Diffusion Equation for Multivalued Stochastic Differential Equations

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Deterministic oscillations with bilinear hysteresis are governed by a multivalued differential equation of the type \( \xi' + k \xi \ni b(\xi) + g \), where \( k \) is maximal monotonic and \( b \) is Lipschitzian. An existence and uniqueness result is proven for corresponding stochastic equation. The diffusion equation satisfied by the laws of \( \xi(t) \) is established. In the particular case \( k = 0 \), this equation is equivalent to the Fokker–Planck equation.

Il est montré que les oscillations déterministes avec hystérésis bilinéaire, sont solutions d'équations différentielles multivoques du type \( \xi' + k \xi \ni b(\xi) + g \), où \( k \) est un opérateur maximal monotone de \( \mathbb{R}^n \), où \( b \) est lipschitzien. Un théorème d'existence et d'unicité est donné pour les équations stochastiques correspondantes. On écrit l'équation de diffusion satisfaite par les lois des variables aléatoires \( \xi(t) \). Si \( k = 0 \), cette équation est équivalente à l'équation de Fokker–Planck.

The antiseismic design of structures and buildings uses a numerical analysis of an informal Fokker–Planck Equation (F.P.E.) concerning a stochastic differential equation with very singular coefficients \([6, 7]\). At the beginning, the goal of the present work was to find a mathematical justification of these methods. Since steel and reinforced concrete have an elastic and not perfectly plastic behaviour \([4]\), the mathematical problem concerns random oscillations with bilinear hysteresis generated by a colored Gaussian input. Using Markovianization techniques \([8]\) we may assume the input is a Gaussian white noise. In these conditions, a mechanical analysis shows that the number of constraints is time-varying. Therefore the equation of motion is not a usual stochastic differential equation of Ito-type,

\[
\frac{d\xi(t)}{dt} = b(\xi(t), t) \, dt + \sigma(\xi(t), t) \, dW(t), \quad \text{(St. Diff. Eq.)}
\]

but a multivalued stochastic differential equation,

\[
\frac{d\xi(t)}{dt} + k(\xi(t)) \, dt \ni b(\xi(t), t) \, dt + \sigma(\xi(t), t) \, dW(t), \quad \text{(M. St. Diff. Eq.)}
\]
where \( \xi \) denotes an \( \mathbb{R}^n \)-valued process, and where \( k \) denotes a given multivalued and maximal monotonic [11] operator of the Euclidean space \( \mathbb{R}^n \). The corresponding deterministic differential equations are well known [2, 5] and of the type

\[
D\xi(t) + k(\xi(t)) \ni b(\xi(t), t) + g(t) \quad \text{(M. Diff. Eq.)}
\]

with \( D = d/dt \), and with a given continuous function \( g \). The problem is to generalize F.P.E. to (M. St. Diff. Eq.) in order to compute moments, distribution function, fatigue criteria. ... In view of the general form of (M. St. Diff. Eq.), the present problem is also of practical interest for the design of all control chains, automatic guidance systems ... involving elements with hysteresis or Coulomb friction, and submitted to random perturbations. The problem will be solved in four steps:

I. Analysis of deterministic oscillations. Oscillations are studied on a finite time interval \( J = [0, T] \). The existence and uniqueness of the solution of (M. Diff. Eq.) on \( J \) with given initial data follow from [2, 5] (see Theorem 1.3). The result is then applied to oscillators with bilinear hysteresis.

II. Existence and uniqueness of stochastic oscillations. Theorem II.6 is proved combining previous results with the techniques used in the free case \( k = 0 \), i.e., for usual stochastic differential equations. There is a connection with the theory of degenerate diffusions with boundary conditions in the particular case where \( -k \) is the sub-derivative [12] of the indicator function \( Z_G \) of a closed and convex subset \( G \) of \( \mathbb{R}^n \), with smooth boundary \( \Sigma \). We recall that \( Z_G = 0 \) on \( G \) and \( Z_G = +\infty \) outside \( G \). Theorem 4.1 of [15] gives an existence and uniqueness result concerning the stochastic equation

\[
d\xi(t) - d\varphi(t) = b(\xi(t), t) \, dt + \sigma(\xi(t), t) \, dW(t),
\]

where \( \varphi(t) \) is an "associated" process of \( \xi(t) \) [15]. A comparison with (M. Diff. Eq.) shows that the process \( \varphi(t) \) is derivable under the hypothesis of Theorem II.6.

III. Diffusion operator of (M. St. Diff. Eq.) and boundary condition. Let \( \text{Dom } k \) be the domain of \( k \), where the process \( \xi \) is living. By hypothesis, \( \text{Dom } k \) is closed. For arbitrary \( s \in J \), let \( \xi_{s,y}(t) \) be the solution of (M. St. Diff. Eq.) starting from \( y \in \text{Dom } k \) at time \( s \). The diffusion operator at time \( s \) is computed by analogy with the free case \( k = 0 \), i.e., computing for arbitrary real and \( C^2 \) function \( \Psi \) on \( \text{Dom } k \) with compact support:

\[
\lim_{h \to 0} E h^{-1}(\Psi^0 \xi_{s,y}(s + h) - \Psi(y)).
\]
In view of application in Section IV, $\mathcal{P}$ is replaced by $\varphi_s = \varphi(\cdot, s)$ with $\varphi \in C^2_0$ $(\text{Dom } k \times \text{int } J)$ (see III.8). In the particular case considered previously where $k = -Z_G$, the Fichera partition of $\Sigma$ in four parts $\Sigma_i$ can be used, $i = 0, 1, 2, 3$ [13]. Since the normal diffusion is vanishing, $\Sigma_3$ is empty; hence the boundary is not reflecting. And since the probabilistic particle never reaches $\Sigma_1$, there is no boundary condition on $\Sigma_1$. Finally, for $k = -Z_G$ we prove that (M. St. Diff. Eq.) defines an initial value problem for the stochastic differential equation (St. Diff. Eq.) with the following boundary condition: the Neumann condition on $\Sigma_2$, and no condition on the complementary subset $\Sigma \setminus \Sigma_2$ of $\Sigma$.

