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On Simple Anti-Flexible Rings*

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1. INTRODUCTION

Anti-flexible algebras were introduced by Kosier [*3*], and a subclass of anti-flexible rings was studied earlier by Kleinfeld [*1*]. Subsequent investigations into the structure of these algebras were made in [*2*] and [*4*]. But the classification of the simple algebras is yet to be determined.

The purpose of this paper is to show that if R is a simple, not associative, anti-flexible, power-associative ring of characteristic not 2 or 3, then R is obtained by introducing a "commutator" in an appropriate commutative associative ring P, and the identity $((x, y), z) = 0$ holds in R.

While, in some sense, this result characterizes simple anti-flexible rings, it also shows that a complete determination of even the simple finite-dimensional algebras will be most difficult. Indeed, the corresponding associative commutative algebras turn out to be somewhat "mixed," that is, they are neither nilpotent nor separable.

The result of Rodabaugh that a simple, anti-flexible, power-associative, finite-dimensional, algebra of characteristic not 2 has a unit element provided it is not nil [*4*] is extended by showing that such an algebra with the additional assumption of characteristic not 3 cannot be nil.

2. PRELIMINARY DEFINITIONS AND IDENTITIES

The associator (x, y, z) and commutator (x, y) are defined by:

$$(x, y, z) = xy \cdot z - x \cdot yz \qquad \text{and} \qquad (x, y) = xy - yx.$$

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The ring R is said to be anti-flexible if

$$A(x, y, z) = (x, y, z) - (z, y, x) = 0 \tag{2.1}$$

is an identity in R.

Throughout the remainder of this paper we assume that R is anti-flexible, $2x = 0$ implies $x = 0$ in R, and that

$$(x, x, x) = 0 \tag{2.2}$$

is an identity in R.

With the aid of (2.1), we obtain, as a linearization of (2.2), the identity

$$B(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0. \tag{2.3}$$

We shall also require the Teichmüller identity (which holds in any ring):

$$\begin{aligned} C(w, x, y, z) = {} & (wx, y, z) - (w, xy, z) \\ & + (w, x, yz) - w(x, y, z) - (w, x, y)\, z = 0 \end{aligned} \tag{2.4}$$

In any ring

$$(xy, z) = x(y, z) + (x, z)\, y + (x, y, z) + (z, x, y) - (x, z, y),$$

from which we subtract $A(x, z, y) + B(x, y, z) = 0$ to obtain

$$D(x, y, z) = (xy, z) - x(y, z) - (x, z)\, y + 2(x, z, y) = 0. \tag{2.5}$$

We let $x \circ y = xy + yx$; then it can be verified that in any ring,

$$\begin{aligned} (x \circ y) \circ z - x \circ (y \circ z) = {} & (x, y, z) + (x, z, y) + (y, x, z) \\ & - (y, z, x) - (z, x, y) - (z, y, x) + (y, (x, z)), \end{aligned}$$

so that from (2.1) we get

$$(x \circ y) \circ z - x \circ (y \circ z) = (y, (x, z)). \tag{2.6}$$

If we retain the additive group of R but replace the product xy of R by the product $x \circ y$, then we obtain a commutative ring R^+, and it follows from (2.2) and (2.6) that R^+ is a Jordan ring. Equally, we could get an anti-commutative ring R^- by replacing the product xy by the commutator product (x, y), and from $0 = D(x, y, z) - D(y, x, z)$ and $0 = A(x, z, y)$, it would follow that

$$((x, y), z) + ((y, z), x) + ((z, x), y) = 0, \tag{2.7}$$

so that R^- would be a Lie ring.

