evaluations are generally consistent with the ones by cardinal evaluations. In consequence of ranking inconsistency, the contradiction and inversion rates have an increasing trend with the number of alternatives, whereas no apparent features have been found with regard to the number of attributes.

6.5. Second-order regression model

In order to get an understanding of the shape of the graph in the above figures, we further conduct a regression analysis. We use a second-order regression model to capture the relationship of the number of alternatives, number of attributes, and



(a) The inversion rates of (Scenario I, Scenario II).



(b) The inversion rates of (Scenario III, Scenario IV).

Fig. 5. Experimental results: the inversion rates. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

different scenario pairs to average Spearman correlation coefficients. From the previous analysis, it is found that there is a negative relationship between the number of alternatives and Spearman correlation coefficients, and no obvious relationship between the number of attributes and Spearman correlation coefficients. However, it is also found that the relationships are not linear: the effects seem to be decreasing as the number of alternatives increases. In addition, the effects also little differ in non-LP and LP approaches. Let $z_{(\mathrm{III,IV})}$ be a dummy variable which is equal to 1 if the correlation coefficient is obtained from pair (Scenario III, Scenario IV) (i.e., LP approach). The second-order regression model relates a dependent variable ρ to a set of independent variables involving m, m^2, n, mn, $z_{(\mathrm{III,IV})}$, $z_{(\mathrm{III,IV})}m$, and $z_{(\mathrm{III,IV})}n$. Let ϵ be the random term representing the effects caused by other factors that are not considered in this model. We assume that ϵ is an independent random variable with finite mean and variance. In sums, we consider the following regression model:

$$\rho = \beta_0 + \beta_1 m + \beta_2 n + \beta_3 m^2 + \beta_4 mn + \beta_5 z_{(\mathrm{III,IV})} + \beta_6 z_{(\mathrm{III,IV})} m + \beta_7 z_{(\mathrm{III,IV})} n + \epsilon. \tag{22}$$

The total sample size is 128 (i.e., 8 different number of alternatives ×8 different number of attributes ×2 different pairs of scenarios). The results are listed in Table 3. Most of the coefficients are significant under 95% significant level except for

Table 3
The second-order regression model of Spearman correlation coefficients.

Variable	Coefficient (standard deviation)	Standardized coefficient	t-statistic	p-value
constant	1.216 (0.032)		38.025	0.000
m	−0.130 (0.008)	−1.935	−16.562	0.000
n	0.005 (0.003)	0.069	1.343	0.182
m^2	0.006 (0.001)	1.105	10.402	0.000
mn	−0.001 (0.000)	−0.090	−1.341	0.182
$z_{(\text{III,IV})}$	0.023 (0.021)	0.076	1.133	0.259
$z_{(\text{III,IV})}m$	−0.007 (0.002)	−0.169	−3.303	0.001
$z_{(\text{III,IV})}n$	−0.003 (0.002)	−0.070	−1.374	0.172

$F = 529.904$; $F(p - \text{value}) = 0.000$; $R^2 = 0.969$; and $\text{adj} - R^2 = 0.967$.

the variables of n, mn, $z_{(\text{III,IV})}$, and $z_{(\text{III,IV})}n$. Overall speaking, the model is significant in terms of F-test and the explanatory power is so high. The R^2 and the adjusted-R^2 are 0.969 and 0.967, respectively.

In order to examine the effects of the number of alternatives (m) and the number of attributes (n) on the correlation between cardinal and ordinal cases, the partial derivatives of estimated Spearman correlation coefficient ($\hat{\rho}$) with respect to m and n are obtained as follows:

$$\frac{\partial \hat{\rho}}{\partial m} = -0.130 + 0.012 \cdot m - 0.001 \cdot n - 0.007 \cdot z_{(\text{III,IV})}, \tag{23}$$

$$\frac{\partial \hat{\rho}}{\partial n} = 0.005 - 0.001 \cdot m - 0.003 \cdot z_{(\text{III,IV})}. \tag{24}$$

From Eqs. (23) and (24), it is found that the effect of the number of alternatives is negative in reference to a reasonable size of the choice set, and the effect of the number of attributes is mixed in general. That is, more alternatives make higher dissimilarity of the preference orders of alternatives under different given data and more attributes get the hybrid results. However, the negative effect of m decreases gradually when the numbers of alternatives become large. The interaction between m and n is negative which means that large n will enhance the effects of m on $\hat{\rho}$ but large m will obstruct the effects of n on $\hat{\rho}$. On the other hand, the effects of m and n are different between different given data. The negative effect of m on $\hat{\rho}$ is higher in the pair of (Scenario III, Scenario IV) relative to the pair of (Scenario I, Scenario II), but the relative effect of n on $\hat{\rho}$ is mitigated. From the magnitude of coefficients in Eqs. (23) and (24), the effect of the number of alternatives becomes greater and have more deviations.

In this study, the uniform distribution from the interval $[0, 1]$ was selected because it is the simplest and most widely statistical distribution used in numerous simulation investigations. However, it should be emphasized that the present simulation results might be contingent on how the random data, including the decision matrix and attribute importance, were generated. Other possibilities, such as assigning interval-valued fuzzy data from a normal distribution, would probably have slightly different computational results.

7. Conclusions

In this study, we have proposed a new decision method for multiattribute decision making under interval-valued fuzzy environments. In addition, we conducted computational experiments to analyze the difference between cardinal evaluations of each alternative with respect to each attribute and ordinal ones. The comparison results represent that the solutions based on ordinal and cardinal evaluations have median to high correlation coefficients and low contradiction rates. In addition, the cases when the number of alternatives is small have high consistency rates and low inversion rates. The above analysis demonstrates that the IVFP results based on ordinal evaluations can approximate the solution based on cardinal evaluations. Thus, based on a set of attributewise rankings (not necessarily numerical values) and a set of attribute weights, the IVFP method performed acceptable results in the computational experiments. This weaker information requirement is very attractive in that we do not need to scale the alternatives in terms of attributes.

Besides being able to determine the approximation of the best order for alternatives, the proposed method has certain advantages in real life applications. For data collection, all that is required is the attributewise rankings. Thus, we eliminate the tedious requirements of the existing compensatory MCDA models such as lengthy scaling procedures. Even though a lengthy data gathering effort is eliminated, the method does have satisfactory results through computational experimental analysis. However, with the increase of the number of alternatives, the number of permutations increases drastically. Fortunately, this implementation difficulty can be moderately surmounted with the help of powerful computer hardware.

The IVFP method presented in this paper is different from previous studies in a number of significant aspects. First, it can treat ordinal or cardinal evaluations of alternatives with respect to each attribute. Second, it can deal with ordering which are not necessarily linear orderings because there are alternatives which are noncomparable in some attributes. Third, it is originally designed for the cardinal preferences of attributes given, but it is also to be used for the ordinal preferences given if the decision maker is willing make an acceptable sacrifice in accuracy. Finally, comparing most of the MCDA methods, the proposed method does not require that the evaluation information of alternatives in each attribute be in numerical form. For each attribute, the alternatives can be merely ranked in terms of their performance; then a simplified version of the interval-valued fuzzy decision matrix can be constructed correspondingly. Because of its limited information requirements, the IVFP method is anticipated to have application values in MCDA reality.

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