Pointwise convergence of cone-like restricted two-dimensional $(C, 1)$ means of trigonometric Fourier series

György Gát

College of Nyíregyháza, Institute of Mathematics and Computer Science, Nyíregyháza, P.O. Box 166, H-4400, Hungary

Received 31 October 2005; received in revised form 8 August 2006; accepted 11 August 2006

Communicated by Vilmos Totik
Available online 12 June 2007

Abstract

The aim of this work is to generalize the more than 60 year old celebrated result of Marcinkiewicz and Zygmund on the convergence of the two-dimensional restricted $(C, 1)$ means of trigonometric Fourier series. They proved for any integrable function $f \in L^1(T^2)$ the a.e. convergence

$$\sigma_{(n_1,n_2)} f \to f$$

provided $\frac{n_1}{\beta} \leq n_2 \leq \beta n_1$, where $\beta > 1$ is fixed constant. That is, the set of indices $(n_1, n_2)$ remains in some positive cone around the identical function. We not only generalize this theorem, but give a necessary and sufficient condition for cone-like sets (of the set of indices) in order to preserve this convergence property.

© 2007 Elsevier Inc. All rights reserved.

MSC: 42B08

Keywords: Two-dimensional trigonometric Fourier series; $(C, 1)$ means; Cone-like restriction; Convergence; Divergence

1. Introduction

The question

What kind of restriction implies the convergence of the two-dimensional $(C, 1)$ means of trigonometric Fourier series of integrable functions?

E-mail addresses: gatgy@zeus.nyf.hu, gatgy@nyf.hu


0021-9045/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2006.08.006
The only example is due to Marcinkiewicz and Zygmund [6]. They proved the a.e. convergence of the two-dimensional \( \sigma_n \) (i.e. \((C, 1)\)) means of trigonometric Fourier series of integrable functions, where the set of indices is inside a cone around the identical function. The result of Marcinkiewicz and Zygmund was also proved in the book of Weisz [11]. We mention that Jessen, Marcinkiewicz and Zygmund also proved in [5] the a.e. convergence \( \sigma_n f \to f \) without any restriction on the indices, but not for functions in \( L^1 \). They proved this for a proper subspace.

This section contains a preliminary result and notions that are needed in formalizing the main theorems, given at the end of this section. The result presented here is an easy observation and the proof is tedious. Let \( \alpha : [1, +\infty) \to [1, +\infty) \) be a strictly monotone increasing continuous function with property \( \lim_{+\infty} \alpha = +\infty, \alpha(1) = 1, \) and \( \beta : [1, +\infty) \to [1, +\infty) \) be a monotone increasing function with property \( \beta(1) > 1 \).

**Definition 1.1.** Define the cone-like restriction sets of \( \mathbb{N}^2 \) as follows:

\[
\mathbb{N}_{\alpha, \beta, 1} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha(n_1)}{\beta(n_1)} \leq n_2 \leq \alpha(n_1)\beta(n_1) \right\},
\]

\[
\mathbb{N}_{\alpha, \beta, 2} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha^{-1}(n_2)}{\beta(n_2)} \leq n_1 \leq \alpha^{-1}(n_2)\beta(n_2) \right\}.
\]

For \( \alpha(x) = x, \beta(x) = \beta \in (1, +\infty) \) we have

\[
\mathbb{N}_{\alpha, \beta, 1} = \mathbb{N}_{\alpha, \beta, 2} = \left\{ n \in \mathbb{N}^2 : \frac{1}{\beta} \leq \frac{n_2}{n_1} \leq \beta \right\}
\]

the “ordinary” restriction set used by Marcinkiewicz and Zygmund (and others).

Now, let \( \tilde{\beta}(x) = \beta \in (1, +\infty) \) be a constant function. It is obvious that \( \mathbb{N}_{\alpha, \beta, 1} \subset \mathbb{N}_{\alpha, \beta, 2} \) and \( \mathbb{N}_{\alpha, \beta, 1} \subset \mathbb{N}_{\alpha, \beta, 2} \) for any \( \beta_1 \leq \beta_2 \). Let

\[
\mathbb{N}_{\alpha, i} := \left\{ \mathbb{N}_{\alpha, \beta, i} : \beta > 1 \right\}
\]

for \( i = 1, 2 \). Let \( i \in \{1, 2\} \). We say that \( \mathbb{N}_{\alpha, i} \) is weaker than \( \mathbb{N}_{\alpha, 3-i} \), if for all \( L \in \mathbb{N}_{\alpha, i} \) there exists an \( \tilde{L} \in \mathbb{N}_{\alpha, 3-i} \) such that:

\[
L \subset \tilde{L}.
\]

This will be abbreviated by

\[
\mathbb{N}_{\alpha, i} \prec \mathbb{N}_{\alpha, 3-i}.
\]

If \( \mathbb{N}_{\alpha, 1} \prec \mathbb{N}_{\alpha, 2} \), and \( \mathbb{N}_{\alpha, 2} \prec \mathbb{N}_{\alpha, 1} \), then we call \( \mathbb{N}_{\alpha, 1} \) and \( \mathbb{N}_{\alpha, 2} \) equivalent. We abbreviate this by

\[
\mathbb{N}_{\alpha, 1} \sim \mathbb{N}_{\alpha, 2}.
\]

We say that \( \alpha \) is a cone-like restriction function (CRF), if

\[
\mathbb{N}_{\alpha, 1} \sim \mathbb{N}_{\alpha, 2}.
\]

Now let \( \mathbb{N}_{\alpha} := \mathbb{N}_{\alpha, 1} \cup \mathbb{N}_{\alpha, 2} \). We say that the cone-like set \( L \in \mathbb{N}_{\alpha} \) is based by the function \( \alpha \). We study the a.e. convergence of the \((C, 1)\) means \( \sigma_n f \) of functions integrable that is, \( f \in L^1(T^2), \)
where $T := [-\pi, \pi] \times [-\pi, \pi]$. We study the convergence restricted by $n \in L, L \in \mathbb{N}_x$, where $x$ is CRF and $\land n \to +\infty$. It is natural to ask: How does a CRF look like? First we prove:

**Proposition 1.2.** Function $x$ is a CRF if and only if there exists $\zeta, \gamma_1, \gamma_2 > 1$ such that:

$$\gamma_1 x(x) \leq x(\zeta x) \leq \gamma_2 x(x)$$

holds for each $x \geq 1$.

**Proof.** First suppose (1), that is $\gamma_1 x(x) \leq x(\zeta x) \leq \gamma_2 x(x)$ holds for each $x \geq 1$. We prove $\mathbb{N}_{x,2} < \mathbb{N}_{x,1}$.

Let $L \in \mathbb{N}_{x,2}$, and $n \in L$. Then $L = \mathbb{N}_{x,\beta_1,2}$ for some $\beta_1 > 1$. This means

$$\frac{x^{-1}(n_2)}{\beta_1} \leq n_1 \leq \frac{x^{-1}(n_2)}{\beta_1}.$$

This inequality is equivalent to

$$x \left( \frac{n_1}{\beta_1} \right) \leq n_2 \leq x(n_1 \beta_1).$$

Since $\zeta > 1$, then there exists a $j \in \mathbb{N}$ such that $\zeta^j > \beta_1$. Thus,

$$n_2 \leq x(n_1 \zeta^j) \leq \gamma_2^j x(n_1)$$

and

$$n_2 \geq x(n_1 / \beta_1) > x(n_1 / \zeta^j) \geq \frac{1}{\gamma_2^j} x(n_1).$$

This implies $L \subset \mathbb{N}_{x,\gamma_2^j,1}$. Thus $\mathbb{N}_{x,2} < \mathbb{N}_{x,1}$. Next, let $n \in L \in \mathbb{N}_{x,1}$. Then $L = \mathbb{N}_{x,\beta_1,1}$ for some $\beta_1 > 1$. This means

$$\frac{x(n_1)}{\beta_1} \leq n_2 \leq x(n_1) \beta_1$$

that is

$$x^{-1}(n_2 / \beta_1) \leq n_1 \leq x^{-1}(n_2 \beta_1).$$

The inequality

$$\gamma_1 x(x) \leq x(\zeta x) \leq \gamma_2 x(x)$$

gives

$$\zeta x^{-1}(x) \leq x^{-1}(\gamma_2 x), \quad x^{-1}(\gamma_1 x) \leq \zeta x^{-1}(x).$$

Take $j \in \mathbb{N}$ such that $\gamma_1^j > \beta_1$.

$$n_1 \leq x^{-1}(n_2 \beta_1) \leq x^{-1}(n_2 \gamma_1^j) \leq \zeta^j x^{-1}(n_2)$$
and
\[ n_1 \geq \bar{x}^{-1}(n_2/\beta_1) \geq \bar{x}^{-1}(n_2/\gamma_1^j) \geq \frac{1}{\bar{x}} \bar{x}^{-1}(n_2). \]

That is, \( n \in \mathbb{N}_{x, \gamma^j, 2} \) and \( L \subset \mathbb{N}_{x, \gamma^j, 2} \). Thus, \( \mathbb{N}_{x, 1} \prec \mathbb{N}_{x, 2} \). Therefore the equivalence \( \mathbb{N}_{x, 1} \sim \mathbb{N}_{x, 2} \) is proved. Next, on the other hand, suppose that \( \mathbb{N}_{x, 1} \sim \mathbb{N}_{x, 2} \) for some CRF \( x \). \( \mathbb{N}_{x, 1} \prec \mathbb{N}_{x, 2} \) means that for all \( \beta > 1 \) there exists a \( \gamma > 1 \) such that \( \mathbb{N}_{x, \beta, 1} \subset \mathbb{N}_{x, \gamma, 2} \). Let \( n \in \mathbb{N}_{x, \beta, 1} \) that is,
\[ \frac{x(n_1)}{\beta} \leq n_2 \leq \beta x(n_1). \]

Then there follows
\[ x^{-1}(n_2/\beta) \leq n_1 \leq x^{-1}(n_2\beta). \]

Since \( n \in \mathbb{N}_{x, \gamma, 2} \), therefore
\[ \frac{x^{-1}(n_2)}{\gamma} \leq x^{-1}(n_2/\beta) \quad \text{and} \quad x^{-1}(n_2\beta) \leq \gamma x^{-1}(n_2). \]

Let \( x \geq 1 \) be an arbitrary real number. Then
\[
\bar{x}^{-1}(\beta x) \leq \bar{x}^{-1}(\beta 2\lfloor x \rfloor) \leq \bar{x}^{-1}(\beta^{2+1/\log_2 \beta} \lfloor x \rfloor) \leq \gamma^{2+1/\log_2 \beta} x^{-1}(\lfloor x \rfloor)
\[
\leq \gamma^{2+1/\log_2 \beta} x^{-1}(x).
\]

Hence \( \mathbb{N}_{x, 1} \prec \mathbb{N}_{x, 2} \) implies the existence of the real numbers \( \beta_1, \gamma_1 > 1 \) for which \( \bar{x}^{-1}(\gamma_1 x) \leq \beta_1 x^{-1}(x) \) for \( x \geq 1 \). Similarly, \( \mathbb{N}_{x, 2} \prec \mathbb{N}_{x, 1} \) implies the existence of the real numbers \( \beta_2, \gamma_2 > 1 \) for which \( \bar{x}(\beta_2 x) \leq \gamma_2 \bar{x}(x) \) for \( x \geq 1 \). Let \( s := \bar{x}^{-1}(x) \). Thus, \( \gamma_1 \bar{x}(s) \leq \bar{x}(\beta_1 s) \). Since \( \bar{x}^{-1}(1) = 1 \) and \( \bar{x}^{-1} \) is strictly monotone increasing we have for all \( x \geq 1 \) that \( \gamma_1 \bar{x}(x) \leq \bar{x}(\beta_1 x) \). Choose \( j \in \mathbb{N} \) such that \( \beta_2^j > \beta_1 \):
\[ \gamma_1 \bar{x}(x) \leq \bar{x}(\beta_1 x) \leq \bar{x}(\beta_2^j x) \leq \gamma^j \bar{x}(x). \]