IV. Diffusion equation for (M. St. Diff. Eq.). This equation (Dif." Eq.) is deduced by a transposition argument from Section III (see IV 4). In the free case, (Dif." Eq.) is equivalent to F.P.E. In the constrained case $k \neq 0$, the equation is different from F.P.E. The essential reason is that F.P.E. concerns a differential operator on the manifold without boundary $\mathbb{R}^n \times \text{int } J$, but (Dif." Eq."') involves in general differential operators on a manifold with boundary. In the particular case considered in Section III, (Dif. Eq."') is equivalent to a boundary value problem of a new type.

These results were announced in [8, 9]. The author thanks Paul Malliavin for useful advice during the elaboration of the present work.

I. Deterministic Oscillations with Hysteresis

(1.1) DEFINITION OF A SOLUTION OF (M. DIFF. EQ.). Let $X$ be a real separable Hilbert space identified with its dual. Let $A$ be a maximal monotonic operator of $X$. Let $b$ be a continuous mapping $X \times J \rightarrow X$, Lipschitzian with respect to $x$, i.e., s.t. for some $C > 0\ |b(x, t) - b(x', t)| \leq C|x - x'|$ for all $x, x' \in X$ and $t \in J$. For a given $g \in L^1(J, X)$, a function $\xi \in C^0(J, X)$ is called a solution of (M. Diff. Eq.) if the following three conditions are satisfied:

(a) for all $t \in J$, $\xi(t) \in \text{Dom } k$;
(b) the distribution derivative $Du = u'$ of $u$ belongs to $L^1(J, X)$;
(c) for almost all $t \in J$,

$$\xi'(t) \in b(\xi(t), t) + g(t) - k(\xi(t)).$$

(I.2)

For an arbitrary closed and convex subset $Q$ of $X$, the projection of the origin of $X$ on $Q$ is denoted $Q^\infty$. The following result follows combining several methods and results of [2, 5].
THEOREM. Assuming:

\[ \text{Dom } k \text{ is closed, and defining the map } k^\infty : \text{Dom } k \to X \text{ by } k^\infty(x) = (kx)^\infty, k^\infty \text{ is bounded on any compact subset of Dom } k. \]

Then, there exists one and only one solution of (M. Diff. Eq.) with a given initial data \( u(0) = u_0 \in \text{Dom } k \).

Moreover, the distribution derivative \( u' \in L^\infty(J, X) \). For all \( t \in [0, T] \), the map \( \xi : J \to X \) weak, is derivable on the right and the value of this right derivative is

\[ D^+\xi(t) = (b(\xi(t), t) + g(t) - k(\xi(t)))^\infty. \]  

The steps of the proof follow.

(a) Unicity follows from monotonicity and \textit{a priori} estimates.

(b) Existence of the solution of \( u' + ku \ni g \) such that \( u(0) = u_0 \) for a given step function \( g \) follows from the resolution of \( u' + ku \ni 0 \) with a given initial data [2].

(c) Since arbitrary \( g \in L^1(J, X) \) is the limit of a sequence of step functions, the existence of a weak solution [1] of \( u' + ku \ni g \) such that \( u(0) = u_0 \) follows from (b).

(d) This weak solution is a solution [1].

(e) Existence of a weak solution \( u \) of \( u' + ku \ni b(t)u + g \text{ s.t. } u(0) = u_0 \) is proven by considering the sequence \( (u_n) \) in \( C^0(J, X) \) s.t. \( u_0(t) = U_0 \) and \( \forall n, u_{n+1} \) is the solution of \( u'_{n+1} + ku_{n+1} \ni b(t)u_n + g \text{ s.t. } u_{n+1}(0) = u_0 \).

(f) An adaptation of the argument of point (d) shows that this weak solution is in fact a solution of (M. Diff. Eq.) and that (I.4) holds.

(I.5) Deterministic oscillation with bilinear hysteresis. The general result concerning these oscillations is presented in the language of seismic design; the method can be extended to multilinear hysteresis. We begin with a deterministic analysis. For example, suppose that an earthquake generates a given acceleration \( g_3(t) \in C^0(J) \) acting in some horizontal direction \( Ox \). A building is modeled by an oscillator with one degree of freedom moving on the \( x'0x \) axis. The equation of motion is

\[ x(t)^{''} + 2hx'(t) + c(t) = g_2(t), \]  

where the given constant \( h > 0 \) characterizes the structural damping and where \( -c(t) \) is the time-varying force generated by the structural stiffness. In order to analyze this term, a static analysis is first realized assuming \( x'(t) = x^{''}(t) = g_2(t) = 0 \) and starting with the value \( c = 0 \) of the internal force. If \( c \) has sufficiently small increments, then the deformation \( x \) is elastic and hence
proportional to \( c \). Hence with an appropriate choice of units the point \( m = (x, c) \) in the \((x, c)\) plane moves first on the line \( c = x \) and the value of the elastic limit is 1. When \( c \) surpasses this elastic limit, the structure has a "nonperfect" plastic behaviour: the deformations are larger and permanent. This means that \( m \) is moving now on the line \( L^{\text{SUP}} \) where \( c = (1 - r) + rx \) with \( 0 \leq r < 1 \). The rupture is realized if \( c \) reaches a certain level. But if \( c \) decreases before this event, \( m \) is moving on a line \( L' \) with angular coefficient one (see Fig. 1). Physically, this means that you have a new plastic behaviour, but with a different origin. If \( c \) decreases sufficiently, a new plastic behaviour appears when \( m \) reaches the intersection of \( L' \) with the line \( L^{\text{INF}} \) where \( c = -(1 - r) + x \) etc. In all cases, \( m(t) \) is moving between the two parallel lines \( L^{\text{SUP}} \) and \( L^{\text{INF}} \) of \( \mathbb{R}^2 \). Hence introducing the function

\[
z(t) = (1 - r)^{-1}(c(t) - rx(t))
\]  

the evolution is subjected to the constraint \(|z(t)| \leq 1\). This constraint introduces a memory: hence the evolution cannot be described by a
differential equation involving $x$ and $x'$ only. Adding the memory variable $z$, the evolution can be described by the differential equation
\[
D \begin{bmatrix} x(t) \\ x'(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -r & -2h & -(1 - r) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x'(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g_2(t) \\ 0 \end{bmatrix}
\]
(1.8)

with a singular term $l(t)$ (see [6, 7]).

