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Affine surfaces with trivial Makar-Limanov invariant

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Abstract

We study the class of 2-dimensional affine $\mathbf{k}$-domains R satisfying $\mathrm{ML}(R) = \mathbf{k}$, where $\mathbf{k}$ is an arbitrary field of characteristic zero. In particular, we obtain the following result: *Let R be a localization of a polynomial ring in finitely many variables over a field of characteristic zero. If* $\mathrm{ML}(R) = K$ *for some field* $K \subset R$ *such that* $\operatorname{trdeg}_K R = 2$*, then R is K-isomorphic to $K[X, Y, Z]/(XY - P(Z))$ for some nonconstant* $P(Z) \in K[Z]$.

Keywords: Locally nilpotent derivations; Group actions; Danielewski surfaces; Affine surfaces; Makar-Limanov invariant; Absolute constants

1. Introduction

Let us recall the definition of the Makar-Limanov invariant:

1.1. Definition. If R is a ring of characteristic zero, a derivation $D : R \to R$ is said to be *locally nilpotent* if for each $r \in R$ there exists $n \in \mathbb{N}$ (depending on r) such that $D^n(r) = 0$. We use the following notations:

$$\mathrm{LND}(R) = \text{set of locally nilpotent derivations } D : R \to R,$$
$$\mathrm{KLND}(R) = \{\ker D \mid D \in \mathrm{LND}(R) \text{ and } D \neq 0\},$$

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$$\mathrm{ML}(R) = \bigcap_{D \in \mathrm{LND}(R)} \ker(D).$$

We are interested in the class of 2-dimensional affine $\mathbf{k}$-domains R satisfying $\mathrm{ML}(R) = \mathbf{k}$, where $\mathbf{k}$ is a field of characteristic zero. The corresponding class of affine algebraic surfaces was studied by several authors ([1,2,7–9,14,17], in particular), but almost always under the assumption that $\mathbf{k}$ is algebraically closed, or even $\mathbf{k} = \mathbb{C}$. In this paper we obtain some partial results valid when $\mathbf{k}$ is an arbitrary field of characteristic zero. We are particularly interested in the following subclass:

1.2. Definition. Given a field $\mathbf{k}$ of characteristic zero, let $\mathfrak{D}(\mathbf{k})$ be the class of $\mathbf{k}$-algebras isomorphic to $\mathbf{k}[X, Y, Z]/(XY - \varphi(Z))$ for some nonconstant polynomial in one variable $\varphi(Z) \in \mathbf{k}[Z] \setminus \mathbf{k}$, where X, Y, Z are indeterminates over $\mathbf{k}$.

The class $\mathfrak{D}(\mathbf{k})$ was studied in [4,5,16], in particular. It is well known that if $R \in \mathfrak{D}(\mathbf{k})$ then R is a 2-dimensional normal affine domain satisfying $\mathrm{ML}(R) = \mathbf{k}$. It is also known that the converse is not true, which raises the following:

Question. *Suppose that R is a 2-dimensional affine $\mathbf{k}$-domain with $\mathrm{ML}(R) = \mathbf{k}$. Under what additional assumptions can we infer that $R \in \mathfrak{D}(\mathbf{k})$?*

Section 3 completely answers this question in the case where R is a smooth $\mathbf{k}$-algebra. This is achieved by reducing to the case $\mathbf{k} = \mathbb{C}$, which was solved by Bandman and Makar-Limanov. This reduction is nontrivial, and makes essential use of the main result of Section 2. Also note Corollary 3.8, which gives a pleasant answer to the above question in the factorial case. Then we derive several consequences from Section 3, for instance consider the following special case of Theorem 4.1:

> *Let R be a localization of a polynomial ring in finitely many variables over a field of characteristic zero. If $\mathrm{ML}(R) = K$ for some field $K \subset R$ such that $\mathrm{trdeg}_K R = 2$, then $R \in \mathfrak{D}(K)$.*

In turn, this has consequences in the study of G_a-actions on $\mathbb{C}^n$.

Conventions. All rings and algebras are commutative, associative and unital. If A is a ring, we write A^* for the units of A; if A is a domain, $\mathrm{Frac}\, A$ is its field of fractions. If $A \subseteq B$ are rings, "$B = A^{[n]}$" means that B is A-isomorphic to the polynomial algebra in n variables over A. If L/K is a field extension, "$L = K^{(n)}$" means that L is a purely transcendental extension of K and $\mathrm{trdeg}_K L = n$ (transcendence degree).

In [5], one defines a Danielewski surface to be a pair $(R, \mathbf{k})$ such that $R \in \mathfrak{D}(\mathbf{k})$. In the present paper we avoid using the term "Danielewski surface" in that sense, because it is incompatible with accepted usage. The reader should keep this in mind when consulting [5] (our main reference for Section 2).

2. Base extension

Let $\mathbf{k}$ be a field of characteristic zero. It is clear that if $R \in \mathfrak{D}(\mathbf{k})$ then $K \otimes_{\mathbf{k}} R \in \mathfrak{D}(K)$ for every field extension $K/\mathbf{k}$. However, if $K \otimes_{\mathbf{k}} R \in \mathfrak{D}(K)$ for some K, it does not follow that $R \in \mathfrak{D}(\mathbf{k})$ (see Example 2.2, below).

2.1. Remark. If $R \in \mathfrak{D}(\mathbf{k})$ then $\operatorname{Spec} R$ has infinitely many $\mathbf{k}$-rational points. (Indeed, if $R = \mathbf{k}[X, Y, Z]/(XY - \varphi(Z))$ then there is a bijection between the set of $\mathbf{k}$-rational points of $\operatorname{Spec} R$ and the zero-set in $\mathbf{k}^3$ of the polynomial $XY - \varphi(Z)$.)