The proof of Proposition 1.2 now is complete. \( \square \)

The system of functions
\[ e^{inx} \quad (n = 0, \pm 1, \pm 2, \ldots) \]

\((x \in \mathbb{R}, \imath = \sqrt{-1})\) is called the trigonometric system. It is orthogonal over any interval of length \( 2\pi \), specially over \( T := [-\pi, \pi] \). Let \( f \in L^1(T) \), that is integrable on \( T \). The \( k \)th Fourier coefficient of \( f \) is
\[
\hat{f}(k) := \frac{1}{2\pi} \int_T f(t)e^{-ikt} \, dt,
\]

where \( k \) is any integer number. The \( n \)th \((n \in \mathbb{N})\) partial sum of the Fourier series of \( f \) is
\[ S_n f(y) := \sum_{k=-n}^{n} \hat{f}(k)e^{iky}. \]
The \(n\)th \((n \in \mathbb{N})\) Fejér or \((C, 1)\) mean of function \(f\) is defined in the following way:

\[
\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y).
\]

It is known that

\[
\sigma_n f(y) = \frac{1}{\pi} \int_{T} f(x) K_n(y - x) \, dx,
\]

where the function \(K_n\) is known as the \(n\)th Fejér kernel; we will now find an appropriate expression for it (see e.g. the book of Bary [1]):

\[
K_n(u) = \frac{1}{2(n+1)} \left( \frac{\sin \left( \frac{\pi (n+1)}{2} \right)}{\sin \left( \frac{\pi}{2} \right)} \right)^2. \tag{2}
\]

From this expression one immediately derive the following properties of the kernel. They will play an essential role later:

\[
K_n(u) \geq 0,
\]

\[
K_n(u) \leq \frac{\pi^2}{2(n+1)u^2} \quad (0 < |u| \leq \pi). \tag{3}
\]

Let \(f\) be an integrable function, that is let \(f \in L^1(T^2)\). The \(k = (k_1, k_2)\)th Fourier coefficient of \(f\) is

\[
\hat{f}(k) = \hat{f}(k_1, k_2) := \frac{1}{(2\pi)^2} \int_{T \times T} f(t_1, t_2) e^{-i(k_1 t_1 + k_2 t_2)} \, dt_1 dt_2,
\]

where \(k_1, k_2\) are integers. The \(n\)th \((n \in \mathbb{N}^2)\) partial sum of the Fourier series of \(f\) is

\[
S_n f(y) = S_{n_1,n_2} f(y_1, y_2) := \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \hat{f}(k_1, k_2) e^{i(k_1 y_1 + k_2 y_2)}.
\]

The \(n\)th \((n \in \mathbb{N}^2)\) two-dimensional Fejér or \((C, 1)\) mean of function \(f\) is defined in the following way:

\[
\sigma_n f(y) = \sigma_{n_1,n_2} f(y) := \frac{1}{(n_1 + 1)(n_2 + 1)} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} S_k f(y),
\]

where \(y \in T^2\). In 1939 Marcinkiewicz and Zygmund [6] proved their celebrated theorem on the convergence of the two-dimensional restricted \((C, 1)\) means of trigonometric Fourier series. They proved for any integrable function \(f \in L^1(T^2)\) the a.e. convergence

\[
\sigma_{(n_1,n_2)} f \rightarrow f
\]

provided \(n_1/\beta \leq n_2 \leq \beta n_1\), where \(\beta > 1\) is fixed constant. So, the set of indices \((n_1, n_2)\) remains in some positive cone around the identical function. Actually, their proof is not a simple one. (We remark that their theorem is also valid for the two-dimensional Walsh–Paley system. For the proof of this see [3,10].) For the time being there is no other restriction set for the indices, which
preserves this a.e. convergence relation, is known. We remark that in 1935 Jessen et al. [5] proved
the unrestricted convergence
\[ \lim_{n \to \infty} \sigma_n f = f \]
(note \( n = \min(n_1, n_2) \)) a.e. But, it is proved for functions in \( L^1 \log^+ L \), which is a proper
subspace of \( L^1(T^2) \). It is quite natural to ask what kind of cone-like restriction sets can be given
preserving the a.e. convergence of the two-dimensional Fejér means of integrable functions. The
aim of this paper is to prove the following two main results

**Theorem 1.3 (The convergence).** Let \( \chi \) be CRF, \( L \in \mathbb{N}_2 \). Then for any \( f \in L^1(T^2) \) the a.e.

\[ \lim_{n \to \infty} \sigma_n f = f \]

holds.

**Theorem 1.4 (The divergence).** Let \( \chi \) be CRF, \( \beta : [1, +\infty) \to [1, +\infty) \) be a monotone increasing
function with property \( \lim_{+\infty} \beta = +\infty \), and \( \delta : [1, +\infty) \to [0, +\infty) \) be a measurable
function with property \( \lim_{+\infty} \delta = 0 \). Let \( L := \mathbb{N}_{\chi, \beta, 1} \) or \( L := \mathbb{N}_{\chi, \beta, 2} \). Then there exists a function
\( f \in L^1 \log^+ L \delta(L) \) such that:
\[ \limsup_{n \to \infty} \sigma_n f = +\infty \]

a.e.

One might think that if we enlarge the cone based by \( \chi \), then the convergence space from \( L^1 \) to
\( L^1 \log^+ (L) \) (no restriction) changes somehow continuously. Theorems 1.3 and 1.4 show that—in
the point of view of spaces of the form \( L^1 \phi(L) \)—there does not exist an interim space between
\( L^1 \) and \( L^1 \log^+ (L) \). We also remark that Móricz proved [7] for functions belonging to certain
Hardy spaces the theorem of Marcinkiewicz and Zygmund [6].

Theorems 1.3 and 1.4 immediately give

**Corollary 1.5.** Let \( \chi \) be CRF, \( \beta : [1, +\infty) \to [1, +\infty) \) be a monotone increasing function with property \( \beta(1) > 1 \), and \( L := \mathbb{N}_{\chi, \beta, 1} \) or \( L := \mathbb{N}_{\chi, \beta, 2} \). Then
\[ \lim_{n \to \infty} \sigma_n f = f \]

holds a.e. for all \( f \in L^1(T^2) \) if and only if the function \( \beta \) is bounded.

Corollary 1.5 shows that the theorem of Marcinkiewicz and Zygmund on the convergence of the
two-dimensional restricted \((C, 1)\) means of trigonometric Fourier series cannot be improved,
that is the cone based by the identical function cannot be enlarged infinitely preserving the a.e.
convergence for each integrable function. Corollary 1.6 below also provides a simpler proof of
the theorem of Marcinkiewicz and Zygmund.

**Corollary 1.6.** Let \( \beta : [1, +\infty) \to [1, +\infty) \) be a monotone increasing function with property
\( \beta(1) > 1 \), then
\[ \lim_{n \to \infty} \sigma_n f = f \]

holds a.e. for all \( f \in L^1(T^2) \) if and only if the function \( \beta \) is bounded.
The “divergence part” of this corollary for the two-dimensional Walsh–Paley system can be read in [4], and the “convergence part” in [3,10]. For an introductory on the trigonometric series see also the book of Zygmund [12], or the book of Bary [1], or Edwards [2].

We denote by $C$ and $\tilde{C}$ constants which may depend only on $\zeta, \gamma_1, \gamma_2$, and can vary at different occurrences. The lower and the upper integer part of real $x$ are denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively.

2. A decomposition lemma

The dyadic subintervals of $T$ are defined in the following way:

$$I_0 := \{T\}, \quad I_1 := \{(-\pi, 0), [0, \pi]\},$$

$$I_2 := \{(-\pi, -\pi/2), [-\pi/2, 0), [0, \pi/2), [\pi/2, \pi]\}, \ldots$$

$$I := \bigcup_{n=0}^{\infty} I_n.$$

The elements of $I$ are said to be dyadic intervals. If $F \in I$, then there exists a unique $n \in \mathbb{N}$ such that $F \in I_n$, and consequently $\text{mes}(F) = \frac{2^{n}}{2^n}$. Each $I_n$ has $2^n$ disjoint elements ($n \in \mathbb{N}$).

$I \times I$ is the set of dyadic rectangles.

Let functions $\phi_j : [1, +\infty) \rightarrow [1, +\infty)$ be monotone increasing and continuous with property $\lim_{j \rightarrow \infty} \phi_j = +\infty$ ($j = 1, 2$). Set $\psi_j = \lfloor \phi_j \rfloor$ ($j = 1, 2$).

The aim of this section is to prove the following decomposition lemma on $T^2$ which will play a prominent role in the proof of Theorem 1.3.

**Lemma 2.1.** Let $f \in L^1(T^2)$, and $\lambda > \|f\|_1/(2\pi)^2$. Then there exists a sequence of integrable functions $(f_i)$ such that:

$$f = \sum_{i=0}^{\infty} f_i,$$

$$\|f_0\|_\infty \leq C\lambda, \quad \|f_0\|_1 \leq C\|f\|_1$$

and

$$\text{supp} f_i \subset I^{i,1} \times I^{i,2},$$

where $I^{i,j} \in I$ are dyadic intervals,

$$\text{mes}(I^{i,j}) = \frac{2\pi}{2^{\psi_j(s_i)}} \quad \text{for some}$$

$s_i \geq 1$ ($j = 1, 2$, $i \in \mathbb{N} \setminus \{0\}$). Moreover, $\int_{T^2} f_i(x) \, dx = 0$ ($i \geq 1$), the dyadic rectangles $I^{i,1} \times I^{i,2}$ are disjoint ($i \in \mathbb{N} \setminus \{0\}$), and for

$$F := \bigcup_{i=1}^{\infty} (I^{i,1} \times I^{i,2}) \quad \text{we have mes}(F) \leq C\|f\|_1/\lambda.$$

**Proof.** Let $s_1 := 1$ and

$$\Omega_1 := \left\{ J = J_1 \times J_2 \in I_{\phi_1(s_1)} \times I_{\phi_2(s_1)} : \text{mes}(J)^{-1} \int_J |f(x)| \, dx > \lambda \right\}.$$
Since for each \( J \in \Omega_1 \), we have
\[
\text{mes}(J)^{-1} = \frac{2\psi_1(1)+\psi_2(1)}{4\pi^2},
\]
then we also have
\[
\lambda < \text{mes}(J)^{-1} \int_J |f(x)| \, dx \leq 2\psi_1(1)+\psi_2(1) \frac{1}{4\pi^2} \int_{T^2} |f(x)| \, dx < 2\psi_1(1)+\psi_2(1) \lambda \leq C \lambda.
\]
Let \( s_2 := \inf \{ s \in [s_1, +\infty) : \sum_{j=1}^2 |\psi_j(s) - \psi_j(s_1)| \geq 1 \} \). Since the functions \( \psi_1, \psi_2 \) are continuous from the right then we have the following three cases:

**Case 1:** \( \psi_1(s_2) = \psi_1(s_1) + 1 \) and \( \psi_2(s_2) = \psi_1(s_1) + 1 \).

**Case 2:** \( \psi_1(s_2) = \psi_1(s_1) \) and \( \psi_2(s_2) = \psi_1(s_1) + 1 \).

**Case 3:** \( \psi_1(s_2) = \psi_1(s_1) + 1 \) and \( \psi_2(s_2) = \psi_1(s_1) + 1 \).

We decompose the dyadic rectangles contained in
\[
[I_{\psi_1(s_1)} \times I_{\psi_2(s_1)}] \setminus \{ J : J \in \Omega_1 \}.
\]
That is,
\[
\Omega_2 := \left\{ J \in I_{\psi_1(s_2)} \times I_{\psi_2(s_2)} : \text{mes}(J)^{-1} \int_J |f(x)| \, dx > \lambda \text{ and } \exists K \in \Omega_1 \text{ such as } J \subset K \right\}.
\]
Consequently, for all \( J \in \Omega_2 \) we get
\[
\lambda < \text{mes}(J)^{-1} \int_J |f(x)| \, dx \leq 4 \lambda.
\]
(In cases 1 and 2 we even have \( 2 \lambda \), but it makes no problem to take \( 4 \lambda \), instead.) Generally, for \( \mathbb{N} \ni n \geq 3 \), \( s_n := \inf \{ s \in [s_{n-1}, +\infty) : \sum_{j=1}^2 |\psi_j(s) - \psi_j(s_{n-1})| \geq 1 \} \). That is, \( \psi_j(s_n) = \psi_j(s_{n-1}) + 1 \) for at least one \( j \) (\( j = 1, 2 \)). If for a \( j \) this is not valid, then \( \psi_j(s_n) = \psi_j(s_{n-1}) \). Also take
\[
\Omega_n := \left\{ J \in I_{\psi_1(s_n)} \times I_{\psi_2(s_n)} : \text{mes}(J)^{-1} \int_J |f(x)| \, dx > \lambda \text{ and } \exists K \in \bigcup_{i=1}^{n-1} \Omega_i \text{ such as } J \subset K \right\}.
\]
Similarly, as in the case of \( \Omega_2 \) we have that for each \( J \in \Omega_n \) the inequalities
\[
\lambda < \text{mes}(J)^{-1} \int_J |f(x)| \, dx \leq 4 \lambda
\]
hold. Denote by \( l_n \in \mathbb{N} \) the number of elements of \( \Omega_n \), and the elements of \( \Omega_n \) by \( J_{n,k} (k = 1, \ldots, l_n, n \in \mathbb{N}) \). Since \( I_{\psi_1(s_n)} \times I_{\psi_2(s_n)} \) has \( 2\psi_1(s_n)+\psi_2(s_n) \) (disjoint) elements, then \( l_n \leq 2\psi_1(s_n)+\psi_2(s_n) \) (\( n \in \mathbb{N} \)). For an arbitrary set \( B \subset T^2 \) the characteristic function of \( B \) is denoted by \( 1_B \). Let
\[
f_{n,k} := \left( f - \text{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x) \, dx \right) 1_{J_{n,k}},
\]
\( k = 1, \ldots, l_n, \ n \in \mathbb{N} \) and \( F := \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{l_n} J_{n,k} \). Since the dyadic rectangles \( J_{n,k} \) are disjoint, then we have the following decomposition of the function \( f \):

\[
\begin{align*}
    f &= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f 1_{J_{n,k}} + f 1_{T^2 \setminus F} \\
    &= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left( f - \text{mes} (J_{n,k})^{-1} \int_{J_{n,k}} f(x) \, dx \right) 1_{J_{n,k}} \\
    &\quad + \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left[ \text{mes} (J_{n,k})^{-1} \int_{J_{n,k}} f(x) \, dx \right] 1_{J_{n,k}} + f 1_{T^2 \setminus F} \\
    &= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f_{n,k} + f_0.
\end{align*}
\]

This means that \( f_0 = \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left[ \text{mes} (J_{n,k})^{-1} \int_{J_{n,k}} f(x) \, dx \right] 1_{J_{n,k}} + f 1_{T^2 \setminus F} \) and the functions \( f_i \) \((i = 1, 2, \ldots)\) in the statement of Lemma 2.1 will be the functions \( f_{n,k} \) \((k = 1, \ldots, l_n, \ n \in \mathbb{N})\), \( \text{supp} f_{n,k} \subset J_{n,k} \) are disjoint dyadic rectangles,

\[
\text{mes} (J_{n,k}) = \frac{4\pi^2}{2^{\psi_1(s_n) + \psi_2(s_n)}},
\]

\[
\int_{T^2} f_{n,k}(x) \, dx = \int_{J_{n,k}} f(x) \, dx - \text{mes} (J_{n,k})^{-1} \int_{J_{n,k}} f(x) \, dx \cdot \text{mes} (J_{n,k}) = 0,
\]

\[
\| f_{n,k} \|_1 \leq \| f 1_{J_{n,k}} \|_1 + \text{mes} (J_{n,k})^{-1} \int_{J_{n,k}} |f(x)| \, dx \| 1_{J_{n,k}} \|_1 = 2 \| f 1_{J_{n,k}} \|_1.
\]

Consequently,

\[
\left\| \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f_{n,k} \right\|_1 \leq 2 \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \| f 1_{J_{n,k}} \|_1 = 2 \int_{F} |f(x)| \, dx \leq 2 \| f \|_1.
\]

This immediately gives

\[
\| f_0 \|_1 = \left\| f - \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f_{n,k} \right\|_1 \leq 3 \| f \|_1.
\]

Since \( F \) is the disjoint union of the dyadic rectangles \( J_{n,k} \), then for the two-dimensional Lebesgue measure of \( F \) we get

\[
\text{mes} (F) = \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \text{mes} (J_{n,k})< \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \frac{1}{2^\psi_1(s_n)} \int_{J_{n,k}} |f(x)| \, dx.
\]
There remains to prove \( \| f_0 \| \leq C \lambda \). The construction of \( \Omega_n \) gives the inequality

\[
\operatorname{mes} (J_{n,k})^{-1} \int_{J_{n,k}} |f(x)| \, dx \leq C \lambda
\]

(in the case of \( n = 1 \) we have \( 2^{\psi_1(1) + \psi_2(1)} \), and in the case of \( n \geq 2 \) we have number 4 as constant \( C \)). That is,

\[
\| f_0 \|_{\infty} \leq C \lambda \left( \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} 1_{J_{n,k}} \right)_{\infty} + \| f 1_{T^2 \setminus F} \|_{\infty} \leq C \lambda + \| f 1_{T^2 \setminus F} \|_{\infty}.
\]

Let \( \mathcal{A}_n \) be the \( \sigma \)-algebra generated by the elements of \( \mathcal{I}_{\psi_1(s_n)} \times \mathcal{I}_{\psi_2(s_n)} \) \((n \in \mathbb{N})\). Then we have an increasing sequence of \( \sigma \) algebras

\[ \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots. \]

The conditional expectation operator of the function \( f \) with respect to \( \mathcal{A}_n \) at a given point \( x \in T^2 \) is

\[
\operatorname{mes} (J)^{-1} \int_{J} f(t) \, dt,
\]

where \( J \) is the unique element of \( \mathcal{I}_{\psi_1(s_n)} \times \mathcal{I}_{\psi_2(s_n)} \) such that \( x \in J \). Since \( \lim_{s \to \infty} \psi_1 = \lim_{s \to \infty} \psi_2 = +\infty \), then the martingale convergence theorem (see e.g. the book of Neveau [8]) gives that this integral mean value converges to \( f(x) \) for almost all \( x \) in \( T^2 \).

Now let \( x \in T^2 \setminus F \). Then the construction of the set \( \Omega_n \) gives for each \( J \in \mathcal{I}_{\psi_1(s_n)} \times \mathcal{I}_{\psi_2(s_n)} \) that \( \operatorname{mes} (J)^{-1} \int_{J} |f(t)| \, dt \leq \lambda \) (for all \( n \in \mathbb{N} \)). From the lines above there follows:

\[
|f(x)| \leq \lambda
\]

for almost all \( x \in T^2 \setminus F \), so

\[
\| f 1_{T^2 \setminus F} \|_{\infty} \leq \lambda, \quad \| f_0 \|_{\infty} \leq C \lambda.
\]

With this the proof of Lemma 2.1 is complete. \( \square \)

3. The convergence

The aim of this section is to prove Theorem 1.3, the convergence theorem. To perform this we need several lemmas. Mainly, we prove that the maximal operator \( \sigma_L f := \sup_{n \in L} |\sigma_n f| \) \((L \in \mathbb{N}_2, \, \lambda \) is CRF) is of weak type \((1, 1)\). This means

\[
\sup_{\lambda > 0} \lambda \operatorname{mes} \{ x : \sigma_L^x f(x) > \lambda \} \leq C \| f \|_1
\]

for all \( f \in L^1(T^2) \). We give further details later. First, apply Lemma 2.1 for functions \( \psi_1(s) := \lfloor \log_2(s) \rfloor \) \((\lfloor x \rfloor \) denotes the lower integer part of \( x \) and \( \psi_2(s) := \lfloor \log_2(x(s)) \rfloor \), where \( x \) is CRF. Then we prove
Lemma 3.1. Let $\alpha$ be CRF, $L \in \mathbb{N}_\alpha$, $f \in L^1(T^2)$, and $\text{supp} f \subset J_1 \times J_2 \in I \times I$ with $\text{mes} (J_j) = \frac{2\pi}{2^{\psi_j(s)}}$ for some $s \geq 1$ ($j = 1, 2$). Suppose that
\[
\int_T f(x_1, x_2) \, dx_j = 0 \quad \text{(for each)} \quad x_3 - j \in T, \quad j = 1, 2.
\]
Then it follows that
\[
\int_{(T \setminus 2J_1) \times (T \setminus 2J_2)} \sigma_L^* f(x_1, x_2) \, d(x_1, x_2) \leq C \| f \|_1.
\]

Proof. We remark that $2J_1$ means the double of the interval $J_1$ with the same center. Let $u_j \in T$ be the center of the dyadic interval $J_j$ ($j = 1, 2$). Then we have
\[
\int_{(T \setminus 2J_1) \times (T \setminus 2J_2)} \sigma_L^* f(x_1, x_2) \, d(x_1, x_2)
\]
\[
= \int_{T \setminus [u_1-2\pi/2^{\psi_1(s)}, u_1+2\pi/2^{\psi_1(s)}]} \int_{T \setminus [u_2-2\pi/2^{\psi_2(s)}, u_2+2\pi/2^{\psi_2(s)}]}
\]
\[
x \sup_{n \in L} \int_{u_1-\pi/2^{\psi_1(s)}}^{u_1+\pi/2^{\psi_1(s)}} \int_{u_2-\pi/2^{\psi_2(s)}}^{u_2+\pi/2^{\psi_2(s)}} f(x_1, x_2) K_{n_1}(y_1-x_1) K_{n_2}(y_2-x_2) \, d(x_1, x_2) \, d(y_1, y_2)
\]
\[
= \int_{T \setminus [-2\pi/2^{\psi_1(s)}, 2\pi/2^{\psi_1(s)}]} \int_{T \setminus [-2\pi/2^{\psi_2(s)}, 2\pi/2^{\psi_2(s)}]}
\]
\[
x \sup_{n \in L} \int_{-\pi/2^{\psi_1(s)}}^{\pi/2^{\psi_1(s)}} \int_{-\pi/2^{\psi_2(s)}}^{\pi/2^{\psi_2(s)}} f(x_1+u_1, x_2+u_2) K_{n_1}(y_1-x_1) K_{n_2}(y_2-x_2) \, d(x_1, x_2) \, d(y_1, y_2).
\]
This equality shows that without loss of generality we can suppose the center of both intervals $J_1$ and $J_2$ be 0. That is, we suppose
\[
J_j = \left[ -\frac{\pi}{2^{\psi_j(s)}}, \frac{\pi}{2^{\psi_j(s)}} \right] \quad (j = 1, 2).
\]
Then we have
\[
\int_{(T \setminus 2J_1) \times (T \setminus 2J_2)} \sigma_L^* f(x_1, x_2) \, d(x_1, x_2)
\]
\[
= \sum_{i=-2^{\psi_1(s)}}^{2^{\psi_1(s)}-1} \sum_{j=-2^{\psi_2(s)}}^{2^{\psi_2(s)}-1} \int_{pi/2^{\psi_1(s)}}^{pi+(1+1)/2^{\psi_1(s)}} \int_{pj/2^{\psi_2(s)}}^{pj+(1+1)/2^{\psi_2(s)}} \sup_{n \in L} \left| \int_{-\pi/2^{\psi_1(s)}}^{\pi/2^{\psi_1(s)}} \int_{-\pi/2^{\psi_2(s)}}^{\pi/2^{\psi_2(s)}} f(x_1, x_2) K_{n_1}(y_1-x_1) K_{n_2}(y_2-x_2) \, d(x_1, x_2) \right| \, d(y_1, y_2).
\]
Let the real number \( s_i \geq 1 \) be defined later. By the help of the following inequality we discuss the maximal function \( \sigma^*_L \).