(1.9) Now, the dependence of $l$ on $x(t)$, $x'(t)$, $z(t)$ and $g_2(t)$ is analyzed by a mechanical analysis:

(a) If $|z(t)| < 1$ then $l(t) = 0$. In fact $\Delta c = Ax$ gives $c' = x'$. But derivation of (1.7) gives $c' = (1 - r) z' + rx'$. Elimination of $c'$ between the two last relations gives $z' = x'$ hence $l(t) = 0$.

(b) If $z(t) = 1$, two subcases are considered.

(b') If $x'(t) \leq 0$ then $l(t) = 0$. In fact if $x'(t) \leq 0$, the previous mechanical analysis shows that $m(t)$ leaves the upper line $L^{\text{sup}}$ hence $z'(t) \leq 0$. This is compatible with $l = 0$ since the third equation of (1.8) then gives $z' = x' \leq 0$.

(b'') If $x'(t) > 0$ then $l(t) = x'(t)$. In fact the third equation of (1.8) gives $z' = x' - l$, but the previous mechanical analysis shows that $m(t)$ stays on the line $L^{\text{sup}}$; hence $z'(t) = 0$, i.e., $l = x'(t)$.

(c) A similar analysis can be realized for $z(t) = 1$.

The mechanical discussion (1.9) can be summarized by the following formula introducing the Heaviside function $U(s) = 1$ for $s \geq 0$, $U(s) = 0$ for $s < 0$:

\[
l(t) = x' \quad x' U(x') = x' U(x') \quad \text{if} \quad z = 1,
\]
\[
= x' = 0 \quad \text{if} \quad |z| < 1,
\]
\[
= x' - x' U(x') = x' U(-x') \quad \text{if} \quad z = -1.
\]
(1.10)

The existence of an $\mathbb{R}^3$-valued function $\xi(t) = [x(t), x'(t), z(t)]^T$ satisfying (1.8) and (1.10) for a given driving force $g_2 \subset C^0(J)$ is not evident. Moreover, (1.10) shows that the velocity $D\xi = \xi'$ is discontinuous when the moving point $\xi(t) \in \mathbb{R}^3$ reaches the boundary of the permitted domain $G = \mathbb{R}^2 \times [-1, +1]$. Therefore, what can be the mathematical meaning of (1.8)? For these reasons, the multivalued equation (M. Diff. Eq.) is considered in the particular case where $X = \mathbb{R}^3$ and

\[
b = \begin{bmatrix} 0 & 1 & 0 \\ -r & -2h & -(1 - r) \\ 0 & 1 & 0 \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ g_2(t) \\ 0 \end{bmatrix},
\]
(1.11)
$k$ denoting the multivalued vector field on $\mathbb{R}^3$ confining on $G$. Theorem (I.3) shows that for an arbitrary initial data $u_0 \in G$ there exists one and only one solution $u \in H^1(J, \mathbb{R}^3)$ with initial value $u_0$ satisfying (M. Diff. Eq.). Hence $x(t) \in H^2(J)$ but in general $x \notin C^2(J)$. Formula (I.4) specifies the meaning of (I.8). More precisely, denoting $k"$ the subderivative of $Z_G$ where $G$ denotes the subset $[-1, +1]$ of the line, (I.4) gives

$$D^+ z(t) = (x'(t) - k"(z(t)))^\infty, \quad \forall t \in [0, T].$$

(I.12) Remarks. (a) In the particular case $r = 0$, the $\mathbb{R}^2$-valued function $u(t) = [x'(t), z(t)]'$ satisfies a multivalued differential equation independent from the first component $x(t)$ of $\zeta(t)$; and $x(t) = \int_0^t x'(s) \, ds$.

(b) As is well known, oscillations on the line with a Coulomb friction can be described by a multivalued differential equation in $x(t)$ and $x'(t)$. Also in this case, formula (I.4) specifies the expression of the friction force.

II. MULTIVALUED STOCHASTIC DIFFERENTIAL EQUATIONS

In general, $k$ denotes a maximal monotonic operator of $\mathbb{R}^n$, and notations of [3] will be used.

(II.1) Probabilistic data. A probability space $(\Omega, \mathcal{F}, P)$, a second order r.v. $\xi_0$ with probability law $\mathcal{L}(\xi_0)$ concentrated on $\text{Dom} \ k$, a Wiener process $W$ defined on $J = [0, T]$ and an increasing family of complete $\sigma$-fields $\mathcal{F}_t$ ($t \in J$) are given s.t.

(a) $\xi_0$ is $\mathcal{F}_0$-measurable.

(b) $\forall t \in J$, $\mathcal{F}_t$ contains the $\sigma$-fields generated by the r.v. $W(s)$ for $0 \leq s \leq t$.

(c) $\forall t \in J$, $\mathcal{F}_t$ is independent from the r.v. $W(t + \lambda) - W(t)$ for $\lambda > 0$.

Let $b: \mathbb{R}^n \times J \rightarrow \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \times J \rightarrow \text{End} \mathbb{R}^n$ be two continuous functions s.t. for some $c > 0$, for arbitrary $x, x' \in \mathbb{R}^n$ and $t \in J$,

$$|b(x, t) - b(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq C |x - x'|, \quad |b(x, t)| + |\sigma(x, t)| \leq C(1 + |x|).$$

(II.2)

(II.3) Definition of a Solution of (M. St. Diff. Eq.). A solution on $J$ on (M. St. Diff. Eq.) is defined as a pair $(\xi, \eta) \in M^b_p(J, \mathbb{R}^n) \times L^p_{\eta}(J, \mathbb{R}^n)$ s.t.