2.2. Example. Let $A = \mathbb{R}[X, Y, Z]/(f)$, where $f = X^2 + Y^2 + Z^2$. Viewing f as an element of $\mathbb{C}[X, Y, Z]$ we have $f = (X + iY)(X - iY) + Z^2$ (where $i^2 = -1$), so $\mathbb{C} \otimes_{\mathbb{R}} A \cong \mathbb{C}[U, V, W]/(UV + W^2) \in \mathfrak{D}(\mathbb{C})$. As $\operatorname{Spec} A$ has only one $\mathbb{R}$-rational point, $A \notin \mathfrak{D}(\mathbb{R})$ by Remark 2.1. Thus

$$A \notin \mathfrak{D}(\mathbb{R}) \quad \text{and} \quad \mathbb{C} \otimes_{\mathbb{R}} A \in \mathfrak{D}(\mathbb{C}).$$

Note[2] that Theorem 2.3 (below) implies that $\operatorname{ML}(A) = A$. Moreover, if we define $A' = \mathbb{R}[U, V, W]/(UV + W^2) \in \mathfrak{D}(\mathbb{R})$ then $A \ncong A'$ but $\mathbb{C} \otimes_{\mathbb{R}} A \cong \mathbb{C} \otimes_{\mathbb{R}} A'$.

2.3. Theorem. *For an algebra R over a field $\mathbf{k}$ of characteristic zero, the following conditions are equivalent*:

(a) $R \in \mathfrak{D}(\mathbf{k})$;
(b) $\operatorname{ML}(R) \neq R$ *and there exists a field extension $K/\mathbf{k}$ such that* $K \otimes_{\mathbf{k}} R \in \mathfrak{D}(K)$.

We shall prove this after some preparation.

2.4. Some facts. Refer to [11] or [13] for background on locally nilpotent derivations. Statement (c) is due to Rentschler [20] and (d) to Nouazé and Gabriel [19] and Wright [21].

(a) If $A \in \operatorname{KLND}(B)$ where B is a domain of characteristic zero then A is *factorially closed* in B (i.e., if $x, y \in B \setminus \{0\}$ and $xy \in A$ then $x, y \in A$). It follows that $\operatorname{ML}(B)$ is factorially closed in B. Any factorially closed subring A of B is in particular *algebraically closed* in B (i.e., if $x \in B$ is a root of a nonzero polynomial with coefficients in A then $x \in A$) and satisfies $A^* = B^*$ (in particular, any field contained in B is contained in A).
(b) Let B be a noetherian domain of characteristic zero. If $0 \neq D \in \operatorname{LND}(B)$ then $D = \alpha D_0$ for some $\alpha \in \ker(D)$ and $D_0 \in \operatorname{LND}(B)$ where D_0 is *irreducible* (i.e., the only principal ideal of B which contains $D_0(B)$ is B).
(c) Let $B = \mathbf{k}^{[2]}$ where $\mathbf{k}$ is a field of characteristic zero. If $D \in \operatorname{LND}(B)$ is irreducible then there exist X, Y such that $B = \mathbf{k}[X, Y]$ and $D = \partial/\partial Y$.
(d) Let B be a $\mathbb{Q}$-algebra. If $D \in \operatorname{LND}(B)$ and $s \in B$ satisfy $Ds \in B^*$ then $B = A[s] = A^{[1]}$ where $A = \ker D$.

[2] A different proof that $\operatorname{ML}(A) = A$ is given in [13, 9.21].

2.5. Lemma. *Let* $\mathbf{k}$ *be a field of characteristic zero and* R *a* $\mathbf{k}$*-algebra satisfying*:

$$\text{there exists a field extension } \bar{\mathbf{k}}/\mathbf{k} \text{ such that } \bar{\mathbf{k}} \otimes_{\mathbf{k}} R \in \mathfrak{D}(\bar{\mathbf{k}}).$$

Then R *is a two-dimensional normal affine domain over* $\mathbf{k}$ *and* $R^* = \mathbf{k}^*$.

Proof. This is rather simple but it will be convenient to refer to this proof later. Choose a field extension $\bar{\mathbf{k}}/\mathbf{k}$ such that $\bar{\mathbf{k}} \otimes_{\mathbf{k}} R \in \mathfrak{D}(\bar{\mathbf{k}})$ and let $\bar{R} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} R$. As R is a flat $\mathbf{k}$-module, the canonical homomorphism $\mathbf{k} \otimes_{\mathbf{k}} R \to \bar{\mathbf{k}} \otimes_{\mathbf{k}} R$ is injective, so we may regard R as a subring of $\bar{R}$. In particular, R is an integral domain and we have the diagram:

$$\begin{array}{ccccccc}
\bar{\mathbf{k}} & \hookrightarrow & \bar{R} & \hookrightarrow & S^{-1}\bar{R} & \hookrightarrow & \operatorname{Frac}\bar{R} \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbf{k} & \hookrightarrow & R & \hookrightarrow & \operatorname{Frac} R & &
\end{array}$$

where $S = R \setminus \{0\}$. Let $\mathcal{B}$ be a basis of $\bar{\mathbf{k}}$ over $\mathbf{k}$ such that $1 \in \mathcal{B}$. Note that $\mathcal{B}$ is also a basis of the free R-module $\bar{R}$ and of the vector space $S^{-1}\bar{R}$ over $\operatorname{Frac} R$. It follows:

$$\bar{\mathbf{k}} \cap R = \mathbf{k} \quad \text{and} \quad \bar{R} \cap \operatorname{Frac} R = R. \tag{1}$$

As $\bar{R} \in \mathfrak{D}(\bar{\mathbf{k}})$, [5, 2.3] implies that $\bar{R}^* = \bar{\mathbf{k}}^*$ and that $\bar{R}$ is a normal domain; so (1) implies that $R^* = \mathbf{k}^*$ and that R is a normal domain. Also:

$$\text{If } E \text{ is a subset of } R \text{ such that } \bar{\mathbf{k}}[E] = \bar{R}, \text{ then } \mathbf{k}[E] = R. \tag{2}$$

Indeed, $\mathcal{B}$ is a basis of the R-module $\bar{R}$ and a spanning set of the $\mathbf{k}[E]$-module $\bar{R}$; as $\mathbf{k}[E] \subseteq R$, it follows that $\mathbf{k}[E] = R$.

Note that R is affine over $\mathbf{k}$, by (2) and the fact that $\bar{R}$ is affine over $\bar{\mathbf{k}}$. Let $n = \dim R$ then, by Noether Normalization Lemma, there exists a subalgebra $R_0 = \mathbf{k}^{[n]}$ of R over which R is integral. Then $\bar{R} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} R$ is integral over $\bar{\mathbf{k}} \otimes_{\mathbf{k}} R_0 = \bar{\mathbf{k}}^{[n]}$, so $n = \dim \bar{R} = 2$. □

We borrow the following notation from [5, 2.1].