\[
\sup_{n \in L} |\sigma_n f| \leq \sup_{n \in L} |\sigma_n f| + \sup_{n \in L} |\sigma_n f|.
\]

See the first part on the right-hand side. In the book of Bary [1] one can find that

\[
0 \leq K_n(u) \leq \frac{\pi^2}{2(n + 1)u^2} \quad (0 < |u| \leq \pi, n \in \mathbb{N}).
\]

Since

\[
y_2 \in \left[ \frac{\pi j}{2\psi_2(s)}, \frac{\pi(j + 1)}{2\psi_2(s)} \right] \quad \text{and} \quad x_2 \in \left[ -\frac{\pi}{2\psi_2(s)}, \frac{\pi}{2\psi_2(s)} \right],
\]

then we have

\[
\frac{1}{|y_2 - x_2|} \leq C \frac{2\psi_2(s)}{|f|},
\]

thus

\[
0 \leq K_{n_2}(y_2 - x_2) \leq C \frac{4\psi_2(s)}{n_2 j^2} \quad \text{and similarly} \quad 0 \leq K_{n_1}(y_1 - x_1) \leq C \frac{4\psi_1(s)}{n_1 i^2}.
\]

Since \( L \in \mathbb{N}_x, x \) is CRF, then without loss of generality, \( L = \mathbb{N}_x, \beta, 1 \) can be supposed for some \( \beta > 1 \). Let \( n_1 \geq s_i, n \in L \). Then \( n_1 \geq 2^{\log_2(s_i)} \geq 2\psi_1(s_i) \) and \( n_2 \geq \frac{\pi(s_i)}{\beta} \geq \frac{\pi(s_i)}{\beta} \geq \frac{1}{2}\psi_2(s_i) \). This gives

\[
\sup_{n \in L} \left| \frac{1}{n_1 \geq s_i} K_{n_1}(y_1 - x_1) K_{n_2}(y_2 - x_2) \right| \leq C \frac{4\psi_1(s) + \psi_2(s)}{2\psi_1(s_i) + \psi_2(s_i) i^4 j^2}.
\]

This gives

\[
\sum_{i = -2\psi_1(s)}^{2\psi_1(s)} \sum_{j = -2\psi_2(s)}^{2\psi_2(s)} \int_{\pi i/2\psi_1(s)}^{\pi(i+1)/2\psi_1(s)} \int_{\pi j/2\psi_2(s)}^{\pi(j+1)/2\psi_2(s)} \sup \left\{ |\sigma_n f(x)| : n \in L, n_1 \geq s_i \right\} dx
\]

\[
\leq C \| f \|_1 \sum_{i = -2\psi_1(s)}^{2\psi_1(s)} \sum_{j = -2\psi_2(s)}^{2\psi_2(s)} \frac{2\psi_1(s) + \psi_2(s)}{2\psi_1(s_i) + \psi_2(s_i) i^2 j^2}
\]

\[
=: C \| f \|_1 A.
\]

Later, we give an upper bound for \( A \). Now, we discuss the second part, that is,

\[
\sup \left\{ |\sigma_n f(x)| : n \in L, n_1 < s_i \right\}.
\]

It is well known that

\[
\sigma_n f(y_1, y_2) = \sum_{|k| = 0}^{n_1} \sum_{|l| = 0}^{n_2} \left( 1 - \frac{|k|}{n_1 + 1} \right) \left( 1 - \frac{|l|}{n_2 + 1} \right) \hat{f}(k,l) e^{iky_1 + ily_2}.
\]
We give upper bounds for the Fourier coefficients \( \hat{f}(k, l) \):

\[
\hat{f}(k, l) = \frac{1}{4\pi^2} \int_{J_1} \int_{J_2} f(x_1, x_2) e^{-i(kx_1 + lx_2)} d(x_1, x_2) = \frac{1}{4\pi^2} \int_{J_1} \int_{J_2} f(x_1, x_2) \left( e^{-ikx_1} - 1 \right) \left( e^{-ilx_2} - 1 \right) d(x_1, x_2).
\]

This follows from the equalities \( \int_{J_1} f(x_1, x_2) dx_1 = 0 \) (\( x_2 \in T \)) and \( \int_{J_2} f(x_1, x_2) dx_2 = 0 \) (\( x_1 \in T \)). It is simple to have

\[
\left| \left( e^{-ikx_1} - 1 \right) \left( e^{-ilx_2} - 1 \right) \right| \leq Ck \lambda_1 x_1 \lambda_2 \leq Ck \frac{1}{2\psi_1(s) + \psi_2(s)}.
\]

That is, for the Fourier coefficients we have

\[
| \hat{f}(k, l) | \leq C \| f \|_1 \frac{kl}{2\psi_1(s) + \psi_2(s)},
\]

which implies

\[
| \sigma_n f(y_1, y_2) | \leq \sum_{|k| = 0}^{n_1} \sum_{|l| = 0}^{n_2} \left( 1 - \frac{|k|}{n_1 + 1} \right) \left( 1 - \frac{|l|}{n_2 + 1} \right) C \| f \|_1 \frac{kl}{2\psi_1(s) + \psi_2(s)} \leq C \| f \|_1 \frac{n_1 n_2}{2\psi_1(s) + \psi_2(s)}.
\]

If \( n_1 < s_{ij} \) and \( n \in L \), then \( n_2 \leq \beta \varepsilon(n_1) \leq \beta \varepsilon(s_{ij}) \). That is, \( n_1 < s_{ij} \) and \( n_2 \leq \beta \varepsilon(s_{ij}) \). This gives

\[
\sum_{i = -2^\psi_1(s)}^{2^\psi_1(s) - 1} \sum_{j = -2^\psi_2(s)}^{2^\psi_2(s) - 1} \int_{\pi i / 2\psi_1(s)} \int_{\pi j / 2\psi_2(s)} \sup \left\{ | \sigma_n f(x) | : n \in L, n_1 < s_{ij} \right\} dx \leq C \| f \|_1 \sum_{i = -2^\psi_1(s)}^{2^\psi_1(s) - 1} \sum_{j = -2^\psi_2(s)}^{2^\psi_2(s) - 1} \frac{4\varepsilon(s_{ij}) + \psi_2(s_{ij})}{4\varepsilon(s_{ij}) + \psi_2(s_{ij})}.
\]

Let \( 1/2 < \delta < 1 \) be an arbitrary real number. Since the function \( s \varepsilon(s) \) is a continuous monotone strictly increasing function, \( \varepsilon(1) = 1 \), and \( \lim_{s \to \infty} s \varepsilon(s) = +\infty \) then for each \( i, j \in \mathbb{Z} \setminus \{0\} \) we have an \( s_{ij} \geq 1 \) so that

\[
\frac{2^\psi_1(s) + \psi_2(s)}{|ij|^\delta} \leq s \varepsilon(s) = s_{ij} \varepsilon(s_{ij}) \leq 4 \frac{2^\psi_1(s) + \psi_2(s)}{|ij|^\delta}
\]

because \( |i| < 2^\psi_1(s) \), \( |j| < 2^\psi_2(s) \). Consequently,

\[
1 < \frac{2^\psi_1(s) + \psi_2(s)}{|ij|^\delta} \leq \frac{2^\psi_1(s) + \psi_2(s)}{|ij|^\delta}.
\]
that is
\[
\frac{2^2\psi_1(s) + \psi_2(s)}{2^2\psi_1(s_j) + \psi_2(s_j)} \leq C|ij|^\delta \quad \text{and} \quad \frac{2^2\psi_1(s_j) + \psi_2(s_j)}{2^2\psi_1(s) + \psi_2(s)} \leq \frac{C}{|ij|^\delta}.
\]

By these inequalities (taking also account the line where the double sum \( A \) is defined) we get
\[
\int_{(T, 2J_1) \times (T, 2J_2)} \sigma^*_L f(x_1, x_2) \ d(x_1, x_2)
\leq C \|f\|_1 \sum_{i = -2^2\psi_1(s)}^{2^2\psi_1(s) - 1} \sum_{i \neq -2^2\psi_1(s)}^{2^2\psi_2(s) - 1} \sum_{j = -2^2\psi_2(s)}^{2^2\psi_2(s) - 1} \int_{\pi(i+1)/2^2\psi_1(s)}^{\pi(i+1)/2^2\psi_1(s)} \int_{\pi(j+1)/2^2\psi_2(s)}^{\pi(j+1)/2^2\psi_2(s)} \int_{\pi(j+1)/2^2\psi_2(s)}^{\pi(j+1)/2^2\psi_2(s)} \int_{\pi(i+1)/2^2\psi_1(s)}^{\pi(i+1)/2^2\psi_1(s)} \left( |ij|^{-2} + |ij|^{-2\delta} \right) \ d(x_1, x_2)
\leq C \|f\|_1.
\]

We recall that, \( 1/2 < \delta < 1 \) thus \( \delta - 2, -2\delta < -1 \). This completes the proof of Lemma 3.1. \(
\)

**Lemma 3.2.** Let \( x \) be CRF, \( L \in \mathbb{N}_2 \), \( g \in L^1(T) \), and \( \text{supp } g \subset J_2 \in I, J_1 \in I \), \( \text{mes } (J_j) = \frac{2\pi}{2^2\psi_{j(s)}} \) for some \( s \geq 1 \) (\( j = 1, 2 \)). Suppose that
\[
\int_T g(x) \ dx = 0.
\]

Then there follows
\[
\int_{(T, 2J_1) \times (T, 2J_2)} \sigma^*_L \left( \frac{2^2\psi_1(s)}{2\pi} 1_{J_1} \times g \right)(x_1, x_2) \ d(x_1, x_2) \leq C \|g\|_1.
\]

**Proof.** Do the same procedure as in the proof of Lemma 3.1. We can suppose that the center of \( J_1 \) and \( J_2 \) is 0. Besides,
\[
\int_{(T, 2J_1) \times (T, 2J_2)} \sigma^*_L \left( \frac{2^2\psi_1(s)}{2\pi} 1_{J_1} \times g \right)(x_1, x_2) \ d(x_1, x_2)
\leq C \|g\|_1 \sum_{i = -2^2\psi_1(s)}^{2^2\psi_1(s) - 1} \sum_{i \neq -2^2\psi_1(s)}^{2^2\psi_2(s) - 1} \sum_{j = -2^2\psi_2(s)}^{2^2\psi_2(s) - 1} \sum_{j \neq -2^2\psi_2(s)}^{2^2\psi_2(s) - 1} \left( |ij|^{-2} + |ij|^{-2\delta} \right) \ d(x_1, x_2)
\leq C \|g\|_1 A.$
The real numbers \( s_{ij} \geq 1 \) will be defined later in the proof of this lemma. What can be said in the case of \( n_1 < s_{ij} \)?

\[
\sigma_n \left( \frac{2\psi_1(s)}{2\pi} 1_{J_1} \times g \right) = \frac{2\psi_1(s)}{2\pi} \sigma_n (1_{J_1}) \times \sigma_n g.
\]

\[
\sigma_{n_1} (1_{J_{n_1}}) (y_1) = \int_{-\pi/2\psi_1(s)}^{\pi/2\psi_1(s)} K_{n_1} (y_1 - x_1) \, dx_1 \leq \frac{Cn_1}{2\psi_1(s)} \leq \frac{C s_{ij}}{s}.
\]

On the other hand,

\[
\hat{g}(l) = \int_{J_2} g(x) \left( e^{-ilx_2} - 1 \right) \, dx_2.
\]