Expanding $0 = C(w, x, y, z) - C(z, y, x, w)$, and using

$$0 = A(z, y, xw) = A(z, yx, w) = A(zy, x, w) = zA(y, x, w) = A(z, y, x)\, w,$$

we get

$$0 = E(w, x, y, z) = ((w, x), y, z) - (w, (x, y), z) + (w, x, (y, z)) - (w, (x, y, z)) - ((w, x, y), z). \tag{2.8}$$

Then we expand

$$0 = E(w, x, y, z) + E(x, y, z, w) + E(y, z, w, x) + E(z, w, x, y) - B((w, x), y, z) - B((x, y), z, w) - B((y, z), w, x) - B((z, w), x, y)$$

to get

$$\begin{aligned} 0 &= F(w, x, y, z) \\ &= (w, (x, y), z) + (x, (y, z), w) + (y, (z, w), x) + (z, (w, x), y). \end{aligned} \tag{2.9}$$

Now we are able to derive the important identity

$$(w, (x, y), z) = 0. \tag{2.10}$$

Expanding

$$0 = E(x, x, y, x) + E(y, x, x, x) - B(x, x, (y, x)) + (B(x, x, y), x),$$

we get $0 = (x, (x, y, x))$, hence from $0 = (x, B(x, y, x))$ and $0 = (x, A(x, x, y))$, we have

$$0 = (x, (x, y, x)) = (x, (x, x, y)) = (x, (y, x, x)). \tag{2.11}$$

Then it follows from (2.11) and $0 = E(y, x, x, x)$ that $((y, x), x, x) = 0$, hence $(x, x, (y, x)) = 0$ by (2.1), and then from $0 = B(x, x, (y, x))$,

$$0 = (x, (y, x), x). \tag{2.12}$$

Substituting $x + z$ for x in (2.12) and subtracting

$$0 = A(z, (y, x), x) + A(z, (y, z), x),$$

we obtain

$$2(x, (y, x), z) + 2(x, (y, z), z) + (x, (y, z), x) + (z, (y, x), z) = 0. \tag{2.13}$$

Substituting $-z$ for z in (2.13) and then adding to (2.13) yields

$$2(x, (y, z), z) + (z, (y, x), z) = 0. \tag{2.14}$$

Next, linearize (2.14) and add $A(w, (y, x), z) = 0$ to get

$$G(w, x, y, z) = (x, (y, z), w) + (x, (y, w), z) + (w, (y, x), z) = 0.$$

Computing

$$0 = F(w, x, y, z) + G(w, x, y, z) + G(x, y, w, z) - A(z, (w, x), y),$$

we get

$$H(w, x, y, z) = (x, (y, z), w) + (y, (w, x), z) = 0$$

Now, identity (2.10) follows from the expansion of

$$0 = H(w, x, y, z) + H(x, w, y, z) - A(w, (y, z), x).$$

With the aid of (2.10), we can improve (2.8) to obtain

$$\begin{aligned} 0 = J(w, x, y, z) = ((w, x), y, z) &+ (w, x, (y, z)) \\ &- (w, (x, y, z)) - ((w, x, y), z). \qquad (2.15) \end{aligned}$$

Finally, applying (2.10) to $0 = B((w, x), y, z) - A(y, z, (w, x))$, we have

$$0 = K(w, x, y, z) = ((w, x), y, z) + ((w, x), z, y). \qquad (2.16)$$

3. Simple Rings

Lemma 3.1. *If R is simple and not associative, then R has no proper one-sided ideals.*

Proof. Suppose, for example, that I is a non-zero right ideal of R. Then $(I, R, R) \subseteq I$; hence from $0 = A(I, R, R)$, we have $(R, R, I) \subseteq I$, and then $(R, I, R) \subseteq I$ because of $0 = B(I, R, R)$. Now we can show that $I + RI$ is a two-sided ideal of R;

$$(I + RI) R \subseteq IR + (RI) R \subseteq I + R(IR) + (R, I, R) \subseteq I + RI,$$

and

$$R(I + RI) \subseteq RI + R(RI) \subseteq RI + (RR) I + (R, R, I) \subseteq RI + I.$$

Since $I \neq 0$ and R is simple, $R = I + RI$. Then from the identity $(w, (x, y), z) = 0$, we have

$$\begin{aligned} (R, R, R) &\subseteq (R, I + RI, R) \subseteq (R, I, R) + (R, RI, R) \\ &\subseteq (R, I, R) + (R, IR, R) \subseteq (R, I, R) \subseteq I. \end{aligned}$$

Since $IR \subseteq I$, $(R, R, R) + (R, R, R) R \subseteq I$. But it is known [1] that $(R, R, R) + (R, R, R) R$ is a two-sided ideal of R, hence it is equal to R since R is not associative. Therefore $I = R$.