(a) For almost all $\omega \in \Omega$, $\xi$ has continuous trajectories $\xi(\cdot, \omega)$ within $\text{Dom} \ k$. 

580-491-6
(b) For almost all $\omega \in \Omega$, for almost all $t \in J$, then $\eta(t, \omega) \in k(\xi(t, \omega))$.

(c) The following equality of Ito's differentials holds:

$$d\xi(t) = (b(\xi(t), t) - \eta(t)) dt + \sigma(\xi(t), t) dW(t).$$  \hspace{1cm} (11.4)

(II.5) There exists at most one solution of (M. St. Diff. Eq.) with initial data $\xi_0$. In fact let $(\xi^1, \eta^1), i = 1, 2$ be two solutions with initial data $\xi_0$,

$$d(\xi^1 - \xi^2) = [-(\eta^1 - \eta^2) + b(\xi^1) - b(\xi^2)] + [\sigma(\xi^1 - \sigma(\xi^2)] dW.$$  

Hence using Ito's formula and monotonicity,

$$\frac{1}{2} |\xi^1(t) - \xi^2(t)|^2 \leq \int_0^t \langle b(\xi^1) - b(\xi^2), \xi^1 - \xi^2 \rangle ds + \int_0^t \langle \xi^1 - \xi^2, \sigma(\xi^1) - \sigma(\xi^2) \rangle dW$$

$$+ \int_0^t \text{tr}(\sigma(\xi^1) - \sigma(\xi^2))(\sigma^T(\xi^1) - \sigma^T(\xi^2)) ds.$$  

Hence taking the expectations of both members and using $2ab \leq a^2 + b^2$,

$$E|\xi^1(t) - \xi^2(t)|^2 \leq C \int_0^t E|\xi^1 - \xi^2|^2 ds.$$  

Hence by the Gronwall lemma,

$$E|\xi^1(t) - \xi^2(t)|^2 = 0, \forall t \in J.$$  

Hence $\xi^2$ is a version of $\xi^1$: this proves (II.5).

The following result, combined with [8], proves existence and uniqueness of random oscillations with bilinear hysteresis, generated by a Gaussian input, with an arbitrary rational spectral density.

(II.6) Theorem. The following hypotheses are assumed on (M. St. Diff. Eq.).

(a) There exists $p \in \{0, 1, \ldots, n - 1\}$ and a maximal monotonic operator $k''$ on $\mathbb{R}^p$ satisfying $H(k'')$ s.t. Dom $k = \mathbb{R}^{n-p} \times$ Dom $k''$ and $\forall x = (x', x'') \in \text{Dom} k, k(x) = (0, k''(x''))$.

(b) Using the block decomposition of the matrices $\sigma(x, t)$ associated with the decomposition $\mathbb{R}^{n-p} \oplus \mathbb{R}^p$ of $\mathbb{R}^n$ as a direct sum, then

$$\sigma(x, t) = \begin{bmatrix} \sigma_1(x, t) & \sigma_2(x, t) \\ 0 & 0 \end{bmatrix}, \forall (x, t) \in \mathbb{R}^n \times J.$$  \hspace{1cm} (II.7)
Then there exists one and only one solution of \((M. \ St. \ Diff. \ Eq.)\) on \(J\) with initial data \(\xi_0\). Moreover, for almost all \(\omega\), \(\xi''(\cdot, \omega) \in H^1(J, \mathbb{R}^p)\) is derivable on the right and \(\forall t \in J,\)
\[
D^+ \xi''(t, \omega) = (b''(\xi(t, \omega), t - k''(\xi''(t, \omega))))^\circ.
\]  

**Proof.** Vectors, drift, processes are systematically denoted below in the following way using the decomposition on direct sum \(\mathbb{R}^{n-F} \oplus \mathbb{R}^p\) of \(\mathbb{R}^n:\)
\[
x = \begin{pmatrix} x' \\ x'' \end{pmatrix}; \quad b = \begin{pmatrix} b' \\ b'' \end{pmatrix}; \quad \xi = \begin{pmatrix} \xi' \\ \xi'' \end{pmatrix}; \quad \eta = \begin{pmatrix} \eta' \\ \eta'' \end{pmatrix}; \quad W = \begin{pmatrix} W' \\ W'' \end{pmatrix}.
\]

(a) We first prove that for arbitrary \(\xi \in M^2_w(J, \mathbb{R}^n)\) there exists one and only one process \(\zeta \in M^2_w(J, \mathbb{R}^n)\) s.t.
\[
d\zeta(t) + k(\zeta(t)) \, dt \equiv b(\zeta(t), t) \, dt + \sigma(\zeta(t), t) \, dW(t)
\]
and with initial date \(\zeta_0\). In view of hypotheses (a) and (b), this equation is equivalent to the system
\[
\begin{align*}
(a) \quad &d\zeta'(t) = b'(\zeta(t), t) \, dt + \sigma'(\zeta(t)) \, dW(t), \\
(b) \quad &d\zeta''(t) + k''(\zeta''(t)) \, dt \equiv b''(\zeta(t), t) \, dt.
\end{align*}
\]  

Hence the part \(\zeta'\) of the process \(\zeta\) is given by an Itô integral; (II.9-b) means that there exists \(\eta'' \in L^2_w(J, \mathbb{R}^p)\) s.t. for arbitrary \(t \geq 0,\)
\[
\xi''(t) = \xi''(0) + \int_0^t (b''(\xi(u), u) - \eta''(u)) \, du.
\]
Hence for almost all \(\omega \in \Omega\), there exists \(\xi''(\cdot, \omega) \in H^1(J, \mathbb{R}^p)\) which is a solution of the deterministic multivalued equation
\[
D\xi''(t, \omega) + k''(\xi''(t, \omega)) \equiv b''(\xi(t, \omega), t), \quad \text{a.e. on } J.
\]
The proof of (I.3) shows that \(\xi''\) is adapted and measurable. But the last equation gives, by monotonicity,
\[
|\xi''(t, \omega)|^2 - |\xi''(0, \omega)|^2 \leq 2 \int_0^t b''\xi'' \, du
\]
\[
\leq 2 \int_0^t |b''(\xi(u), u)|^2 \, du + \int_0^t |\zeta''(t, \omega)|^2 \, du.
\]
Hence Gronwall’s lemma gives \(\xi'' \in M^2_w(J, \mathbb{R}^p).\)