This gives

\[
|\hat{g}(l)| \leq \frac{Cl}{2\psi_2(s)} \|g\|_1,
\]

therefore

\[
\left| \sigma_n \left( \frac{2\psi_1(s)}{2\pi} 1_{J_1} \times g \right) (y_1, y_2) \right| \leq C \sum_{|l|=0}^{n_2} \left( 1 - \frac{|l|}{n_2 + 1} \right) \frac{l}{2\psi_2(s)} \|g\|_1 2\psi_1(s) s_{ij} \frac{s}{s} \leq C \|g\|_1 \frac{2\psi_1(s)}{2\psi_2(s)} s_{ij}.
\]

Since \( L \in \mathbb{N}_x \), where \( x \) is CRF then, without loss of generality, \( L = \mathbb{N}_x \beta, 1 \) can be supposed for some \( \beta > 1 \). If \( n_1 < s_{ij} \) and \( n \in L \), then \( n_2 \leq \beta x(n_1) \leq \beta x(s_{ij}) \leq C 2\psi_2(s_{ij}) \). This gives

\[
\sum_{j=2\psi_1(s)}^{2\psi_1(s)-1} \sum_{i=2\psi_2(s)}^{2\psi_2(s)-1} \int_{\pi/2\psi_1(s)}^{\pi(j+1)/2\psi_2(s)} \int_{\pi(i+1)/2\psi_1(s)}^{\pi j/2\psi_2(s)} \times \sup \left\{ \left| \sigma_n \left( \frac{2\psi_1(s)}{2\pi} 1_{J_1} \times g \right) (x) \right| : n \in L, n_1 < s_{i,j} \right\} \, d(x_1, x_2)
\]

\[
\leq C \|g\|_1 \sum_{i=2\psi_1(s)}^{2\psi_1(s)-1} \sum_{j=2\psi_2(s)}^{2\psi_2(s)-1} \frac{4\psi_2(s_{ij})}{4\psi_2(s)} \frac{s_{ij}}{s} \leq C \|g\|_1 B.
\]

We give the construction of a double sequence \( (s_{ij}) \) such that both sums \( A \) and \( B \) will be finite. Let \( 0 < \varepsilon < \delta < 1 \) be real numbers defined later. \( \varepsilon \) “will be near” \( 0 \) and \( \delta \) “will be near” \( 1 \). Define \( (s_{ij}) \) in a way that (recall that \( x(1) = 1 \))

\[
s x(s) = s_{ij} x(\varepsilon||j||^\delta
\]

(4)
for all $i$ and $j$. This can be done since $\alpha$ is continuous and strictly monotone increasing with property $\lim_{+\infty} \alpha = +\infty$, and since with $s_{ij} = 1$ we have on the right side

$$|ij|^{\delta} \leq |ij| \leq 2^{\psi_1(s)} 2^{\psi_2(s)} \leq s \alpha(s).$$

We give an $\varepsilon > 0$ such that:

$$\frac{\alpha(s_{ij})}{\alpha(s)} \leq |ij|^{\varepsilon-1+\delta, \tau}$$

for all $i$ and $j$ $(\tau \in \mathbb{N})$ discussed later, and depends on $\gamma_2$ and $\zeta$ for $\gamma_2$ and $\zeta$ see (1) in Proposition 1.2). On the contrary, suppose that for all $\varepsilon > 0$ there exist an $i$ and $j$ such that:

$$\alpha(s_{ij}) > |ij|^{\varepsilon-1+\delta, \tau} \alpha(s).$$

Let $\eta := -\varepsilon - 1 + \delta$. Then since $\alpha$ is CRF we have

$$\alpha(s_{ij}) > |ij|^\eta \log_{\gamma_2} |ij| \gamma_2 \alpha(s) \gamma_2 \geq \alpha(s) \gamma_2 \geq \alpha(s_\zeta^\eta \log_{\gamma_2} |ij| + \tau).$$

This implies

$$s_{ij} \geq s_\zeta^\eta \log_{\gamma_2} |ij| + \tau \geq s_\zeta^\eta \log_{\gamma_2} |ij| + \tau - 1,$$

$$s_{ij} \alpha(s_{ij}) \geq \zeta^\eta \log_{\gamma_2} |ij| + \tau - 1 \gamma_2 |ij| \eta \alpha(s).$$

Thus,

$$|ij|^{-\delta} = \frac{s_{ij} \alpha(s_{ij})}{\alpha(s)} \geq \zeta^\eta \log_{\gamma_2} |ij| + \tau - 1 \gamma_2 |ij| \eta = |ij|^{-\delta} \eta \log_{\gamma_2} \zeta^{-1} \gamma_2.$$

Let $\tau$ be defined in a way that $1 = \alpha(1) < \zeta^{-1} \gamma_2$. This gives $|ij|^{-\delta} \geq |ij| \eta(1 + \log_{\gamma_2} \zeta), \delta \leq (\varepsilon + 1 - \delta)(1 + \log_{\gamma_2} \zeta)$. This does not hold for all $\delta$ and $\varepsilon$. To see this let $\delta \not\nearrow 1$ and $\varepsilon \searrow 0$. We found that there exists a $\varepsilon > 0$ such that:

$$\frac{s_{ij} \alpha(s_{ij})}{\alpha(s)} \leq |ij|^{-\varepsilon-1+\delta, \tau} \leq C |ij|^{-\varepsilon-1+\delta}$$

for all $i$ and $j$. Discuss expressions $A$ and $B$ and recall that $\frac{s \alpha(s)}{s_{ij} \alpha(s_{ij})} = |ij|^{\delta}$. Therefore:

$$A = \sum_{i=-2^{\psi_1(s)}}^{2^{\psi_1(s)}-1} \sum_{j=-2^{\psi_2(s)}}^{2^{\psi_2(s)}-1} \frac{2^{\psi_1(s)} 2^{\psi_2(s)} 2^{\psi_1(s)+\psi_2(s)}}{2^{\psi_1(s)} 2^{\psi_2(s)} (s_{ij}) 2^{\psi_2(s)}}$$

$$\leq C \sum_{i=-2^{\psi_1(s)}}^{2^{\psi_1(s)}-1} \sum_{j=-2^{\psi_2(s)}}^{2^{\psi_2(s)}-1} \frac{|ij|^{\delta}}{|ij|^2}$$

$$< \infty.$$
because $0 < \delta < 1$, and

$$
B = \sum_{i=-2^{j(s)}}^{2^{j(s)}-1} \sum_{j=-2^{j(s)}}^{2^{j(s)}-1} \frac{4\psi_2(s_{ij}) s_{ij}}{4\psi_2(s)} \frac{s_{ij}}{s}.
$$

\[
\leq C \sum_{i=-2^{j(s)}}^{2^{j(s)}-1} \sum_{j=-2^{j(s)}}^{2^{j(s)}-1} \frac{s_{ij} \mathcal{A}(s_{ij}) \mathcal{A}(s)}{\mathcal{A}(s)} - 1
\]

\[
\leq C \sum_{i=-2^{j(s)}}^{2^{j(s)}-1} \sum_{j=-2^{j(s)}}^{2^{j(s)}-1} |ij|^{-\delta - 1}
\]

\[
< \infty
\]

(recall that $\frac{s_{ij} \mathcal{A}(s_{ij})}{\mathcal{A}(s)} = |ij|^{-\delta}$ (4)). The proof of Lemma 3.2 is complete. \(\square\)

**Lemma 3.3.** Let $\mathcal{A}$ be CRF, $L \in \mathbb{N}_\mathcal{A}$, $g \in L^1(T)$, and $\text{supp} \ g \subset J_1 \in \mathcal{I}$, $J_2 \in \mathcal{I}$, with $\text{mes}(J_j) = \frac{2\pi}{2^{j(s)}}$ for some $s \geq 1$ ($j = 1, 2$). Suppose that

$$
\int_T g(x) \, dx = 0.
$$

Then there follows

$$
\int_{(T \setminus J_1) \times (T \setminus J_2)} \sigma_L^*(g \times \frac{2^{j(s)} \mathcal{A}_1 \mathcal{A}_2}{2\pi} 1_{J_2})(x_1, x_2) \, d(x_1, x_2) \leq C \|g\|_1.
$$

**Proof.** The proof is similar to the proof of Lemma 3.2, and therefore it is left to the reader. \(\square\)

Later on we will need the following lemma corresponding to the maximal function of the one-dimensional Fejér kernels.

**Lemma 3.4.**

$$
\int_{T \setminus [-\pi/a, \pi/a]} \sup_{n \geq b} K_n(t) \, dt \leq C \frac{a}{b} \quad (a, b \in \mathbb{N} \setminus \{0\}).
$$

**Proof.** Once again we refer to Bary’s book [1], one can find there (3)

$$
0 \leq K_n(u) \leq \frac{\pi^2}{2(n + 1)u^2} \quad (0 < |u| \leq \pi, n \in \mathbb{N}).
$$

\[
\left[\frac{\pi}{a}, \pi\right] \subset \left[\frac{\pi}{a}, \frac{2\pi}{a}\right] \cup \left[\frac{2\pi}{a}, \frac{4\pi}{a}\right] \cup \ldots \cup \left[\frac{2^j\pi}{a}, \pi\right],
\]
where \( j = \lfloor \log_2(a) \rfloor \). Then

\[
\int_{[2^j \pi/a, 2^{j+1} \pi/a)} \sup_{n \geq b} K_n(t) \, dt \leq \frac{Ca^2}{b} \frac{a}{4^j} \frac{\operatorname{mes} \left( \left[ \frac{2^j \pi}{a}, \frac{2^{j+1} \pi}{a} \right) \right)}{b} \leq \frac{Ca}{b2^j}.
\]

This gives,

\[
\int_{[\pi/a, \pi)} \sup_{n \geq b} K_n(t) \, dt \leq \sum_{l=0}^{\infty} \frac{Ca}{b2^{l}} \leq \frac{Ca}{b}.
\]

In the same way we also get

\[
\int_{[-\pi, -\pi/a)} \sup_{n \geq b} K_n(t) \, dt \leq \sum_{l=0}^{\infty} \frac{Ca}{b2^{l}} \leq \frac{Ca}{b}.
\]

This completes the proof of Lemma 3.4. □

Next we prove that the operator \( \sigma_{L}^{\ast} \) \( (L \in \mathbb{N}_{x}, x \) is CRF) is a quasi-local-like (for the exact definition of local and quasi-local operators see e.g. the book of Schipp et al. [9]) one. That is, we prove

Lemma 3.5. Let \( x \) be CRF, \( L \in \mathbb{N}_{x}, f \in L^{1}(T^{2}) \), and \( \operatorname{supp} f \subset J_{1} \times J_{2} \in \mathcal{I} \times \mathcal{I} \), with \( \operatorname{mes} \left( J_{j} \right) = \frac{2\pi}{2^{\psi_{j}(s)}} \) for some \( s \geq 1 \) \( (j = 1, 2) \). Suppose that

\[
\int_{T^{2}} f(x_{1}, x_{2}) \, dx_{1} \, dx_{2} = 0.
\]

Then there follows

\[
\int_{T^{2} \setminus (2J_{1} \times 2J_{2})} \sigma_{L}^{\ast} f(y_{1}, y_{2}) \, dy_{1} \, dy_{2} \leq C \|f\|_{1}.
\]

Proof. Since \( L \in \mathbb{N}_{x}, x \) is CRF, then without loss of generality, \( L = \mathbb{N}_{x, \beta, 1} \) can be supposed for some \( \beta > 1 \).

\[
T^{2} \setminus (2J_{1} \times 2J_{2}) = [(T \setminus 2J_{1}) \times (T \setminus 2J_{2})] \cup [2J_{1} \times (T \setminus 2J_{2})] \cup [(T \setminus 2J_{1}) \times 2J_{2}]
\]

\[
= A_1 \cup A_2 \cup A_3.
\]