A similar argument shows R has no proper left ideals as well.

The middle nucleus M of R is defined as

$$M = \{m \in R \mid (R, m, R) = 0\}.$$

It follows from the linearity of the associator and from (2.4) that M is a subring of R. We note (2.10) expresses $(R, R) \subseteq M$.

LEMMA 3.2. *If R is simple, not associative, and if T is a subset of M such that $(T, R, R) \subseteq T$, then $(T, R) = 0$.*

Proof. We show first that $(T, R) + (T, R) R$ is a right ideal of R. Evidently, it is sufficient to sow that $((T, R), R, R) \subseteq (T, R)$, since

$$(T, R) R \cdot R \subseteq (T, R)(RR) + ((T, R), R, R) \subseteq (T, R) R + ((T, R), R, R).$$

Thus consider $((t, x), y, z)$, where $t \in T$, $x, y, z \in R$. We have

$$\begin{aligned} 0 = J(t, x, y, z) = ((t, x), y, z) + (t, x, (y, z)) \\ - (t, (x, y, z)) - ((t, x, y), z), \end{aligned}$$

and since $t, (y, z) \in M$, it follows from $0 = B(t, x, (y, z))$ that $0 = (t, x, (y, z))$, so that $((t, x), y, z) = (t, (x, y, z)) + ((t, x, y), z) \in (T, R)$.

According to Lemma 3.1, either $(T, R) = 0$ or $R = (T, R) + (T, R) R$. We show that in the latter case, R would be associative. Indeed, let $t \in T$, $y, z \in R$. Expanding $0 = D(t, y, z)$, we get

$$(t, z) y = (ty, z) - t(y, z) + 2(t, z, y).$$

Since $(R, R) \subseteq M$ and since M is a subring, it follows from our assumption on T that $(t, z) y \in M$. Therefore $R = (T, R) + (T, R) R \subseteq M$, and R would be associative. Hence $(T, R) = 0$.

LEMMA 3.3 *If in the ring R the mapping $x \to 3x$ is onto, then*

$$(((R, R), R), R, R) \subseteq ((R, R), R).$$

Proof. Since $0 = B((a, b), c, (x, y))$, it follows from (2.10) that $((a, b), c, (x, y)) = 0$. Hence from $0 = J((a, b), c, x, y)$ we have

$$(((a, b), c), x, y) = (((a, b), c, x), y) + ((a, b), (c, x, y)),$$

from which it follows that

$$(((a, b), c), x, y) \equiv (((a, b), c, x), y) \qquad (\text{mod } T) \tag{3.4}$$

where $T = ((R, R), R)$. We obtain

$$(((a, b), c), x, y) \equiv -(((a, b), x), c, y) \qquad (\text{mod } T) \tag{3.5}$$

from $0 = (K(a, b, c, x), y)$ and (3.4). Also,

$$(((a, b), c), x, y) \equiv -(((c, x), b), a, y) \qquad (\text{mod } T). \tag{3.6}$$

Indeed, expanding $0 = J(a, b, c, x)$, we see that

$$(((a, b), c, x), y) \equiv -((a, b, (c, x)), y) \qquad (\text{mod } T),$$

and since $0 = (A(a, b, (c, x)), y)$, we get

$$(((a, b), c, x), y) \equiv -(((c, x), b, a), y) \qquad (\text{mod } T),$$

from which (3.6) follows immediately, using (3.4).