(b) Let \(Y = M^2_w(J, \mathbb{R}^n)\) be the completion of \(M^2_w(J, \mathbb{R}^n)\). In general if a map \(S\) of a complete metric space \(Y\) is s.t. \(S^k\) is a strict contraction for
some integer \( k \), then \( S \) has one and only one fixed point. We now prove that for some \( k \), \( S^k \) is a strict contraction. For \( i = 1, 2 \) let \( \xi^i \in M^2_{\text{pr}}(J, \mathbb{R}^n) \) and \( \xi^i = S^k \xi^i \):

\[
dt \xi^i + k(\xi^i) dt \in b(\xi^i) du + \sigma(\xi^i) dW.
\]

Hence combining Itô's lemma with a monotonicity argument,

\[
\frac{1}{2} |\xi^1(t) - \xi^2(t)|^2 \leq \int_0^t \langle b(\xi^1) - b(\xi^2), \xi^1 - \xi^2 \rangle du + \int_0^t \langle \sigma(\xi^1) - \sigma(\xi^2), \xi^1 - \xi^2 \rangle dW
\]

\[
+ \frac{1}{2} \int_0^t \text{tr}(\sigma(\xi^1) - \sigma(\xi^2))(\sigma^T(\xi^1) - \sigma^T(\xi^2)) du.
\]

Hence considering the expectations of both members and using Gronwall's lemma,

\[
E |\xi^1(t) - \xi^2(t)|^2 \leq C \int_0^t E |\xi^1(u) - \xi^2(u)|^2 du, \quad \forall t \subset J
\]

Hence recursively on \( k \),

\[
\|S^k \xi^1 - S^k \xi^2\|^2 \leq \frac{C^{k+1} T^{k+1}}{(k+1)!} \|\xi^1 - \xi^2\|^2.
\]

Hence \( S^k \) is a strict contraction if \( k \) is big enough.

(c) Let \( \xi \) be a fixed point of \( S \) in \( Y \). Hence \( \|\xi - S(\xi)\| = 0 \) and \( S \xi \in M^2_{\text{pr}}(J, \mathbb{R}^n) \) is also a fixed point of \( S \). Since \( S \xi \) has almost surely continuous trajectories, \( \xi \) is a solution of (M. St. Diff. Eq.) and \( \xi(0) = \xi_0 \). Part (a) of the proof in the particular case \( \zeta = \xi \) shows that for almost all \( \omega \in \Omega \), \( \xi''(\cdot, \omega) \in H^1(J, \mathbb{R}^p) \) is a solution of the multivalued equation

\[
D\xi''(t, \omega) + k''(\xi''(t, \omega)) \in b''(\xi(t, \omega), t).
\]

Hence (II.8) follows from (I.4).

III. THE DIFFUSION OPERATOR OF (M. ST. DIFF. EQ.)

Notations and hypothesis of Theorem (II.6) are used.

(III.1) Transition probabilities \( p \) and \( P^F \). For arbitrary \( s \in J \) and \( y \in \text{Dom } k \), \( \xi_{sy} \) denotes the unique continuous and \( \text{Dom } k \)-valued process...
defined on $J_s = [s, T]$, which is a solution of (M. St. Diff. Eq.) starting from $y$ at time $s$. The methods used in the theory of diffusion on $\mathbb{R}^n$ show that the set of probability laws

$$p(s, y, t, dx) = \mathcal{L}(\xi_{s,y}(t)) \quad \text{for} \quad y \in \text{Dom } k, s \text{ and } t \in J, s \leq t,$$

is a probability of transition $p$ on $\text{Dom } k$, and that the process $\xi(t)$ has the Markov property with respect to $p$ and $\{\mathcal{F}_t, t \in J\}$. In the particular case $k = 0$, the previous notations $\xi_{s,y}, p(\cdots), \xi(t)$ are replaced by the notations $\xi^F_{s,y}, p^F(\cdots)$, and $\xi^F(t)$.

**III.3 Some linear operators.** For any integer $r$, $C^r_c(\text{Dom } k \times \text{int } J)$ denotes the space of restrictions to $V = \text{Dom } k \times \text{int } J$ of all real $C^r$ functions $\varphi = \varphi(x, s)$ on $\mathbb{R}^n \times \text{int } J$ with compact support. For arbitrary fixed $s \in \text{int } J$, the function $\varphi(\cdot, x) \in C^r(\text{Dom } k)$ is denoted $\varphi_s = \varphi_s(x)$. In the free case, for arbitrary $s \in J$ and $\varphi \in C^r_0 (\mathbb{R}^n \times \text{int } J)$,

$$y \mapsto \lim_{h \downarrow 0} h^{-1} \left[ \int p^F(s, y, s + h, dx) \varphi_s(x) - \varphi_s(y) \right]$$

on $\mathbb{R}^n$ is defined and continuous. The limit is denoted $L^F_s(y, \partial_y) \varphi_s(y)$, where $L^F_s$ is the diffusion operator at time $s$.

$$L^F_s(y, \partial_y) = \frac{1}{2} \sum_{i,j} a_{ij}(y, s) \partial_{ij} + \sum_j b_j(y, s) \partial_j,$$

(III.4)

with $a_{ij} = \sum_k \sigma_{ik} \sigma_{jk}$. In the constrained case, the following differential operators are defined on $\text{Dom } k$ for arbitrary $s \in J$:

$$L_s(y, \partial_y) = \frac{1}{2} \sum_{i,j \leq n-p} a_{ij}(y, s) \partial_{ij} + \sum_{l \leq n-p} b_l(y, s) \partial_j.$$

(III.5)

Note that $L_s$ contains only derivation with respect to $y_1, y_2, \ldots$ and $y_{n-p}$.