First we discuss the integral on the set \( A_1 \). Let

\[
g(x_{1}, x_{2}) := f(x_{1}, x_{2}) - 1_{J_1}(x_{1}) \frac{2\psi_{1}(s)}{2\pi} \int_{J_1} f(t, x_{2}) \, dt - 1_{J_2}(x_{2}) \frac{2\psi_{2}(s)}{2\pi} \int_{J_2} f(x_{1}, t) \, dt
\]

for \( (x_{1}, x_{2}) \in T^{2} \). Then Lemma 3.1 can be applied for function \( g \):

\[
\int_{A_1} \sigma_{L}^{\ast} g(x_{1}, x_{2}) \, dx_{1} \, dx_{2} \leq C \|g\|_{1} \leq C \|f\|_{1}.
\]

Similarly, for the function

\[
1_{J_1}(x_{1}) \frac{2\psi_{1}(s)}{2\pi} \int_{J_1} f(t, x_{2}) \, dt
\]
we apply Lemma 3.2, and for the function
\[ 1_{J_2}(x_2) \frac{2^{\psi_2(s)}}{2\pi} \int_{J_2} f(x_1, t) \, dt \]
apply Lemma 3.3. This, and the sublinearity of the operator \( \sigma_L^* \) gives
\[ \int_{A_1} \sigma_L^* f(x_1, x_2) d(x_1, x_2) \leq C \| f \|_1. \]

We discuss the integral of \( \sigma_L^* f(x_1, x_2) \) on the set \( A_3 \). As in the proof of Lemma 3.1 we can suppose that the center of \( J_1 \) and \( J_2 \) is 0. First, investigate the integral
\[ \int_{A_3} \sup \{ |\sigma_n f(y_1, y_2)| : n \in L, n_1 < s \} d(y_1, y_2). \]
This integral is less or equal than
\[ \sum_{j=0}^{[\log_\zeta(s)]} \int_{A_3} \sup \left\{ |\sigma_n f(y_1, y_2)| : n \in L, n_1 \in \left[ \frac{s}{\zeta^{j+1}}, \frac{s}{\zeta^j} \right] \right\} d(y_1, y_2) =: \sum_{j=0}^{[\log_\zeta(s)]} B_{3,j}. \]

Give an upper bound for \( B_{3,j} \). Let \( \tau := [\log_\zeta(4)] \) (in the proof of this lemma, only). For any \( j \in \{0, \ldots, [\log_\zeta(s)] \} \) we have
\[ A_3 = (T \setminus 2J_1) \times 2J_2 \subseteq T \times 2J_2 \]
\[ = \left( T \setminus \left[ -\frac{\pi \zeta^{j+\tau}}{s}, \frac{\pi \zeta^{j+\tau}}{s} \right] \right) \times 2J_2 \cup \left( \left[ -\frac{\pi \zeta^j}{s}, \frac{\pi \zeta^j}{s} \right] \right) \times 2J_2. \]
\[ =: A_{3,1,j} \cup A_{3,2,j}. \]

In order to give an upper bound for \( B_{3,j} \) first discuss the following integral:
\[ B_{3,j}^1 := \int_T \left[ -\pi \zeta^{j+\tau}/s, \pi \zeta^{j+\tau}/s \right] \int_{2J_2} \sup_{n_1 \in \left[ \frac{x}{\zeta^{j+\tau}}, \frac{y}{\zeta^j} \right]} |\sigma_n f(y)| \, dy. \]
Since \( x_1 \in J_1, x_2 \in J_2, y_1 \in T \setminus \left[ -\frac{\pi \zeta^{j+\tau}}{s}, \frac{\pi \zeta^{j+\tau}}{s} \right] \) and \( y_2 \in 2J_2, y_2 - x_2 \in 4J_2, y_1 - x_1 \notin \left[ -\frac{\pi \zeta^j}{s}, \frac{\pi \zeta^j}{s} \right] \), because e.g. for \( y_1 \geq \frac{\pi \zeta^{j+\tau}}{s} \) we have \( y_1 - x_1 \geq \frac{\pi \zeta^{j+\tau}}{s} - \frac{\pi}{2^{\psi_1(s)}} \geq \frac{\pi}{s}. \) By this and by the theorem of Fubini we have
\[ B_{3,j}^1 \leq \| f \|_1 \int_T \left[ -\pi \zeta^j/s, \pi \zeta^j/s \right] \int_{4J_2} \sup_{n_1 \in \left[ \frac{x}{\zeta^{j+\tau}}, \frac{y}{\zeta^j} \right]} K_{n_1}(t_1) K_{n_2}(t_2) \, d(t_1, t_2). \]
Since \( n_1 \leq \frac{x}{\zeta^j} \) and \( n \in L \), then we have \( n_2 \leq \beta \left( \frac{\beta}{\zeta^j} \right) \), and
\[ \int_{4J_2} \sup_{n_2 \leq \beta \left( \frac{\beta}{\zeta^j} \right)} K_{n_2}(t_2) \, dt_2 \leq C \frac{\beta \left( \frac{\beta}{\zeta^j} \right)}{\beta(s)} \cdot \]
On the other hand, by Lemma 3.4 it follows that
\[
\int_{T} \left[ -\frac{\pi \zeta^{j}}{s}, \frac{\pi \zeta^{j}}{s} \right] \sup_{n_{1} \geq \frac{s}{\zeta^{j}}} K_{n_{1}}(t_{1}) \, dt_{1} \leq C.
\]
Thus,
\[
B_{3,j}^{1} \leq C \| f \|_{1} \frac{\beta \alpha\left( \frac{s}{\zeta^{j}} \right)}{\alpha(s)}.
\]
Meanwhile,
\[
\int_{T} \left[ -\frac{\pi \zeta^{j+1}}{s}, \frac{\pi \zeta^{j+1}}{s} \right] j_{1} \sup_{n \in L} |\sigma_{n} f| \leq C \| f \|_{1} \int_{T} \left[ -\frac{\pi \zeta^{j+1}}{s}, \frac{\pi \zeta^{j+1}}{s} \right] \int_{T} \sup_{n \in L} n_{1} n_{2}
\]
\[
\leq C \| f \|_{1} \frac{\zeta^{j}}{s} \frac{1}{\alpha(s)} \frac{s}{\zeta^{j}} \beta \alpha\left( \frac{s}{\zeta^{j}} \right)
\]
\[
\leq C \| f \|_{1} \frac{\alpha\left( \frac{s}{\zeta^{j}} \right)}{\alpha(s)}.
\]
That is,
\[
B_{3,j} \leq C \| f \|_{1} \frac{\alpha\left( \frac{s}{\zeta^{j}} \right)}{\alpha(s)}.
\]
Consequently,
\[
\int_{A_{3}} \sup \{ |\sigma_{n} f(y_{1}, y_{2})| : n \in L, n_{1} < s \} \, d(y_{1}, y_{2})
\]
\[
\leq C \sum_{j=0}^{\left\lceil \log_{1}(s) \right\rceil} \frac{\alpha\left( \frac{s}{\zeta^{j}} \right)}{\alpha(s)}
\]
\[
\leq C \| f \|_{1} \frac{\alpha(s) + \frac{1}{\gamma_{1}} \alpha(s) + \frac{1}{\gamma_{1}^{2}} \alpha(s) + \cdots}{\alpha(s)}
\]
\[
\leq C \| f \|_{1}
\]
(recall that \( \alpha(s/\zeta^{j}) \leq \frac{1}{\gamma_{1}} \alpha(s) \)). Thus,
\[
\int_{A_{3}} \sigma_{L}^{s} f \leq C \| f \|_{1} + \int_{A_{3}} \sup \{ |\sigma_{n} f(y_{1}, y_{2})| : n \in L, n_{1} \geq s \} \, d(y_{1}, y_{2}).
\]
That is, the rest to prove that the second addable is also bounded by \( C \| f \|_{1} \). If \( n_{1} \in [s \zeta^{j}, s \zeta^{j+1}) \), \( n \in L \), \( j \in \mathbb{N} \), then we have
\[
n_{2} \in \left[ \frac{\alpha(s \zeta^{j})}{\beta}, \beta \alpha(s \zeta^{j+1}) \right].
\]
By Lemma 3.4 we have
\[
\int_{T \setminus 2J_{1}} \sup_{n_{1} \in [s \zeta^{j}, s \zeta^{j+1})} K_{n_{1}}(t_{1}) \, dt_{1} \leq C \frac{s}{\zeta^{j}}
\]
and
\[
\int_{T} \sup_{t_{2} \in \left[\frac{2s_{i,j}}{\beta}, \beta(s_{i,j}+1)\right]} K_{n_{2}}(t_{2}) \, dt_{2}
\]
\[
\leq \int_{-\pi/\beta(s_{i,j}+1)}^{\pi/\beta(s_{i,j}+1)} C \beta z(s_{i,j}+1) \, dt_{2} + \int_{T \setminus \left[\frac{-\pi}{\beta(s_{i,j}+1)}, \frac{\pi}{\beta(s_{i,j}+1)}\right]} \sup_{t_{2} \geq \frac{\beta(s_{i,j})}{\beta}} K_{n_{2}}(t_{2}) \, dt_{2}
\]
\[
\leq C + \frac{C \gamma_{2}}{\beta} \leq C + C_{2} \leq C.
\]

We have proved
\[
\int_{A_{3}} \sigma_{L}^{*} f \leq C \| f \|_{1}.
\]
In the same way one also can have
\[
\int_{A_{2}} \sigma_{L}^{*} f \leq C \| f \|_{1}.
\]
That is, the inequality
\[
\int_{T^{2} \setminus (2J_{1} \times 2J_{2})} \sigma_{L}^{*} f(y_{1}, y_{2}) d(y_{1}, y_{2}) \leq C \| f \|_{1}
\]
is proved. This completes the proof of Lemma 3.5. \(\square\)

Now, we are ready to prove

**Theorem 3.6.** Let \(\gamma\) be CRF, \(L \in \mathbb{N}_{\gamma}\). Then the operator \(\sigma_{L}^{*}\) is of weak type \((1, 1)\).

**Proof.** The fact that the operator \(\sigma_{L}^{*}\) is of type \((\infty, \infty)\) (this means \(\| \sigma_{L}^{*} f \|_{\infty} \leq C \| f \|_{\infty}\) for all \(f \in L^\infty(T^{2})\)) easily follows from the well-known inequality:
\[
\int_{-\pi}^{\pi} |K_{n}(x)| \, dx = \pi \quad (n \in \mathbb{N}).
\]
Let \(f \in L^{1}(T^{2})\), and \(\lambda > \| f \|_{1}/(4\pi^{2})\). By Lemma 2.1 we have a sequence of functions \((f_{j})\) such that:
\[
f = \sum_{i=0}^{\infty} f_{i},
\]
\[
\| f_{0} \|_{\infty} \leq C \lambda, \quad \| f_{0} \|_{1} \leq C \| f \|_{1}
\]
and
\[
supp f_{i} \subset I_{i}^{1,1} \times I_{i}^{1,2},
\]
where \(I_{i,j} \in \mathcal{I}\) are dyadic intervals
\[
\operatorname{mes}(I_{i,j}) = \frac{2\pi}{2^{s_{i,j}}(s_{i})} \quad \text{for some}
\]
\[ s_i \geq 1 \quad (j = 1, 2, \ i \in \mathbb{N} \setminus \{0\}) \]. Moreover, \( \int_{T^2} f_i(x) \, dx = 0 \) \((i \geq 1)\), the dyadic rectangles \( I^{i,1} \times I^{i,2} \) are disjoint \((i \in \mathbb{N} \setminus \{0\})\), and for

\[ F := \bigcup_{i=1}^{\infty} (I^{i,1} \times I^{i,2}) \]

we have \( \text{mes}(F) \leq C \|f\|_1 / \lambda \).