Now we can derive

$$(((a, b), c), x, y) \equiv (((c, a), b), x, y) \qquad (\text{mod } T). \tag{3.7}$$

Computing, mod T, using (3.5), (3.6), and the properties of the commutator,

$$\begin{aligned}(((a, b), c), x, y) &\equiv -(((c, x), b), a, y) + K((c, x), b, a, y)\\ &= (((c, x), b), y, a)\\ &\equiv -(((b, y), x), c, a)\\ &\equiv (((b, y), c), x, a) - K((b, y), c, x, a)\\ &= -(((b, y), c), a, x)\\ &= (((y, b), c), a, x)\\ &\equiv -(((c, a), b), y, x) + K((c, a), b, y, x)\\ &= (((c, a), b), x, y).\end{aligned}$$

Finally,

$$3(((a, b), c), x, y) \equiv 0 \qquad (\text{mod } T),$$

for according to (3.7), $(((a, b), c), x, y)$ remains unchanged mod T when a, b, c are cyclically permuted. Hence

$$\begin{aligned}3(((a, b), c), x, y) \equiv (((a, b), c), x, y) &+ (((c, a), b), x, y)\\ &+ (((b, c), a), x, y) \qquad (\text{mod } T).\end{aligned}$$

But because of the Jacobi identity, (2.7),

$$(((a, b), c), x, y) + (((c, a), b), x, y) + (((b, c), a), x, y) = 0,$$

hence (3.8).

Let a', b, c, x, y be arbitrary elements of R. Since $a' = 3a$ for some $a \in R$,

$$(((a', b), c), x, y) = 3(((a, b), c), x, y) \equiv 0 \qquad (\text{mod } T),$$

which completes the proof of the lemma.

LEMMA 3.9. *If R is simple, not associative, and of characteristic $\neq 2$, 3, then* $0 = (((R, R), R), R) = ((R, R), (R, R))$.

Proof. Since $3R$ is an ideal of R, it follows from our assumptions that $x \to 3x$ is an onto mapping. Let $T = ((R, R), R)$; from Lemma 3.3 we have that $(T, R, R) \subseteq T$. But T is a subset of the middle nucleus because of (2.10), hence from Lemma 3.2, we conclude that $(((R, R), R), R) = 0$, and since R^- is a Lie ring, (2.7), $0 = ((R, R), (R, R))$ as well.

THEOREM 3.10. *If R is a simple, not associative ring of characteristic not 2 or 3, then $((x, y), z) = 0$ is an identity in R, and R^+ is a commutative associative ring.*

Proof. The fact that R^+ is associative will follow from (2.6) and the identity $((x, y), z) = 0$.

First we prove that $((R, R), R)(R, R) \subseteq ((R, R), R)$. Indeed if $w \in ((R, R), R)(R, R)$, then $w = \sum t_i u_i$, where $t_i \in ((R, R), R)$ and $u_i \in (R, R)$. Hence it would be sufficient to prove that $((a, b), c)(r, s) \in ((R, R), R)$ for all $a, b, c, r, s \in R$. Since

$$\begin{aligned} 0 = D(c, (r, s), (a, b)) &= (c(r, s), (a, b)) - c((r, s), (a, b)) \\ &\quad - (c, (a, b))(r, s) + 2(c, (a, b), (r, s)), \end{aligned}$$

we have from (2.10) and Lemma 3.9 that

$$((a, b), c)(r, s) = (c(r, s), (a, b)) \in ((R, R), R).$$

Next, we note that $W = ((R, R), R) + ((R, R), R)\, R$ is a right ideal of R:

$$\begin{aligned} WR \subseteq W + ((R, R), R)\, R \cdot R \subseteq W &+ ((R, R), R)(RR) \\ &+ (((R, R), R), R, R) \subseteq W \end{aligned}$$

because of Lemma 3.3. According to Lemma 3.1, $W = 0$ or $W = R$. Suppose $W = R$. Then