For arbitrary $y \in \text{Dom } k$ and $s \in J$, the components of the vector of $\mathbb{R}^p$

$$b^{1N}(y, s) = (b''(y, s) - k''(y))^{\circ}$$

(III.6)

are denoted $b_j^{1N}(y, s)$ with $n-p < j \leq n$. The following differential operators on $\text{Dom } k$

$$b_j^{1N}(y) \partial'' = \sum_{j=n-p+1, \ldots, n} b_j^{1N}(y, s) \partial_j,$$

(III.7)

contain only derivations with respect to $y_{n-p+1}, \ldots$ and $y_n$. 
(III.8) **Proposition.** For arbitrary \( \varphi \in C^2_0 \), \( s \in \text{int } J \) and \( y \in \text{Dom } k \), the numbers

\[
(L_s \varphi_s)^h(y) = Eh^{-1}[\varphi_s \circ \xi_{s,y}(s + h) - \varphi_s(y)]
\]  

(III.9)

tend to the following limit for \( h \downarrow 0 \),

\[
(L_s \varphi_s)(y) = L_s^{''}(y, \partial_y) \varphi_s(y) + b_s^{1N}(y) \partial'' \varphi_s(y).
\]  

(III.10)

Moreover,

\[
\sup_{h,y,s} |(L_s \varphi_s)^h(y)| < \infty.
\]  

(III.11)

**Proof.** (III.9) can be written as the sum of two terms:

\[
(L_s \varphi_s)^h(y) = I^h(y, s) + II^h(y, s),
\]

with

\[
I^h(y, s) = Eh^{-1}[\varphi_s(\xi_{s,y}(s + h), \xi_{s,y}^{''}(s + h)) - \varphi_s(y', \xi_{s,y}^{''}(s + h))],
\]

\[
II^h(y, s) = Eh^{-1}[\varphi_s(y', \xi_{s,y}^{''}(s + h)) - \varphi_s(y', y'')].
\]

The limit for \( h \downarrow 0 \) of \( I^h(y, s) \) is computed using Itô’s formula:

\[
I^h(y, s) = h^{-1}E \int_0^h \ du \sum_{i < n - p} b_i(\xi_{s,y}^{''}(s + u), \xi_{s,y}(s + u)), s + u)
\]

\[
\times (\partial_y \varphi_s)(\xi_{s,y}(s + u), \xi_{s,y}(s + h))
\]

\[
+ h^{-1}E \int_0^h \ du \sum_{i, j < n - p} a_{ij}(\xi_{s,y}(s + u), \xi_{s,y}(s + u), s + u)
\]

\[
\times (\partial_{ij} \varphi_s)(\xi_{s,y}(s + u), \xi_{s,y}(s + h)).
\]

Hence putting \( u = \nu h \) and applying Lebesgue’s theorem, the above

\[
= \frac{1}{2} \sum_{i, j} a_{ij}(y, s) \partial_{ij} \varphi_s(y) + \sum_{i < n} b_i(y, s) \partial_i \varphi_s(y).
\]

Moreover,

\[
\sup_{h,y,s} |I^h(y, s)| < \infty.
\]

Concerning the second term, (II.8) gives

\[
D^+ \xi_{s,y}^{''}(s + u) = b_s^{1N}(\xi_{s,y}(s + u), s + u).
\]
Hence by ordinary differential calculus,
\[ II^h(y, s) = E h^{-1} \int_0^h du \sum_{j > n-p} \partial_j \phi_s(y', \xi_{s,y}(s + u)) \cdot b^1_{1N}(y', \xi_{s,y}(s + u), s + u) \]
\[ = E \int_0^h dv \sum_{j > n-p} \partial_j \phi_s(y', \xi_{s,y}(s + vh)) b^1_{1N}(y', \xi_{s,y}(s + vh), s + vh). \]

This is the integral on \( \Omega \times [0, 1] \) of functions uniformly bounded for all \( s, y \) and \( h \). Hence \( \sup_{h,y,s} |II^h(y, s)| < \infty \) and (III.11) is proven. Finally, Lebesgue's convergence theorem gives
\[ \lim_{h \downarrow 0} II^h(y, s) = b^1_{1N}(y) \partial'' \phi_s(y). \]

(III.12) Domain of a semi-group \[14]. Let \( G'' \) be a closed convex subset of \( \mathbb{R}^n \) whose boundary \( \Sigma'' \) is a \( C^\infty \) manifold of dimension \( p - 1 \). We consider the particular case where \(-k = (0, -k'')\) is the subderivative of \( Z'' \), where \( G = \mathbb{R}^{n-p} \times G'' \) and where \( h = h(x) \) and \( \sigma = \sigma(x) \) are time independent. Let \( t \to V(t) = \exp tF \) be the semi-group defined by \( p \) in the Banach space of all real and continuous functions \( \psi(x) \) on \( \text{Dom} \ k \) converging to zero if \( |x| \to \infty \). If the space of bounded and Borelian function on \( G \) is denoted by \( B(G) \), the linear operator \( L_x : C^0(G) \to B(G) \) is \( s \)-independent. For arbitrary \( y \in \Sigma = \mathbb{R}^{n-p} \times \Sigma'' \), \( \psi(y) = (0, \psi''(y)) \) denotes the normal outward vector in \( y \). Putting \( \Sigma_2 = \{ y \in \Sigma; b''_s(y) \cdot \psi''(y) > 0 \} \), (III.8) implies that for arbitrary \( \psi \in C^0(G) \),
\[ \psi \in \text{Dom} \ F \iff \partial \psi(y)/\partial v = 0, \forall y \in \Sigma_2. \] (III.14)

(III.15) Connection with boundary conditions for diffusion. We consider (M. St. Diff. Eq.) assuming \( k \) is as in (III.12), but with \( b \) and \( \sigma \) eventually time dependent. Then (III.12) shows that (M. St. Diff. Eq.) describes a degenerate diffusion with a Neumann condition on a variable part \( \Sigma_2(s) \) of the boundary, and no condition on \( \Sigma \setminus \Sigma_2(s) \).