It is obvious that

\[ \text{mes}\left(\sigma_L^* f > \frac{1}{2} \tilde{C} \lambda\right) \leq \text{mes}\left(\sigma_L^* f_0 > \frac{1}{2} \tilde{C} \lambda\right) + \text{mes}\left(\sigma_L^* \left(\sum_{i=1}^{\infty} f_i\right) > \frac{1}{2} \tilde{C} \lambda\right). \]

The inequality

\[ \|\sigma_L^* f_0\|_\infty \leq \pi^2 \|f_0\|_\infty \leq C \lambda \]

shows that if we choose \( \tilde{C} > 2C \), then

\[ \text{mes}\left(\sigma_L^* f_0 > \frac{1}{2} \tilde{C} \lambda\right) = 0. \]

On the other hand, by Lemma 3.5 it follows that

\[ \text{mes}\left(\sigma_L^* \left(\sum_{i=1}^{\infty} f_i\right) > \frac{1}{2} \tilde{C} \lambda\right) \leq \text{mes}\left(\bigcup_{i=1}^{\infty} (2I^{i,1} \times 2I^{i,2})\right) \]

\[ + \text{mes}\left(\left\{ x \in T^2 : \sigma_L^* \left(\sum_{i=1}^{\infty} f_i\right)(x) > \frac{1}{2} \tilde{C} \lambda\right\}\right) \]

\[ \leq C \|f\|_1 / \lambda + \frac{2}{\tilde{C} \lambda} \int_{T^2 \setminus \bigcup_{i=1}^{\infty} (2I^{i,1} \times 2I^{i,2})} \sigma_L^* \left(\sum_{i=1}^{\infty} f_i\right)(x) \, dx \]

\[ \leq C \|f\|_1 / \lambda + \frac{2}{\tilde{C} \lambda} \sum_{i=1}^{\infty} \int_{T^2 \setminus (2I^{i,1} \times 2I^{i,2})} \sigma_L^* f_i(x) \, dx \]

\[ \leq C \|f\|_1 / \lambda + \frac{C}{\lambda} \sum_{i=1}^{\infty} \|f_i\|_1 \leq C \|f\|_1 / \lambda. \]

The proof of Theorem 3.6 is complete. \( \square \)

**Corollary 3.7.** Let \( \lambda \) be CRF, \( L \in \mathbb{N}_2 \). Then the operator \( \sigma_L^* \) is of type \((p, p)\) for all \( 1 < p \leq \infty \).

**Proof.** Apply the interpolation lemma of Marcinkiewicz (see e.g. the book of Schipp et al. [9]), and the fact that the operator \( \sigma_L^* \) is sublinear. \( \square \)

**Proof of Theorem 1.3 (The Convergence).** It is known that the set of two-dimensional trigonometric polynomials is dense in \( L^1(T^2) \). This fact and Theorem 3.6 (\( \sigma_L^* \) is of weak type \((1, 1)\)) by
standard argument (see e.g. [9]) imply the a.e. equality

\[ \lim_{n \to \infty} \sigma_n f = f \]

for each \( f \in L^1(T^2) \). The proof of Theorem 1.3 is complete. \( \square \)

4. The divergence

The main aim of this section is to prove Theorem 1.4, that is to prove the theorem of divergence. Let \( x \) be CRF, \( \beta : [1, +\infty) \to [1, +\infty) \) be a monotone increasing function with property \( \lim_{x \to +\infty} \beta = +\infty \), and let \( \delta : [1, +\infty) \to [0, +\infty) \) be a measurable function with the property \( \lim_{x \to +\infty} \delta = 0 \). Let \( L := \mathbb{N}_x, \beta, 1 \) (or \( L := \mathbb{N}_x, \beta, 2 \)). We prove the existence of such a function \( f \in L^1 \log \log L \), that is

\[ \int_{T^2} |f(x)| \log^+ |f(x)| \delta(|f(x)|) \, dx < \infty \]

such that:

\[ \sup_{n \in L} \sigma_n f = +\infty \]

almost everywhere, that is the relation \( \lim_{n \to \infty} \sigma_n f = f \) may hold on a set of measure zero.

We suppose that \( L = \mathbb{N}_x, \beta, 1 \). The case \( L = \mathbb{N}_x, \beta, 2 \) can be discussed in the same way, therefore it is left to the reader. Let \( x \in T^2 \), \( n \in \mathbb{N}^2 \). Denote by

\[ I_n(x) = I_{n_1}(x_1) \times I_{n_2}(x_2) \in \mathcal{I} \times \mathcal{I} \]

the two-dimensional dyadic rectangle for which \( x \in I_n(x) \) and

\[ \mes(I_{n_j}(x_j)) = \frac{2\pi}{2^{n_j}} \quad (j = 1, 2). \]

For \( n, a \in \mathbb{N}^2 \) define the following subset of \( \mathcal{I} \times \mathcal{I} \):

\[ \mathcal{I}_{n,a}(x) := \{ I_{n_1+j}(x_1) \times I_{n_2+a_2+j}(x_2) : j = 0, 1, \ldots, \wedge a \}. \]

It is easy to get

\[ \bigcap \mathcal{I}_{n,a}(x) = I_{n_1+\wedge a}(x_1) \times I_{n_2+\wedge a_2}(x_2), \]

\[ \mes\left( \bigcap \mathcal{I}_{n,a}(x) \right) = \frac{4\pi^2}{2^{n_1+n_2+\wedge a+\wedge a_2}}. \]

\( F \in \mathcal{I}_{n,a}(x) \) implies \( \mes(F) = \frac{4\pi^2}{2^{n_1+n_2+\wedge a+\wedge a_2}}. \) Next we prove

Lemma 4.1.

\[ \mes\left( \bigcup \mathcal{I}_{n,a}(x) \right) = \frac{4\pi^2(1+\wedge a/2)}{2^{n_1+n_2+\wedge a}}. \]
Proof. Denote (only for the sake of this proof)
\[ \mu_k := \text{mes} \left( \bigcup_{j=0}^{k} (I_{n_1+j}(x_1) \times I_{n_2+a_2-j}(x_2)) \right) \]
for \( k = 0, 1, \ldots, \land a \). Then, \( \mu_0 = \frac{4\pi^2}{2^{n_1+n_2+2}} \), and for \( k > 0 \) we have
\[ \mu_k = \mu_{k-1} + \text{mes} \left( (I_{n_1+k}(x_1) \times I_{n_2+a_2-k}(x_2)) \right) \]
\[ - \text{mes} \left( \bigcup_{j=0}^{k-1} (I_{n_1+j}(x_1) \times I_{n_2+a_2-j}(x_2)) \cap (I_{n_1+k}(x_1) \times I_{n_2+a_2-k}(x_2)) \right) \]
\[ = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2}} - \text{mes} \left( \bigcup_{j=0}^{k-1} (I_{n_1+j}(x_1) \times I_{n_2+a_2-j}(x_2)) \right) \]
\[ = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2}} - \text{mes} \left( I_{n_1+k}(x_1) \times I_{n_2+a_2-k+1}(x_2) \right) \]
\[ = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2}} - \frac{4\pi^2}{2^{n_1+n_2+a_2+1}} \]
\[ = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2+1}}. \]

This gives
\[ \text{mes} \left( \bigcup \mathcal{I}_{n,a}(x) \right) = \mu_{\land a} = \mu_0 + \frac{4 \land a \pi^2}{2^{n_1+n_2+a_2+1}} = \frac{4\pi^2(1 + \land a/2)}{2^{n_1+n_2+a_2}}. \]

This completes the proof of Lemma 4.1. □

Let \( b = (b_1, b_2) \in \mathbb{N}^2 \) be discussed later. Let natural number \( a_1 \) be so large that
\[ 2^{a_1 \log_2(2)} \geq 4. \]  

(5)

Then define the sequence \((d_{k,2})\) (\( 1 \leq k \in \mathbb{N}, d_{0,2} = 0 \)) as
\[ d_{k,2} = \frac{\lfloor \log_2(2^{b_1+ka_1}) \rfloor}{k}. \]

(6)

We have \( 2^{b_1+(k+1)a_1} = 2^{b_1+ka_1} 2^{a_1 \log_2(2)} \geq 2^{a_1 \log_2(2)} 2^{b_1+ka_1} \), and therefore
\[ \lfloor \log_2(2^{b_1+(k+1)a_1}) \rfloor \geq \log_2(42^{b_1+ka_1}) \geq 1 + \lfloor \log_2(2^{b_1+ka_1}) \rfloor. \]

(7)

Consequently, (6) and (7) give that the sequence \((kd_{k,2})\) is strictly monotone increasing. Besides, it is also simple to prove that
\[ Ca_1 \leq kd_{k,2} - (k - 1)d_{k-1,2} \leq \tilde{C}a_1 \]

(8)

with some positive constants \( C, \tilde{C} \) depending only on \( \zeta, \gamma_1 \) and \( \gamma_2 \).
For \( t \in T^2, k \in \mathbb{N}, \)
\[
d_k = (d_{k,1}, d_{k,2}) := (a_1, d_{k,2}), \quad \Delta_k = (\Delta_{k,1}, \Delta_{k,2}) := (a_1, kd_{k,2} - (k - 1)d_{k-1,2})
\]
define the sets \( J_{b,d}^k(t), \Omega_{b,d}^k(t) \) recursively:
\[
J_{b,d}^0(t) := \{t\}, \quad \Omega_{b,d}^0(t) := \bigcup I_{b,1}(t).
\]
Suppose that the sets \( J_{b,d}^j(t) \) and \( \Omega_{b,d}^j(t) \) are defined for \( j < k \). Then consider
\[
(I_{b_1}(t_1) \times I_{b_2}(t_2)) \setminus \bigcup_{j=0}^{k-1} \bigcup \Omega_{b,d}^j(t)
\]
as the disjoint union of dyadic rectangles of the form \( I_{b_1+k_1a_1}(x_1) \times I_{b_2+kd_2}(x_2) \). Take from each rectangle an element as representative. The set of \( x \)'s corresponding to these rectangles is \( J_{b,d}^k(t) \). That is,
\[
(I_{b_1}(t_1) \times I_{b_2}(t_2)) \setminus \bigcup_{j=0}^{k-1} \bigcup \Omega_{b,d}^j(t) = \bigcup_{x \in J_{b,d}^k(t)} \big[ I_{b_1+k_1a_1}(x_1) \times I_{b_2+kd_2}(x_2) \big].
\]
Then take
\[
\Omega_{b,d}^k(t) := \bigcup_{x \in J_{b,d}^k(t)} \bigcup I_{b+k_1a_1, \Delta_{k+1}}(x).
\]
This gives the a.e. equality
\[
I_b(t) = \bigcup_{j=0}^{\infty} \Omega_{b,d}^j(t).
\] (9)
To clarify (9) note that by construction and Lemma 4.1 we have
\[
\frac{\text{mes} \left( \Omega_{b,d}^k(t) \right)}{\text{mes} \left( I_b(t) \setminus \bigcup_{j=0}^{k-1} \Omega_{b,d}^j(t) \right)} = \frac{1 + \Delta_{k+1}/2}{2\Delta_{k+1,2}} \geq C > 0,
\]
since by (8) \( \Delta_{k+1,2} \leq C \alpha_1 \). Consequently, if the monotone decreasing (in \( k \)) sequence \( \text{mes} \left( I_b(t) \setminus \bigcup_{j=0}^{k-1} \Omega_{b,d}^j(t) \right) \) does not converge to 0, then it has a positive lower bound and so does the sequence \( \text{mes} \left( \Omega_{b,d}^k(t) \right) \). This would imply \( \text{mes} \left( \bigcup_{j=0}^{\infty} \Omega_{b,d}^j(t) \right) = \infty \), and that would be a contradiction. This proves (9).