$$(R, R) \subseteq (W, R) \subseteq (((R, R), R)\, R, R)$$

because of Lemma 3.9. Now we prove that $(((R, R), R)\, R, R) \subseteq ((R, R), R)$. Evidently, it is sufficient to show that if x, y, z, b, c are arbitrary elements in R, then $(((x, y), z)\, b, c) \in ((R, R), R)$. Thus, set $a = ((x, y), z)$. Now

$$0 = D(a, b, c) = (ab, c) - a(b, c) - (a, c)\, b + 2(a, c, b).$$

But $(a, c)\, b = 0$ because of Lemma 3.9, and $2(a, c, b) \in ((R, R), R)$ because of Lemma 3.3. Therefore

$$(ab, c) \in ((R, R), R) + ((R, R), R)(R, R) \subseteq ((R, R), R).$$

because of the paragraph above. Altogether we have $(R, R) \subseteq ((R, R), R)$. But then

$$((R, R), R) \subseteq (((R, R), R), R) = 0$$

because of Lemma 3.9. Therefore $W = 0$, and this completes the proof of the theorem.

We should like to point out that Kosier's main result on the structure of simple anti-flexible rings [3] can be obtained as a consequence of Theorem 3.10. Kosier proved that if e is an idempotent of an anti-flexible ring R, then R has a Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ relative to e, and that if R is simple and not associative, then $R_{10} = R_{01} = 0$. This result will follow from Theorem 3.10 in the following way. Let $x_{10} \in R_{10}$. Then $x_{10} = (e, x_{10}) = (e, (e, x_{10})) = 0$. Similarly, $R_{01} = 0$.

THEOREM 3.11. *Let P be a commutative, associative, nontrivial ring in which $2x = 0$ implies $x = 0$ and in which there is defined a bilinear map* $\langle x, y\rangle \colon P \times P \to P$ satisfying for all $x, y, z \in P$

(1) $\langle x, x\rangle = 0$,

(2) $\langle x^2, x\rangle = 0$,

(3) $\langle\langle x, y\rangle, z\rangle = 0$,

(4) $\langle x, y\rangle \neq 0$ *for some x, y in P,*

and

(5) $\langle I, P\rangle \not\subseteq I$ *for each proper ideal I of P.*

Then the ring R obtained by taking the additive group of P and replacing the product xy in P by the new product

$$x \otimes y = xy + \langle x, y \rangle \tag{3.12}$$

is a simple, not associative, anti-flexible ring satisfying $(x, x, x) = 0$. Conversely, each simple, not associative, anti-flexible ring of characteristic not 2 or 3 satisfying $(x, x, x) = 0$ may be obtained in this way.

Proof. The converse is just Theorem 3.10. Let (x, y) be the commutator in R. We take $P = R^+$ and $\frac{1}{2}(x, y)$ for our bilinear map. (Since R is simple and of characteristic not 2, $\frac{1}{2}(x, y)$ is a well-defined element.) Condition (2) is just $(x, x, x) = 0$ and (5) follows from the fact that R is simple.

Now, condition (1) is equivalent to $\langle x, y \rangle = -\langle y, x \rangle$. Moreover, linearizing condition (2) yields

$$\langle xy, z \rangle + \langle yz, x \rangle + \langle zx, y \rangle = 0. \tag{3.13}$$

Using (1), (3), and (3.13) we obtain

$$(x \otimes y) \otimes z - x \otimes (y \otimes z) = \langle x, y \rangle z + \langle z, y \rangle x - \langle xz, y \rangle. \tag{3.14}$$

Interchanging x and z in (3.14) leaves it unchanged, hence R is anti-flexible. Let $x = y = z$ in (3.14) and apply conditions (1) and (2) to obtain $(x, x, x) = 0$ in R.