2.6. Definition. Given a $\mathbf{k}$-algebra R, let $\Gamma_{\mathbf{k}}(R)$ denote the (possibly empty) set of ordered triples $(x_1, x_2, y) \in R \times R \times R$ satisfying:

The $\mathbf{k}$*-homomorphism* $\mathbf{k}[X_1, X_2, Y] \to R$ *defined by*

$$X_1 \mapsto x_1, \quad X_2 \mapsto x_2 \quad \text{and} \quad Y \mapsto y$$

is surjective and has kernel equal to $(X_1X_2 - \varphi(Y))\mathbf{k}[X_1, X_2, Y]$ *for some nonconstant polynomial in one variable* $\varphi(Y) \in \mathbf{k}[Y]$.

Note that $R \in \mathfrak{D}(\mathbf{k})$ if and only if $\Gamma_{\mathbf{k}}(R) \neq \emptyset$.

Proof of Theorem 2.3. That $R \in \mathfrak{D}(\mathbf{k})$ implies $\mathrm{ML}(R) = \mathbf{k}$ is well known (for instance it follows from part (d) of [5, 2.3]), so it suffices to prove that (b) implies (a).

Suppose that R satisfies (b). Note that if $K/\mathbf{k}$ is a field extension satisfying $K \otimes_{\mathbf{k}} R \in \mathfrak{D}(K)$ then for any field extension L/K we have $L \otimes_{\mathbf{k}} R \in \mathfrak{D}(L)$. In particular, there exists a field extension $\bar{\mathbf{k}}/\mathbf{k}$ such that $\bar{\mathbf{k}} \otimes_{\mathbf{k}} R \in \mathfrak{D}(\bar{\mathbf{k}})$ and such that $\bar{\mathbf{k}}$ is an algebraically closed field. We fix such a field $\bar{\mathbf{k}}$. The fact that $\bar{\mathbf{k}}$ is algebraically closed implies that

$$\text{the fixed field } \bar{\mathbf{k}}^G \text{ is equal to } \mathbf{k} \tag{3}$$

where $G = \mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$. We use the notation ($\bar{R}$, $\mathcal{B}$, etc.) introduced in the proof of Lemma 2.5. As $\mathrm{ML}(R) \neq R$, there exists $0 \neq D \in \mathrm{LND}(R)$. Let $\bar{D} \in \mathrm{LND}(\bar{R})$ be the unique extension of D, let $A = \ker D$ and $\bar{A} = \ker \bar{D}$.

It follows from [5] that $\bar{A} = \bar{\mathbf{k}}^{[1]}$ ([5, 2.3] shows that some element of $\mathrm{KLND}(\bar{R})$ is a $\bar{\mathbf{k}}^{[1]}$ and, by [5, 2.7.2], $\mathrm{Aut}_{\bar{\mathbf{k}}}(\bar{R})$ acts transitively on $\mathrm{KLND}(\bar{R})$). Applying the exact functor $\bar{\mathbf{k}} \otimes_{\mathbf{k}} \_$ to the exact sequence $0 \to A \to R \xrightarrow{D} R$ of $\mathbf{k}$-linear maps shows that $\bar{\mathbf{k}} \otimes_{\mathbf{k}} A = \bar{A} = \bar{\mathbf{k}}^{[1]}$, so $A = \mathbf{k}^{[1]}$. Choose $f \in R$ such that $A = \mathbf{k}[f]$, then $\bar{A} = \bar{\mathbf{k}}[f]$.

Consider the nonzero ideals $I = A \cap D(R)$ and $\bar{I} = \bar{A} \cap \bar{D}(\bar{R})$ of A and $\bar{A}$, respectively. Let $\psi \in A$ and $s \in R$ be such that $I = \psi A$ and $D(s) = \psi$. We claim that

$$\bar{I} = \psi \bar{A}. \tag{4}$$

Indeed, an arbitrary element of $\bar{I}$ is of the form $\bar{D}(\sigma)$ where $\sigma \in \bar{R}$ and $\bar{D}^2(\sigma) = 0$. Write $\sigma = \sum_{\lambda \in \mathcal{B}} s_\lambda \lambda$ with $s_\lambda \in R$, then $0 = \bar{D}^2(\sigma) = \sum_{\lambda \in \mathcal{B}} D^2(s_\lambda)\lambda$, so for all $\lambda \in \mathcal{B}$ we have $D^2(s_\lambda) = 0$, hence $D(s_\lambda) \in I = \psi A$, and consequently $\bar{D}(\sigma) \in \psi \bar{A}$, which proves (4).

By 2.4(b), $\bar{D} = \alpha \Delta$ for some $\alpha \in \bar{A} \setminus \{0\}$ and some irreducible $\Delta \in \mathrm{LND}(\bar{R})$. Consider the nonzero ideal $I_0 = \bar{A} \cap \Delta(\bar{R})$ of $\bar{A}$. We claim that

$$I_0 = \Delta(s)\bar{A}. \tag{5}$$

To see this, consider an arbitrary element $\Delta(\sigma)$ of I_0 (where $\sigma \in \bar{R}$, $\Delta^2(\sigma) = 0$). Then $\alpha\Delta(\sigma) = \bar{D}(\sigma) \in \bar{I} = \psi\bar{A} = \bar{D}(s)\bar{A} = \alpha\Delta(s)\bar{A}$, so $\Delta(\sigma) \in \Delta(s)\bar{A}$ and (5) is proved.

Consider the case where $\Delta(s) \in \bar{R}^*$. Then $\bar{R} = \bar{A}[s] = \bar{\mathbf{k}}[f, s]$ by 2.4(d), so (2) implies that $R = \mathbf{k}[f, s] = \mathbf{k}^{[2]}$, so in particular $R \in \mathfrak{D}(\mathbf{k})$ and we are done.