\( T^2 \) is the disjoint union of \( 2^{b_1+b_2} \) pieces of dyadic rectangles of the form \( I_b(t) \), that is denoting
\[
e_{k,m} := -\pi + \frac{2\pi m}{2^k} \quad (m = 0, \ldots, 2^k - 1, k \in \mathbb{N}),
\]
we have
\[
T^2 = \bigcup_{m_j=0,\ldots,2^{b_j}-1}^{\infty} \bigcup_{k=0}^{k_{b,d}(e_{b_1,m_1}, e_{b_2,m_2})} \Omega_{b,d}^k(e_{b_1,m_1}, e_{b_2,m_2}).
\]
Denote \( t_m := (e_{b_1,m_1}, e_{b_2,m_2}) \) the pairs, which determine the decomposition of \( T^2 \). Define the functions \( f_{b,d} : T^2 \rightarrow [0, +\infty) \) \((b, d \text{ as above})\) in the following way:

\[
f_{b,d}(x) = \sum_{m_j=0}^{2^{b_j}} \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} 2^{\Delta k+1} \mathbb{1}_{I_{b_1+k_{a_1}+\Delta k+1} \times I_{b_2+(k+1)\Delta k+1,2}}(y)(x).
\]

Next we prove that the functions \( f_{b,d} \) are in \( L^1 \log^+ L \), that is we prove

**Lemma 4.2.** For all \( b, d \) we have

\[
\int_{T^2} |f_{b,d}(x)| \log^+ |f_{b,d}(x)| \, dx \leq 80.
\]

**Proof.**

\[
\int_{T^2} |f_{b,d}(x)| \log^+ |f_{b,d}(x)| \, dx
\]

\[
= \sum_{m_j=0}^{2^{b_j}} \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} 2^{\Delta k+1} \log(2^{\Delta k+1}) \times \text{mes} \left( \mathbb{1}_{I_{b_1+k_{a_1}+\Delta k+1} \times I_{b_2+(k+1)\Delta k+1,2}}(y)(x) = 1 \right)
\]

\[
= \sum_{m_j=0}^{2^{b_j}} \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} 2^{\Delta k+1} \log(2^{\Delta k+1}) \text{mes} \left( \bigcap_{I_{b+k\Delta k,\Delta k+1}}(y) \right)
\]

\[
= \sum_{m_j=0}^{2^{b_j}} \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} 2^{\Delta k+1} \log(2^{\Delta k+1}) \frac{\text{mes} \left( \bigcup_{I_{b+k\Delta k,\Delta k+1}}(y) \right)}{2^{\Delta k+1}(1 + \Delta k+1/2)}
\]

\[
\leq \log(2^{\Delta k+1}) \frac{\text{mes} \left( T^2 \right)}{1 + \Delta k+1/2} \leq 80. \quad \square
\]

**Lemma 4.3.** Let \( b, d \) as above. Then

\[
\sup_{n \in \mathbb{N}_{\geq 1}} \sigma_n f_{b,d}(y) \geq 2^{-9}
\]

for almost every \( y \in T^2 \).

**Proof.** It is easy to show that for \(-2/n \leq u \leq 2/n \) \((u \neq 0, n \in \mathbb{N} \setminus \{0\})\)

\[
K_n(u) = \frac{1}{2(n+1)} \left( \frac{\sin \left( \frac{u}{2} (n+1) \right)}{\sin \left( \frac{u}{2} \right)} \right)^2 \geq \frac{1}{2(n+1)} \frac{4}{u^2} 2^{-4}(n+1)^2 u^2/4
\]

\[
= 2^{-5}(n+1),
\]

since for \( 0 < |x| \leq 2 \) we have \( |\sin(x)| \geq 0.25|x| \) and \((n+1)|u/2| \leq 2\).
For almost all $y \in T^2$ there exists a unique $t_m \in T^2$, $k \in \mathbb{N}$ such that $y \in \Omega_{b,d}^k(t_m)$ and hence a unique $t \in I_{b,d}^k(t_m)$ with

$$y \in \bigcup I_{b+kd_k, \Delta_{k+1}}(t).$$

Then we have $y \in I_{b_1+ka_1+j}(t_1) \times I_{b_2+(k+1)d_{k+1,2}-j}(t_2)$ for a $j \in \{0, 1, \ldots, \wedge \Delta_{k+1}\}$. Then by the nonnegativity of the function $f_{b,d}$ and the Fejér kernels it is not difficult to give a lower bound for $\sup_{n \in \mathbb{N}_x, \beta, 1} \sigma_n f_{b,d}(y)$:

$$\sup_{n \in \mathbb{N}_x, \beta, 1} \sigma_n f_{b,d}(y) \geq \int_{I_{b_1+ka_1+j} \times I_{b_2+(k+1)d_{k+1,2}-j}} f_{b,d}(x) K_{2b_1+ka_1+j-2}(y_1 - x_1) K_{2b_2+(k+1)d_{k+1,2}-j-2}(y_2 - x_2) \, dx$$

$$= 2^{\wedge \Delta_{k+1}} \int_{I_{b_1+ka_1+j} \times I_{b_2+(k+1)d_{k+1,2}-j}} f_{b,d}(x) K_{2b_1+ka_1+j-2}(y_1 - x_1) K_{2b_2+(k+1)d_{k+1,2}-j-2}(y_2 - x_2) \, dx$$

$$\geq 2^{\wedge \Delta_{k+1}} \frac{4\pi^2}{2b_1+b_2+ka_1+j-2+b_2+(k+1)d_{k+1,2}-j-2}$$

$$= \pi^2 2^{-12} \geq 2^{-9}.$$

The rest is to prove that

$$n = (n_1, n_2) = (2^{b_1+ka_1+j-2}, 2^{b_2+(k+1)d_{k+1,2}-j-2}) \in \mathbb{N}_x, \beta, 1.$$

We verify that

$$n_2 \leq 2^{b_2+(k+1)d_{k+1,2}-2} \leq \alpha(2^{b_1+ka_1-2}) \beta(2^{b_1-2}) \leq \alpha(n_1) \beta(n_1) \tag{10}$$

and

$$n_2 \geq 2^{b_2+kd_{k,2}-2} \geq \alpha(2^{b_1+(k+1)a_1-2}) \geq \frac{\alpha(n_1)}{\beta(n_1)}. \tag{11}$$

First discuss (10). By the definition $(d_{k,2})$, that is by (6) and (1) we have

$$2^{b_1+ka_1+1} \leq \alpha(2^{b_1+(k+1)a_1}) \leq \alpha(2^{b_1+ka_1-2}) \leq \gamma_2^{(a_1-2) \log \gamma_2 + 1} \alpha(2^{b_1+ka_1-2}).$$

We have not given any condition for $b \in \mathbb{N}_x^2$. From now, let it be $b_1 \in \mathbb{N}$ such that $\beta(2^{b_1-2}) \geq 2^{b_1+\gamma_2^{(a_1-2) \log \gamma_2 + 1}}$, and $b_2 \geq 2$. Then (10) is verified. Next, we turn our attention to (11). Also by (6) and (1) we get

$$2^{kd_{k,2}} \geq \alpha(2^{b_1+ka_1}) \geq \frac{1}{\gamma_2^{(a_1-2) \log \gamma_2 + 1}} \alpha(2^{b_1+(k+1)a_1-2}) \geq \frac{\alpha(2^{b_1+(k+1)a_1-2})}{\beta(2^{b_1-2})}.$$

Then (11) is also verified. This completes the proof of Lemma 4.3. □
Finally, we prove Theorem 1.4, that is we give the construction of a counterexample function \( f \). Let \( \omega : \mathbb{N} \to \mathbb{N} \) be a strictly monotone increasing sequence, such that:

\[
\delta(t) \leq \frac{1}{4^n}
\]

for all \( t \geq \omega_n, n \in \mathbb{N} \). This can be done, since \( \lim_{+\infty} \delta = 0 \). Set \( a_1^{(n)} \) in a way that with the constants \( C, \tilde{C} \) (depending only on \( \zeta, \gamma_1 \) and \( \gamma_2 \)) in (8)

\[
2^{a_1^{(n)}} \min(1, C) \geq 2^{a_1^{(n-1)}} \max(1, \tilde{C}), \omega_n, 2^n.
\] (12)

Let \( d^{(n)}, b_1^{(n)} \) and \( b_2^{(n)} \) be defined as above (e.g. \( d_k^{(n)} = a_1^{(n)} \) and see (6) for \( d_{k,2}^{(n)} \)). Then take

\[
f := \sum_{n=0}^{\infty} 2^n f_n := \sum_{n=0}^{\infty} 2^n f_{b_1^{(n)}, d^{(n)}}.
\]

By the construction of the functions \( f_{b,d} \) it is easy to have

\[
\text{mes} \left( x \in \mathbb{T}^2 : f_{b,d} \neq 0 \right) \leq \sup_k \frac{1}{2^{\Delta_k + (1 + \Delta_k/2)}} \leq \frac{1}{2^{Ca_1}}
\]

with some positive constant \( C \) depending only on \( \zeta, \gamma_1 \) and \( \gamma_2 \). (Recall that \( Ca_1 \leq \Delta_{k,2} = kd_{k,2} - (k-1)d_{k-1,2} \leq \tilde{C}a_1 \).) Thus, taking

\[
H_{-1} := \left\{ x \in \mathbb{T}^2 : f_n(x) = 0 \forall n \in \mathbb{N} \right\},
\]

\[
H_n := \left\{ x \in \mathbb{T}^2 : f_n(x) \neq 0, f_{n+1+j}(x) = 0 \ (j = 0, 1, \ldots) \right\} \quad (n \in \mathbb{N}),
\]

\[
\text{mes} \left( \limsup \left\{ f_k \neq 0 \right\} \right) = \text{mes} \left( \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} \left\{ f_k \neq 0 \right\} \right)
\]

\[
\leq \lim_{n \to \infty} \sum_{k \geq n} \text{mes} \left( \left\{ f_k \neq 0 \right\} \right) \leq \lim_{n \to \infty} \sum_{k \geq n} \frac{1}{2^{Ca_{1(k)}}} = 0.
\]

This implies the a.e. equality \( \bigcup_{n=1}^{\infty} H_n = \mathbb{T}^2 \). This will play an important role a couple of lines below in the proof of \( f \in L^1 \log^+ L\delta(L) \). Let \( x \in H_n \). Then by (12) (for the constants \( C, \tilde{C} \) see (8))

\[
2^n 2^{a_1^{(n)}} \min(1, C) \leq 2^{n \inf_k \Delta_k^{(n)}} \leq 2^n f_n(x) \leq f(x) = \sum_{k=0}^{n} 2^k f_k(x) \leq C 2^n f_n(x).
\]

Then \( \delta(f(x)) \leq \frac{1}{4^n} \) on the set \( x \in H_n \). By Lemma 4.2 we easily obtain

\[
\int_{\mathbb{T}^2} f(x) \log^+ (f(x)) \delta(f(x)) \, dx = \sum_{n=0}^{\infty} \int_{H_n} f(x) \log^+ (f(x)) \delta(f(x)) \, dx
\]

\[
\leq \sum_{n=0}^{\infty} \int_{H_n} C 2^n f_n(x) \log^+ (C 2^n f_n(x)) \frac{1}{4^n} \, dx
\]

\[
\leq C \sum_{n=0}^{\infty} \frac{1}{2^n} \int_{\mathbb{T}^2} f_n(x) \log^+ (f(x)) \, dx \leq C.
\]
Now, by Lemma 4.3 we show that $f$ is “a real counterexample” function. By the nonnegativity of the functions $f_n$ and the Fejér kernel functions we have

$$\sup_{n \in \mathbb{N}_{x,y,1}} \sigma_n f(y) \geq \sup_{n \in \mathbb{N}_{x,y,1}} \sigma_n 2^k f_{hk,dk}(y) \geq 2^{-9} 2^k$$

for almost all $y \in T^2$ and for all $k \in \mathbb{N}$. This implies that we have

$$\sup_{n \in \mathbb{N}_{x,y,1}} \sigma_n f(y) = +\infty$$

almost everywhere. This completes the proof of Theorem 1.4. \hfill \Box

**Acknowledgment**

The author wishes to thank the referees.

**References**