(III.16) Application to oscillators with hysteresis. Assuming \( n = 3, p = 1, G'' = [-1, +1] \), the oscillator with bilinear hysteresis of Section I is driven by \( a(t)(DW_2)(t) \), where \( a(t) \) denotes a given continuous and positive function on \( J \). The initial state is the origin of \( \mathbb{R}^3 \). Hence the equation of motion is (M. St. Diff. Eq.) with \( \sigma_{ij}(t) = a(t) \) for \( i = j = 2 \) and otherwise \( \sigma_{ij}(t) = 0 \). By (III.15), this equation is a degenerate differential equation with the Neumann condition on \( \Sigma_2 = \Sigma^+_2 \cup \Sigma^-_2 \) and no condition on \( \Sigma \setminus \Sigma_2 \) with \( \Sigma^\pm = \{(x, x', \pm 1) \in \Sigma; x' \geq 0 \} \). This equation does not enter in the classes of problems considered in [13].
IV. DIFFUSION EQUATION FOR PROBABILITY LAWS

In this section, the functions $b(x, t)$ and $\sigma(x, t)$ on $J \times \mathbb{R}^n$ are supposed $C^\infty$ on $J \times \mathbb{R}^n$.

(IIV.1) **Definition of the Measure $M$ on Dom $k \times J$ Defined by the Probability Laws $\mathcal{L}(\xi(t))$.** Putting $m_t = \mathcal{L}(\xi(t))$, the following notation is used for $\varphi \in C^\infty(Dom k \times J)$ and $t \in J$:

$$
\langle m_t, \varphi \rangle = E(\varphi, t \circ \xi(t)) = \int \varphi(x, t) m_t(x).
$$

The following linear form on $C^\infty_0(G \times J)$,

$$
\varphi \rightarrow \langle M, \varphi \rangle = \int_0^T \langle m_t, \varphi \rangle \, dt,
$$

is positive and continuous: hence $M$ is a positive measure on $G \times J$.

(IIV.4) **Proposition.** For arbitrary $\varphi \in C^\infty_0(G \times \text{int } J)$, the bounded measurable function $x, t \rightarrow (L, \varphi_t)(x)$ on $G \times J$ is denoted $L \varphi$. Then

$$
- \langle M, \partial_t \varphi \rangle = \langle M, L \varphi \rangle, \quad \forall \varphi \in C^\infty_0(G \times \text{int } J). \quad \text{(Diff. Eq.)}
$$

This equation is called the diffusion equation and $L$ is called the diffusion operator.

**Proof.** (a) We assume that $\xi$ starts from some point $y_0 \in G$ at time $t = 0$. Integration of both members of $(L, \varphi_t)(y) = \lim (L, \varphi_t)^h(y)$ with resp. to $M$ gives, using Lebesgue's theorem,

$$
\langle M, L \varphi \rangle = \lim_{h \downarrow 0} h^{-1} \int_0^T dt \int p(0, y_0, t, dy) \left[ \int p(t, y, t + h, dx) \varphi_t(x) - \varphi_t(y) \right].
$$

Hence using the semi-group property of $p$, the above

$$
= \lim_{h \downarrow 0} \int_0^T dt \, h^{-1} \left[ \int p(0, y_0, t + h, dx) \varphi(t, x) - \varphi(t, y) \right].
$$

Since there exists $\varepsilon > 0$ s.t. $\varphi(x, t) = 0$ for $t < \varepsilon$ and for $t > T - \varepsilon$, the change of variable $t + h = t'$ is possible in the first integral in the brackets:
\[
\begin{align*}
&= \lim_{h \to 0} \int_0^T dt \ h^{-1} \int p(0, y_0, t, dx)[\varphi(t-h, x) - \varphi(t, x)] \\
&= -\int_0^T dt \int p(0, y_0, t, dx) \partial_t \varphi(t, x) = -\langle M, \partial_t \varphi \rangle.
\end{align*}
\]

(b) If the probability law \( \mu = \mathcal{L}(\xi_0) \) is arbitrary, the Markov property gives

\[
\langle m_t, \varphi_t \rangle = \int p(0, y_0, t, dx) \varphi(t, y_0) \mu(y_0). \tag{IV.5}
\]

Hence the proposition follows from (a) by linearity.

(IV.6) Corollary. Suppose \( k = k(x) \) is a Lipschitzian and \( C^\infty \) function on some open subset \( W \) of \( \text{Dom} \ k \). If the restriction of \( M \) to \( W \times \text{int} J \) is denoted by \( M' \), (Diff. Eq.) gives, for arbitrary \( \varphi \in C_0^\infty(W \times \text{int} J) \),

\[
\langle M', \partial_t \varphi \rangle = \langle M', L \varphi \rangle, \tag{IV.7}
\]

with

\[
L = L^F - \sum j \kappa_j(x) \partial_j. \tag{IV.8}
\]

In the present case, \( L \) can be transposed in (IV.7); hence

\[
\partial_t M' = L^T M'. \tag{IV.9}
\]

If the differential operator \( \partial_t - L^T \) on \( W \times \text{int} J \) is hypoelliptic \( M' \) has a \( C^\infty \) density \( f(x, t) \) with respect to the Lebesgue measure \( dx \ dt \). In the particular case where \( k = p = 0 \) and \( W = \mathbb{R}^n \), then (IV.9) shows that \( M = M' \) satisfies F.P.E.:

\[
\partial_t M = (L^F)^T M. \tag{IV.10}
\]