From now on assume that $\Delta(s) \notin \bar{R}^*$. By [5, 2.8], $\bar{A} = \bar{\mathbf{k}}[\Delta(y)]$ for some $y \in \bar{R}$. Note that $\Delta(y) \in I_0$, so (5) gives $\Delta(s) \mid \Delta(y)$ in $\bar{A}$. As $\Delta(y)$ is an irreducible element of $\bar{A}$ (because $\bar{\mathbf{k}}[\Delta(y)] = \bar{A} = \bar{\mathbf{k}}^{[1]}$) and $\Delta(s) \notin \bar{A}^*$, we have $\bar{\mathbf{k}}[\Delta(s)] = \bar{A} = \bar{\mathbf{k}}[f]$ and consequently $\Delta(s) = \mu(f - \lambda)$ for some $\mu \in \bar{\mathbf{k}}^*$, $\lambda \in \bar{\mathbf{k}}$. We may as well replace Δ by $\mu^{-1}\Delta$, so

$$\Delta(s) = f - \lambda, \quad \text{for some } \lambda \in \bar{\mathbf{k}}. \tag{6}$$

We claim:

$$\{c \in \bar{\mathbf{k}} \mid \bar{R}/(f - c)\bar{R} \text{ is not an integral domain}\} = \{\lambda\}. \tag{7}$$

Indeed, [5, 2.8] implies that there exists $x_2 \in \bar{R}$ such that $(f - \lambda, x_2, s) \in \Gamma_{\bar{\mathbf{k}}}(\bar{R})$. This means (cf. 2.6) that the $\bar{\mathbf{k}}$-homomorphism $\pi : \bar{\mathbf{k}}[X_1, X_2, Y] \to \bar{R}$ defined by $X_1 \mapsto f - \lambda$, $X_2 \mapsto x_2$,

$Y \mapsto s$, is surjective and has kernel $(X_1X_2 - P(Y))$ for some nonconstant $P(Y) \in \bar{\mathbf{k}}[Y]$ (where X_1, X_2, Y are indeterminates). By (5) and $\Delta(s) \notin \bar{R}^*$, we see that there does not exist $\sigma \in \bar{R}$ such that $\Delta(\sigma) = 1$; as Δ is irreducible, it follows from 2.4(c) that $\bar{R} \neq \bar{\mathbf{k}}^{[2]}$ and hence that $\deg_Y P(Y) > 1$. Thus, for $c \in \bar{\mathbf{k}}$,

$$\bar{R}/(f-c)\bar{R} \cong \bar{\mathbf{k}}[X_1, X_2, Y]/\big(X_1 - (c-\lambda), X_1X_2 - P(Y)\big)$$

is a domain if and only if $c \neq \lambda$. This proves (7).

Let $\theta \in \mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$. Then θ extends to some $\Theta \in \mathrm{Aut}_R(\bar{R})$ and Θ determines a ring isomorphism

$$\bar{R}/(f-\lambda)\bar{R} \cong \bar{R}/\Theta(f-\lambda)\bar{R} = \bar{R}/\big(f-\theta(\lambda)\big)\bar{R}.$$

So $\bar{R}/(f-\theta(\lambda))\bar{R}$ is not a domain and it follows from (7) that $\theta(\lambda) = \lambda$. As this holds for every $\theta \in \mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$, (3) implies that $\lambda \in \mathbf{k}$. To summarize, if we define $x_1 = f - \lambda$ then

$$x_1, s \in R \text{ and there exists } x_2 \in \bar{R} \text{ such that } (x_1, x_2, s) \in \Gamma_{\bar{\mathbf{k}}}(\bar{R}).$$

We now show that x_2 can be chosen in R. Consider the ideals $J = \mathbf{k}[s] \cap x_1R$ of $\mathbf{k}[s]$ and $\bar{J} = \bar{\mathbf{k}}[s] \cap x_1\bar{R}$ of $\bar{\mathbf{k}}[s]$, and choose $\varphi(Y) \in \mathbf{k}[Y]$ such that $J = \varphi(s)\mathbf{k}[s]$. Let $\Phi(s)$ be any element of $\bar{J}$ (where $\Phi(Y) \in \bar{\mathbf{k}}[Y]$). Then $\Phi(s) = x_1G$ for some $G \in \bar{R}$. As $\mathcal{B}$ is a basis of the R-module $\bar{R}$ and also of the $\mathbf{k}[Y]$-module $\bar{\mathbf{k}}[Y]$, we may write $G = \sum_{\lambda\in\mathcal{B}} G_\lambda\lambda$ (where $G_\lambda \in R$) and $\Phi = \sum_{\lambda\in\mathcal{B}} \Phi_\lambda\lambda$ (where $\Phi_\lambda \in \mathbf{k}[Y]$). Then $\sum_{\lambda\in\mathcal{B}}(x_1G_\lambda)\lambda = \Phi(s) = \sum_{\lambda\in\mathcal{B}} \Phi_\lambda(s)\lambda$, so for every $\lambda \in \mathcal{B}$ we have $\Phi_\lambda(s) = x_1G_\lambda$, i.e., $\Phi_\lambda(s) \in J = \varphi(s)\mathbf{k}[s]$. We obtain that $\Phi(s) \in \varphi(s)\bar{\mathbf{k}}[s]$, so:

$$\bar{J} = \varphi(s)\bar{\mathbf{k}}[s].$$

On the other hand, [5, 2.4] asserts that $\bar{J} = x_1x_2\bar{\mathbf{k}}[s]$, so $x_1x_2 = \mu\varphi(s)$ for some $\mu \in \bar{\mathbf{k}}^*$. It is clear that if (x_1, x_2, s) belongs to $\Gamma_{\bar{\mathbf{k}}}(\bar{R})$ then so does $(x_1, \mu^{-1}x_2, s)$; so there exists $x_2 \in \bar{R}$ such that $(x_1, x_2, s) \in \Gamma_{\bar{\mathbf{k}}}(\bar{R})$ and $x_1x_2 = \varphi(s)$. As $x_2 = \varphi(s)/x_1 \in \mathrm{Frac}\, R$, (1) implies that $x_2 \in R$. Thus

$$(x_1, x_2, s) \in \Gamma_{\bar{\mathbf{k}}}(\bar{R}), \quad \text{where } x_1, x_2, s \in R.$$

In particular we have $\bar{R} = \bar{\mathbf{k}}[x_1, x_2, s]$, so (2) gives $R = \mathbf{k}[x_1, x_2, s]$. As $x_1x_2 = \varphi(s)$ where $\varphi(Y) \in \mathbf{k}[Y]$ is nonconstant, it follows that $(x_1, x_2, s) \in \Gamma_{\mathbf{k}}(R)$ and hence that $R \in \mathfrak{D}(\mathbf{k})$. □

3. On a result of Bandman and Makar-Limanov

In this paper we adopt the following:

3.1. Definition. Let R be an affine algebra over a field $\mathbf{k}$ and let $q = \dim R$. We say that R is a *complete intersection over* $\mathbf{k}$ if $R \cong \mathbf{k}[X_1, \dots, X_{p+q}]/(f_1, \dots, f_p)$ for some $p \geqslant 0$ and some $f_1, \dots, f_p \in \mathbf{k}[X_1, \dots, X_{p+q}]$.