Next, it follows from the assumption that $2x = 0$ implies $x = 0$ in P that $x \to 2x$ is an onto mapping. Indeed, $2P$ is an ideal of P. Moreover, by the linearity of $\langle x, y \rangle$, $\langle 2P, P \rangle \subseteq 2P$; hence $2P$ is not a proper ideal of P by (5). Therefore $2P = P$ since $2P \neq 0$. Now, (3.12) and condition (1) imply

$$2xy = x \otimes y + y \otimes x. \tag{3.15}$$

Let I be an ideal of R. Choose $y \in I$, $r \in R$. Then by (3.15) $2yr \in I$ which implies that I is an ideal of P since $x \to 2x$ is onto. But then $\langle I, P \rangle \subseteq I$ by (3.12) which implies that I is not a proper ideal of P by condition (5), and thus I is not a proper ideal of R. Hence R is simple.

Finally, if R were associative, then R would be a simple associative ring such that $((R, R), R) = 0$. This follows from (3) and the fact that the commutator of x, y in R is precisely $2\langle x, y \rangle$. Condition (4) insures the existence of an $a \in R$ such that $(a, x) \neq 0$ for some $x \in R$. Since R is associative, $D(a, a, x) = 0$ yields

$$(a^2, x) = a(a, x) + (a, x)\, a = 2a(a, x)$$

because of $((R, R), R) = 0$. But also, $(a^2, x) \in C$, the center of R. Thus $2a(a, x) \in C$ which implies $a(a, x) \in C$. The center of a simple associative ring is either 0 or a field. Now, $C \neq 0$ because of (4). Moreover, if C is a field, then (a, x) has an inverse in C, thus $a \in C$ which implies $(a, x) = 0$, is a contradiction. Hence R is not associative.

4. Finite-Dimensional Algebras

First we remark that if R is a central simple, power-associative, anti-flexible, not associative, finite-dimensional algebra over a field Φ of characteristic not 2 or 3, then the corresponding algebra P is, of course, not a nil algebra since R has a unit element. Furthermore, P is not separable. For suppose P were semi-simple over Φ. Then $P = P_1 \oplus \cdots \oplus P_t$, where each P_i is an extension of field Φ. Assume first that each P_i is a separable extension of Φ. Then for each i, $P_i = \Phi(\omega_i)$ for some element ω_i which is algebraic over Φ. From power-associativity, we have $\langle \omega_i^n, \omega_i^m \rangle = 0$ for all integers n and m, whence $\langle P_i, P_i \rangle = 0$. Moreover, if we let $x = 1_i$, 1_i the unit of P_i, $y \in P_i$, $z \in P_j$, $i \neq j$, in (3.13), then $\langle y, z \rangle = 0$, that is $\langle P_i, P_j \rangle = 0$ for $i \neq j$. Hence $\langle P, P \rangle = 0$, which would imply that $R = P$, and in particular, R is associative. On the other hand, suppose P_1 is not a separable extension of Φ. Then there exists an extention Λ of Φ such that P_1, when considered as an algebra over Λ, contains a nonzero nilpotent element. Since P_1 is commutative and associative, so P_1 as an algebra over Λ has a nonzero radical. Therefore, P_Λ has a nonzero radical.

Finally the extension of Rodabaugh's result [4, Theorem 6.1] as mentioned in Section 1 is made by

Theorem 4.1. *If R is a simple, finite-dimensional, power-associative, anti-flexible algebra of characteristic not 2 or 3, then R is not nil.*

Proof. Of course, we may assume that R is not associative so that R^+ is commutative and associative by (2.6) and Theorem 3.10. Hence if R is nil, then R^+ is nil from which it follows that R^+ is nilpotent. In particular, there exists $a \neq 0$ in R such that $ax + xa = 0$ for all $x \in R$. Then $(aR, R) = 0$. Indeed, let $x \in R$. Then by Theorem 3.10

$$2(ax, R) = ((a, x), R) + (ax + xa, R) = 0.$$

Now, in any ring,

$$(xy, z) + (yz, x) + (zx, y) = B(x, y, z) = 0,$$

hence from $(aR, R) = 0 = ax + xa$, we obtain $(a, R^2) = 0$. Hence $(a, R) = 0$ since $R^2 = R$. But then $0 = aR = Ra$ since $ax + xa = 0$ for all $x \in R$, which is impossible since R is simple.

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