(IV.11) Equivalence of (Diff. Eq.) with a boundary value problem. In many practical situations Corollary (IV.6) can be applied with

\[
W = (\text{Dom} \ k \ \setminus \Sigma'), \tag{IV.12}
\]

where \( \Sigma' \) denotes some \( C^\infty \) manifold embedded in \( \text{Dom} \ k \). Introducing the restriction \( M'' \) of \( M \) to \( \Sigma' \times (\text{int} J) \) and denoting by \( \text{ext} \) the inverse of restriction operators, we have

\[
M = \text{ext} M' + \text{ext} M''.
\]
In general, $f = f(x, t)$ is continuous on $V$; hence the restriction $\gamma f$ of $f$ to $\Sigma' \times (\text{int } J)$ can be defined. Then (Dif." Eq.) can be converted by a calculus of distribution theory, in a system of coupled p.d.e. satisfied by the unknown measures $M' = f(x, t) \, dx \, dt$ and $M''$. The example given below shows how F.P.E. has to be replaced for oscillators with bilinear hysteresis.

\textbf{(IV.13) PROPOSITION.} The equation (M. St. Diff. Eq.) is considered in the particular case where $p = 1$ and where $-k$ is the multivalued vector field on $\mathbb{R}^n$ confining on $G = \mathbb{R}^{n-1} \times G''$, with $G'' = [-1, +1]$. The notations of (IV.11) are used with $\Sigma' = \Lambda = \partial G$. If $M'$ has a density $f = f(x, t)$ continuous on $V = G \times \text{int } J$, then the measures $M' = f(x, t) \, dx \, dt$ and $M''$, living, resp., on $\text{int } V$ and on $\partial V = \Sigma \times (\text{int } J)$, satisfy the following coupled partial differential equations:

\begin{align*}
\text{(a)} & \quad \partial_t M' = (L^t)^T M' \quad \text{on int } V, \\
\text{(b)} & \quad \partial_t M'' = b_n^{\text{OUT}}(\gamma f) + (L_t)^T M'' \quad \text{on } \partial V, \\
\text{(c)} & \quad M'' = 0 \quad \text{on } \partial \text{IN } V,
\end{align*}

where $b_n^{\text{OUT}} = b_n - b_n^{\text{IN}}$, and where

$$
\partial \text{IN } V = \{(x,t) \in \partial V; b(x, t) \cdot v(x) < 0\}. \quad \text{(IV.15)}
$$

Hence $\partial \text{IN } V = \bigcup_{t \in J} \Sigma_1(t) \times \{t\}$, where $\Sigma_1(t)$ denotes the part of $\Sigma$ where the drift is inward at time $t$. In the case (III.16), $\Sigma_1(t)$ is time independent since

$$
\Sigma_1(t) = \{(x, x', 1) \in \mathbb{R}^3; x' < 0\} \cup \{(x', x', -1) \in \mathbb{R}^3; x' > 0\}.
$$

\textbf{Proof.} Conventionally, (Dif." Eq.) is written suppressing the time integration, i.e., under the form

$$
- \langle m_t, \partial_t \phi \rangle = \langle m_t, L_t \phi \rangle. \quad \text{(IV.16)}
$$

Introducing the functions $t \to m'_t = f(x, t) \, dx$ and $t \to m''_t$ defining, resp., $M'$ and $M''$, the L.H.S. of (IV.16) has two parts:

$$
- \langle m_t, \partial_t \phi \rangle = \langle m'_t, L^t \phi \rangle + \langle m''_t, L_t \phi \rangle,
$$

with $L^t_t = L'_t + b_n \partial_n$; $L_t = L'_t + b_n^{\text{IN}} \partial_n$. Hence

$$
- \langle m'_t, \partial_t \phi \rangle - \langle m''_t, \partial_t (\gamma \phi) \rangle
+ \int_{x \in G} (L'_t \phi) m'_t(x) \, dx + \int_{x \in G} b_n(x, t) \partial_n \phi(x, t) \, m'_t(x)
+ \int_{x \in \Sigma} (L'_t \phi)(x') m''_t(x') + \int_{x' \in \Sigma} b_n^{\text{IN}}(x', t) (\partial_n \gamma \phi)(x') \, m''_t(x').
$$
Performing integration by parts,

\[ 0 = -\langle \partial_t m', \varphi \rangle - \langle \partial_t m", \varphi \rangle, \]

\[ + \langle (L^T) m', \varphi \rangle + \int_{x \in \Sigma} (\gamma b_n)(\varphi)(m'_x) \]

\[ - \int_{x \in \Sigma} (\gamma \varphi)(\partial_n b_n) m'_x(x) + \int_{x \in \Sigma} (L^T \varphi)(x') m''(x') \]

\[ + \int_{x \in \Sigma} \gamma b_n^N(x')(\partial_n \varphi)(x') m''(x'). \]

This can be written \( 0 = \langle \langle T, \varphi \rangle \rangle \) for all \( \varphi \in C_0^\infty(V) \), where \( T \) is the sum of three distributions on \( V \): \( T_1 = \int (x, t) \) \( dx \) \( dt \), one simple sheet \( T_2 \) and one double sheet \( T_3 \) supported by \( \partial V \). Writing \( 0 = T_1 = T_2 = T_3 \), we obtain the following three equations:

\[
\begin{align*}
(a) \quad \partial_t M' &= (L^T) M', \\
(b) \quad \partial_t M'' &= (L^T) M'' + (\pm \gamma M')(\gamma b_n), \\
(c) \quad (\gamma b_n^N) M'' &= 0,
\end{align*}
\]

with the sign + (resp. -) in (b) if \( x_n = +1 \) (resp. \( x_n = -1 \)); hence (IV.14.b) \( \Leftrightarrow \) (IV.17.b).

For arbitrary \( (x, t) \in \partial V \) we have \( (\gamma b_n^N)(x', t) \neq 0 \) iff \( (x', t) \in \partial^N V \). Hence (c) means \( M'' = 0 \) on \( \partial^N V \). Hence (IV.14.c) \( \Leftrightarrow \) (IV.17.c).

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