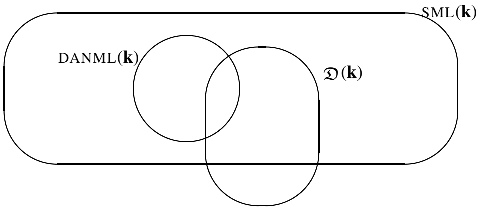
We refer to [18, 28.D] for the definition of a *smooth* **k**-*algebra* and to [18, 26.C] for the definition of the R-module $\Omega_{R/\mathbf{k}}$ (the module of differentials of R over **k**), where R is a **k**-algebra.

3.2. Theorem. *Let* **k** *be a field of characteristic zero and* R *a smooth affine* **k**-*domain of dimension* 2 *such that* $\mathrm{ML}(R) = \mathbf{k}$. *Then the following are equivalent*:

(a) $R \in \mathfrak{D}(\mathbf{k})$;
(b) *R is generated by* 3 *elements as a* **k**-*algebra*;
(c) *R is a complete intersection over* **k**;
(d) $\bigwedge^2 \Omega_{R/\mathbf{k}} \cong R$.

We shall prove this by reducing to the case $\mathbf{k} = \mathbb{C}$, which was proved by Bandman and Makar-Limanov in [1]. That reduction makes essential use of Theorem 2.3.

3.3. Remark. Let **k** be a field of characteristic zero. According to the definition of "Danielewski surface over **k**" given in [10], one has the following situation:



where DANML(**k**) is the class of Danielewski surfaces S over **k** satisfying $\mathrm{ML}(S) = \mathbf{k}$, SML(**k**) is the larger class of smooth affine surfaces S over **k** satisfying $\mathrm{ML}(S) = \mathbf{k}$, and $\mathfrak{D}(\mathbf{k})$ is the class of surfaces corresponding to the already defined class $\mathfrak{D}(\mathbf{k})$ of **k**-algebras. Among other things, paper [10] classifies the elements of DANML(**k**) and characterizes those which belong to $\mathfrak{D}(\mathbf{k})$. In contrast, Theorem 3.2 characterizes the elements of SML(**k**) which belong to $\mathfrak{D}(\mathbf{k})$.

3.4. Remark. Let R be a q-dimensional smooth affine domain over a field **k** of characteristic zero. Then $X = \operatorname{Spec} R$ is in particular an irreducible regular scheme of finite type over the perfect field **k**; so, by [15, ex. 8.1(c), p. 187], the sheaf of differentials $\Omega_{X/\mathbf{k}}$ is locally free of rank q; so the canonical sheaf $\omega_X = \bigwedge^q \Omega_{X/\mathbf{k}}$ is locally free of rank 1, i.e., is an invertible sheaf on X. As ω_X and the structure sheaf $\mathcal{O}_X$ are respectively the sheaves associated to the R-modules $\bigwedge^q \Omega_{R/\mathbf{k}}$ and R, the condition $\bigwedge^q \Omega_{R/\mathbf{k}} \cong R$ is equivalent to $\omega_X \cong \mathcal{O}_X$ (one says that X has trivial canonical sheaf). This is also equivalent to the canonical divisor of X being linearly equivalent to zero (because $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$ by [15, 6.16, p. 145]).

3.5. Remark. Let A' and B be algebras over a ring A and let $B' = A' \otimes_A B$. Then $\Omega_{B'/A'} \cong B' \otimes_B \Omega_{B/A}$ (cf. [18, p. 186]) and, for any B-module M, $\bigwedge^n (B' \otimes_B M) \cong B' \otimes_B \bigwedge^n M$ for every n [3, Chapter 3, §7, No. 5, Proposition 8]. Consequently, $\bigwedge^n \Omega_{B'/A'} \cong B' \otimes_B \bigwedge^n \Omega_{B/A}$.

3.6. Lemma. *Let R be an algebra over a field* **k**. *If R is a complete intersection over* **k** *and a smooth* **k**-*algebra, then* $\bigwedge^q \Omega_{R/\mathbf{k}} \cong R$ *where* $q = \dim R$.

This is the well-known fact that a smooth complete intersection has trivial canonical sheaf, but we do not know a suitable reference so we sketch a proof.

Proof of 3.6. Let $R = \mathbf{k}[X_1, \dots, X_{p+q}]/(f_1, \dots, f_p)$ and let $\varphi_{ij} \in R$ be the image of $\frac{\partial f_j}{\partial X_i}$. Because R is smooth over $\mathbf{k}$, [18, 29.E] implies that the matrix (φ_{ij}) satisfies:

$$\text{the } p \times p \text{ determinants of } (\varphi_{ij}) \text{ generate the unit ideal of } R. \tag{8}$$

By [15, 8.4A, p. 173], there is an exact sequence $R^p \xrightarrow{\varphi} R^{p+q} \to \Omega_{R/\mathbf{k}} \to 0$ of R-linear maps where φ is the map corresponding to the matrix (φ_{ij}). Now if R is a ring and $R^p \xrightarrow{\varphi} R^{p+q} \to M \to 0$ is an exact sequence of R-linear maps such that φ satisfies (8), then $\bigwedge^q M \cong R$. □

3.7. Lemma. *Let R be an integral domain containing a field $\mathbf{k}$ of characteristic zero. If R is normal and* $\mathrm{ML}(R) = \mathbf{k}$*, then for any field extension K of $\mathbf{k}$ we have*:

(a) *$K \otimes_{\mathbf{k}} R$ is an integral domain*;
(b) $\mathrm{ML}(K \otimes_{\mathbf{k}} R) = K$.

Proof. As $\mathbf{k} = \mathrm{ML}(R)$ is algebraically closed in R (2.4(a)) and R is normal, it follows that $\mathbf{k}$ is algebraically closed in $L = \operatorname{Frac} R$. By [22, Corollary 2, p. 198], $K \otimes_{\mathbf{k}} L$ is an integral domain. As K is flat over $\mathbf{k}$ and $R \to L$ is injective, $K \otimes_{\mathbf{k}} R \to K \otimes_{\mathbf{k}} L$ is injective and (a) is proved.

Let $\xi \in \mathrm{ML}(K \otimes_{\mathbf{k}} R)$. Consider a basis $\mathcal{B}$ of K over $\mathbf{k}$; note that $\mathcal{B}$ is also a basis of the free R-module $R' = K \otimes_{\mathbf{k}} R$ and write $\xi = \sum_{\lambda \in \mathcal{B}} x_\lambda \lambda$ (where $x_\lambda \in R$). If $D \in \mathrm{LND}(R)$ then D extends to an element $D' \in \mathrm{LND}(R')$ and the equation $0 = D'(\xi) = \sum_{\lambda \in \mathcal{B}} D(x_\lambda)\lambda$ shows that $D(x_\lambda) = 0$ for all $\lambda \in \mathcal{B}$. As this holds for every $D \in \mathrm{LND}(R)$, we have $x_\lambda \in \mathrm{ML}(R) = \mathbf{k}$ for all λ, so $\xi \in K$. □

Proof of Theorem 3.2. Implications (a) ⇒ (b) ⇒ (c) are trivial and (c) ⇒ (d) is Lemma 3.6, so only (d) ⇒ (a) requires a proof. Assume for a moment that $\mathbf{k} = \mathbb{C}$ and suppose that R satisfies (d). Then Lemmas 4 and 5 of [1] imply that $R \in \mathfrak{D}(\mathbb{C})$, so the theorem is valid in the case $\mathbf{k} = \mathbb{C}$.

Let $\mathbf{k}$ be a field of characteristic zero, consider a smooth affine $\mathbf{k}$-domain R of dimension 2 such that $\mathrm{ML}(R) = \mathbf{k}$, and suppose that R satisfies (d).

We have $R \cong \mathbf{k}[X_1, \dots, X_n]/(f_1, \dots, f_m)$ for some $m, n \geqslant 0$ and some $f_1, \dots, f_m \in \mathbf{k}[X_1, \dots, X_n]$. Also consider $D_1, D_2 \in \mathrm{LND}(R)$ such that $\ker D_1 \cap \ker D_2 = \mathbf{k}$. Each D_i can be lifted to a (not necessarily locally nilpotent) $\mathbf{k}$-derivation δ_i of $\mathbf{k}[X_1, \dots, X_n]$. Let $\mathbf{k}_0$ be a subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$ and which contains all coefficients of the polynomials f_i and $\delta_i(X_j)$. Define $R_0 = \mathbf{k}_0[X_1, \dots, X_n]/(f_1, \dots, f_m)$ and note that $\mathbf{k} \otimes_{\mathbf{k}_0} R_0 \cong R$. As $\mathbf{k}_0 \to \mathbf{k}$ is injective and R_0 is flat over $\mathbf{k}_0$, $\mathbf{k}_0 \otimes_{\mathbf{k}_0} R_0 \to \mathbf{k} \otimes_{\mathbf{k}_0} R_0$ is injective and we may regard R_0 as a subring of R. In particular, R_0 is a domain (a 2-dimensional affine $\mathbf{k}_0$-domain). Also note that $D_i(R_0) \subseteq R_0$ for $i = 1, 2$; if $d_i : R_0 \to R_0$ is the restriction of D_i then $d_1, d_2 \in \mathrm{LND}(R_0)$ and $\ker d_1 \cap \ker d_2 = \mathbf{k} \cap R_0 = \mathbf{k}_0$ (see (1) for the last equality), showing that $\mathrm{ML}(R_0) = \mathbf{k}_0$. As $\mathbf{k}_0$ is a field and $\mathbf{k} \to R$ is obtained from $\mathbf{k}_0 \to R_0$ by base extension, the fact that $\mathbf{k} \to R$ is smooth implies that $\mathbf{k}_0 \to R_0$ is smooth (cf. [18, 28.O]).

Consider the R-module $M = \bigwedge^2 \Omega_{R/\mathbf{k}}$ and the R_0-module $M_0 = \bigwedge^2 \Omega_{R_0/\mathbf{k}_0}$. Consider an isomorphism of R-modules $\theta : R \to M$ and let $\omega = \theta(1)$. We have $R \otimes_{R_0} M_0 \cong M$ by 3.5, so there is a natural homomorphism $M_0 \to R \otimes_{R_0} M_0 \cong M$, $x \mapsto 1 \otimes x$; by adjoining a finite subset of $\mathbf{k}$ to $\mathbf{k}_0$, we may arrange that there exists $\omega_0 \in M_0$ such that $1 \otimes \omega_0 = \omega$. Consider the R_0-linear map $f : R_0 \to M_0$, $f(a) = a\omega_0$. Note that $R = \mathbf{k} \otimes_{\mathbf{k}_0} R_0$ is faithfully flat as an R_0-module and that applying the functor $R \otimes_{R_0} \_$ to f yields the isomorphism θ; so f is an isomorphism, so $\bigwedge^2 \Omega_{R_0/\mathbf{k}_0} \cong R_0$. As $R \in \mathcal{D}(\mathbf{k})$ would follow from $R_0 \in \mathcal{D}(\mathbf{k}_0)$, the problem reduces to proving the case $\mathbf{k} = \mathbf{k}_0$ of the theorem. Now $\mathbf{k}_0$ is isomorphic to a subfield of $\mathbb{C}$, so it suffices to prove the theorem in the case $\mathbf{k} \subseteq \mathbb{C}$.

Assume that $\mathbf{k} \subseteq \mathbb{C}$. As R is smooth over $\mathbf{k}$, the local ring $R_{\mathfrak{p}}$ is regular for every $\mathfrak{p} \in \operatorname{Spec} R$ (by [18, 28.E, F, K]) so in particular R is a normal domain. Then it follows from 3.7 that $R' = \mathbb{C} \otimes_{\mathbf{k}} R$ is an integral domain and that $\mathrm{ML}(R') = \mathbb{C}$. By [18, 28.G], R' is smooth over $\mathbb{C}$. It is clear that $\dim R' = 2$ (for instance see the proof of 2.5) and 3.5 gives $\bigwedge^2 \Omega_{R'/\mathbb{C}} \cong R' \otimes_R \bigwedge^2 \Omega_{R/\mathbf{k}} \cong R' \otimes_R R \cong R'$. As the theorem is valid over $\mathbb{C}$, it follows that $R' \in \mathcal{D}(\mathbb{C})$. As $\mathrm{ML}(R) = \mathbf{k} \neq R$, Theorem 2.3 implies that $R \in \mathcal{D}(\mathbf{k})$. □

3.8. Corollary. *Let R be a 2-dimensional affine domain over a field $\mathbf{k}$ of characteristic zero. If R is a UFD and a smooth $\mathbf{k}$-algebra satisfying $\mathrm{ML}(R) = \mathbf{k}$, then $R \in \mathfrak{D}(\mathbf{k})$.*

Proof. Since R is a UFD, the scheme $X = \operatorname{Spec} R$ has a trivial divisor class group [15, 6.2, p. 131]. By Remark 3.4, it follows that $\bigwedge^2 \Omega_{R/\mathbf{k}} \cong R$ and the desired conclusion follows from Theorem 3.2. □

4. Localizations of nice rings

Throughout this section we fix a field $\mathbf{k}$ of characteristic zero and we consider the class $\mathcal{N}(\mathbf{k})$ of $\mathbf{k}$-algebras B satisfying the following conditions:

B is a geometrically integral affine $\mathbf{k}$-domain which is smooth over $\mathbf{k}$ and satisfies at least one of the following conditions:

- *B is a UFD*; *or*
- *B is a complete intersection over $\mathbf{k}$.*

Note that $\mathbf{k}^{[n]} \in \mathcal{N}(\mathbf{k})$ for every n.

4.1. Theorem. *Suppose that R is a localization of a ring belonging to the class $\mathcal{N}(\mathbf{k})$. If $\mathrm{ML}(R) = K$ for some field $K \subset R$ such that $\operatorname{trdeg}_K R = 2$, then $R \in \mathfrak{D}(K)$.*

4.2. Lemma. *Let $B \in \mathcal{N}(\mathbf{k})$, let E be a finitely generated $\mathbf{k}$-subalgebra of B and let $S = E \setminus \{0\}$. Then $S^{-1}B$ is a smooth algebra over the field $S^{-1}E$.*

Proof. Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$ and define $\bar{E} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} E$ and $\bar{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$. Note that $\bar{B}$ is a domain because B is geometrically integral, and $\bar{E} \to \bar{B}$ is injective because $\bar{\mathbf{k}}$ is flat over $\mathbf{k}$. Let $K = \operatorname{Frac} E$ and $L = \operatorname{Frac} \bar{E}$. As $\bar{B}$ is smooth over $\bar{\mathbf{k}}$, applying [15, 10.7, p. 272] to $\operatorname{Spec} \bar{B} \to \operatorname{Spec} \bar{E}$ implies that $L \to L \otimes_{\bar{E}} \bar{B}$ is smooth. It is not difficult to see that $L \to L \otimes_{\bar{E}} \bar{B}$ is obtained from $K \to K \otimes_E B$ by base extension. As K is a field and $L \to L \otimes_{\bar{E}} \bar{B}$ is smooth, it follows from [18, 28.O] that $K \to K \otimes_E B$ is smooth. □

4.3. Lemma. *Let $B \in \mathcal{N}(\mathbf{k})$, let S be a multiplicative subset of B and suppose that K is a field such that $\mathbf{k} \cup S \subseteq K \subseteq S^{-1}B$. Then $S^{-1}B$ is a smooth K-algebra and some transcendence basis of $K/\mathbf{k}$ is a subset of B.*

Proof. Note that $K/\mathbf{k}$ is a finitely generated field extension and write $K = \mathbf{k}(\alpha_1, \dots, \alpha_m)$. For each i we have $\alpha_i = b_i/s_i$ for some $b_i \in B$ and $s_i \in S$; as $S \subseteq K$, we have $b_i = s_i\alpha_i \in K$. Define $E = \mathbf{k}[b_1, \dots, b_m, s_1, \dots, s_m] \subseteq K$ and $S_1 = E \setminus \{0\}$, then $S_1^{-1}E = K$ and hence $S_1^{-1}B = S^{-1}B$. By Lemma 4.2, $S^{-1}B$ is a smooth K-algebra. Moreover, $\{b_1, \dots, b_m, s_1, \dots, s_m\}$ contains a transcendence basis of $K/\mathbf{k}$. □

Proof of Theorem 4.1. We have $R = S^{-1}B$ for some $B \in \mathcal{N}(\mathbf{k})$ and some multiplicative subset S of B. As $\mathbf{k}^* \cup S \subseteq R^* \subseteq \mathrm{ML}(R) = K$, R is smooth over K by Lemma 4.3. By definition of $\mathcal{N}(\mathbf{k})$, B is a UFD or a complete intersection over $\mathbf{k}$.

If B is a UFD then so is R; in this case we obtain $R \in \mathfrak{D}(K)$ by Corollary 3.8, so we are done.

From now on, assume that B is a complete intersection over $\mathbf{k}$. Let $q = \dim B$ and write $B = \mathbf{k}[X_1, \dots, X_{p+q}]/(G_1, \dots, G_p)$. Using Lemma 4.3 again, choose a transcendence basis $\{f_1, \dots, f_{q-2}\}$ of K over $\mathbf{k}$ such that $f_1, \dots, f_{q-2} \in B$; let $S_0 = \mathbf{k}[f_1, \dots, f_{q-2}] \setminus \{0\}$ and $K_0 = \mathbf{k}(f_1, \dots, f_{q-2})$. We claim:

$$S_0^{-1}B \text{ is a complete intersection over } K_0. \tag{9}$$

Let us prove this. For $1 \leqslant i \leqslant q-2$, choose $F_i \in \mathbf{k}[X_1, \dots, X_{p+q}]$ such that $\pi(F_i) = f_i$ where $\pi : \mathbf{k}[X_1, \dots, X_{p+q}] \to B$ is the canonical epimorphism. Also, let $T_1, \dots, T_{q-2}$ be extra indeterminates. The $\mathbf{k}$-homomorphism $\mathbf{k}[T_1, \dots, T_{q-2}, X_1, \dots, X_{p+q}] \to B$ which maps T_i to f_i and X_i to $\pi(X_i)$ has kernel $(G_1, \dots, G_p, F_1 - T_1, \dots, F_{q-2} - T_{q-2})$, so there is an isomorphism of $\mathbf{k}$-algebras

$$B \cong \mathbf{k}[T_1, \dots, T_{q-2}, X_1, \dots, X_{p+q}]/(G_1, \dots, G_p, F_1 - T_1, \dots, F_{q-2} - T_{q-2}).$$

Localization gives an isomorphism of $\mathbf{k}$-algebras

$$S_0^{-1}B \cong \mathbf{k}(T_1, \dots, T_{q-2})[X_1, \dots, X_{p+q}]/(G_1, \dots, G_p, F_1 - T_1, \dots, F_{q-2} - T_{q-2}) \tag{10}$$

which maps K_0 onto $\mathbf{k}(T_1, \dots, T_{q-2})$. As the right-hand side of (10) is a complete intersection over $\mathbf{k}(T_1, \dots, T_{q-2})$, assertion (9) is proved. Then we obtain

$$\bigwedge^2 \Omega_{S_0^{-1}B/K_0} \cong S_0^{-1}B \tag{11}$$

by Lemma 3.6, because $S_0^{-1}B$ is a smooth K_0-algebra by Lemma 4.2.

Each element of K belongs to $\mathrm{Frac}(S_0^{-1}B)$ and is algebraic over K_0, hence integral over $S_0^{-1}B$; as $S_0^{-1}B$ is normal, $K \subseteq S_0^{-1}B$ and hence $S_0^{-1}B = R$. We may therefore rewrite (11) as:

$$\bigwedge^2 \Omega_{R/K_0} \cong R. \tag{12}$$

Applying [18, 26.H] to $K_0 \subseteq K \subseteq R$ gives the exact sequence of R-modules

$$\Omega_{K/K_0} \otimes_K R \to \Omega_{R/K_0} \to \Omega_{R/K} \to 0,$$

where $\Omega_{K/K_0} = 0$ by [18, 27.B]. So $\Omega_{R/K} \cong \Omega_{R/K_0}$ and hence (12) gives $\bigwedge^2 \Omega_{R/K} \cong R$. So $R \in \mathfrak{D}(K)$ by Theorem 3.2. □

Let $\mathbf{k}$ be a field of characteristic zero, let $B \in \mathcal{N}(\mathbf{k})$ and consider locally nilpotent derivations $D : B \to B$. See 1.1 for the definition of $\mathrm{KLND}(B)$. It is known that if $A \in \mathrm{KLND}(B)$ then $\mathrm{trdeg}_A(B) = 1$, and if A_1, A_2 are distinct elements of $\mathrm{KLND}(B)$ then $\mathrm{trdeg}_{A_1 \cap A_2}(B) \geqslant 2$. We are interested in the situation where $\mathrm{trdeg}_{A_1 \cap A_2}(B) = 2$, i.e., when A_1, A_2 are distinct and have an intersection which is as large as possible.

4.4. Corollary. *Let $B \in \mathcal{N}(\mathbf{k})$, where $\mathbf{k}$ is a field of characteristic zero. If $A_1, A_2 \in \mathrm{KLND}(B)$ are such that $\mathrm{trdeg}_{A_1 \cap A_2}(B) = 2$, then the following hold.*

(a) *Let $R = A_1 \cap A_2$ and $K = \mathrm{Frac}\, R$. Then $K \otimes_R B \in \mathfrak{D}(K)$.*
(b) *If B is a UFD then there exists a finite sequence of local slice constructions which transforms A_1 into A_2.*

Remark. This generalizes results 1.10 and 1.13 of [6]. Local slice construction was originally defined in [12] in the case $B = \mathbf{k}^{[3]}$, and was later generalized in [5].

Proof of Corollary 4.4. Let $S = R \setminus \{0\}$, $\mathcal{A}_i = S^{-1} A_i$ $(i = 1, 2)$ and $\mathcal{B} = S^{-1} B = K \otimes_R B$. If $D_i \in \mathrm{LND}(B)$ has kernel A_i, then $S^{-1} D_i \in \mathrm{LND}(\mathcal{B})$ has kernel $\mathcal{A}_i$; thus $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{KLND}(\mathcal{B})$. Using that A_1, A_2 are factorially closed in B, we obtain $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq K$, so $\mathrm{ML}(\mathcal{B}) \subseteq K$. The reverse inclusion is trivial $(K^* \subseteq \mathcal{B}^* \subseteq \mathrm{ML}(\mathcal{B}))$, so $\mathrm{ML}(\mathcal{B}) = K$. Then $\mathcal{B} \in \mathfrak{D}(K)$ by Theorem 4.1, so assertion (a) is proved.

In [5, 3.3], one defines a graph $\underline{\mathrm{KLND}}(B)$ whose vertex-set is $\mathrm{KLND}(B)$; then, given $A, A' \in \mathrm{KLND}(B)$, one says that A' can be obtained from A "by a local slice construction" if there exists an edge in $\underline{\mathrm{KLND}}(B)$ joining vertices A and A'. So assertion (b) of the corollary is equivalent to the existence of a path in $\underline{\mathrm{KLND}}(B)$ going from A_1 to A_2. Paragraph [5, 3.2.2] also defines a subgraph $\underline{\mathrm{KLND}}_R(B)$ of the graph $\underline{\mathrm{KLND}}(B)$, and clearly A_1, A_2 are two vertices of $\underline{\mathrm{KLND}}_R(B)$; so, to prove (b), it suffices to show that $\underline{\mathrm{KLND}}_R(B)$ is a connected graph. We have $R \in \mathcal{R}^{\mathrm{in}}(B)$ (cf. [5, 5.2]) and consequently (cf. [5, 5.3], using that B is a UFD) we have an isomorphism of graphs $\underline{\mathrm{KLND}}_R(B) \cong \underline{\mathrm{KLND}}_K(\mathcal{B})$. As $\mathcal{B} \in \mathfrak{D}(K)$ by part (a), we may apply [5, 4.8] and conclude that $\underline{\mathrm{KLND}}_K(\mathcal{B})$ is connected. Assertion (b) is proved. □

The following is a trivial consequence of Corollary 4.4.

4.5. Corollary. *Let $B \in \mathcal{N}(\mathbf{k})$, where $\mathbf{k}$ is a field of characteristic zero. Suppose that B has transcendence degree two over $\mathrm{ML}(B)$.*

(1) *Let $R = \mathrm{ML}(B)$ and $K = \mathrm{Frac}\, R$. Then $K \otimes_R B \in \mathfrak{D}(K)$.*
(2) *If B is a UFD then, for any $A_1, A_2 \in \mathrm{KLND}(B)$, there exists a finite sequence of local slice constructions which transforms A_1 into A_2.*

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