Let $F[X]$ be the Pixley–Roy hyperspace of a regular space $X$. In this paper, we prove the following theorem.

**Theorem.** For a space $X$, the following are equivalent:

1. $F[X]$ is a $k$-space;
2. $F[X]$ is sequential;
3. $F[X]$ is Fréchet–Urysohn;
4. Every finite power of $X$ is Fréchet–Urysohn for finite sets;
5. Every finite power of $F[X]$ is Fréchet–Urysohn for finite sets.

As an application, we improve a metrization theorem on $F[X]$.

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1. **Introduction**

In this paper, all spaces are assumed to be regular. The symbol $\mathbb{N}$ is the set of all positive integers. Unexplained notions and terminology are the same as in [4].

For a space $X$, let $F[X]$ be the space of all nonempty finite subsets of $X$ with the Pixley–Roy topology [16]: for $A \in F[X]$ and an open set $U \subset X$, let

$$[A, U] = \{B \in F[X]: A \subset B \subset U\};$$

the family $\{[A, U]: A \in F[X], U \text{ open in } X\}$ is a base for the Pixley–Roy topology. For each $n \in \mathbb{N}$, we put $F_n[X] = \{A \in F[X]: |A| \leq n\}$. Each $F_{n+1}[X] \setminus F_n[X]$ is a discrete space. It is known that $F[X]$ is always zero-dimensional, completely regular and every subspace of $F[X]$ is metacompact: see [3].

The following facts are used in the next section.

**Lemma 1.1.** ([17, Proposition 1.2]) Let $Y$ be a subspace of a space $X$. Then $F[Y]$ is homeomorphic to the closed subspace $\{A \in F[X]: A \subset Y\}$ of $F[X]$.

**Lemma 1.2.** ([13, Theorem 2.8]) For spaces $X_1, \ldots, X_k$, $F[X_1] \times \cdots \times F[X_k]$ can be embedded as a closed subspace of $F[X_1 \times \cdots \times X_k]$. 

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2. The Fréchet–Urysohn property of $\mathcal{F}[X]$

For a space $X$ and a point $x \in X$, a family $\mathcal{P}$ of nonempty subsets of $X$ is said to be a $\pi$-network at $x$ if every neighborhood of $x$ contains some member of $\mathcal{P}$. According to Gruenhage and Szeptycki [10,11], a space $X$ is said to be Fréchet–Urysohn for finite sets (abbr., FUfin) if for every point $x \in X$ and a $\pi$-network $\mathcal{P}$ at $x$ consisting of finite subsets of $X$, there is a sequence $\{P_n: n \in \omega\} \subset \mathcal{P}$ converging to $x$ (i.e., every neighborhood of $x$ contains $P_n$ for all but finitely many $n \in \omega$). This notion was first studied in Reznichenko and Sipacheva [19] systematically. For example, they showed in [19, Propositions 1 and 6] that (1) if $X$ is an FUfin space with a unique non-isolated point, then every finite power of $X$ is also FUfin; (2) if a space $X$ is FUfin, then for every point $x \in X$ and a sequence $\{P_n: n \in \omega\}$ of $\pi$-networks at $x$ consisting of finite subsets of $X$, there are $P_n \in \mathcal{P}_n \subset \mathcal{P}$ such that $\{P_n: n \in \omega\}$ converges to $x$. In particular, every FUfin space is strongly Fréchet–Urysohn, where a space $X$ is said to be strongly Fréchet–Urysohn (22) (or, countably bi-$k$ [14]) if for every point $x \in X$ and a decreasing sequence $\{A_n: n \in \omega\}$ of subsets of $X$, $x \in \cap_{n \in \omega} A_n$ implies that there are points $x_n \in A_n \ (n \in \omega)$ such that $x_n \rightarrow x \ (n \rightarrow \infty)$.

The following is the main theorem of this paper, and it follows from Lemmas 2.2 and 2.3 below.

**Theorem 2.1.** For a space $X$, the following are equivalent:

1. $\mathcal{F}[X]$ is a $k$-space;
2. $\mathcal{F}[X]$ is sequential;
3. $\mathcal{F}[X]$ is Fréchet–Urysohn;
4. every finite power of $X$ is FUfin;
5. every finite power of $\mathcal{F}[X]$ is FUfin.

Let $S_2 = (\{\infty\} \cup \omega \cup (\omega \times \omega))$ be the Arens space [4, 16.19]: each $(n, m) \in \omega \times \omega$ is isolated in $S_2$, for each $n \in \omega \ (n \cup \{m, m\}; m \in \omega)$ is the convergent sequence with the limit $n$ and basic open neighborhoods of $\infty$ are of the form $N(f, k) = (\{\infty\} \cup \omega \cup \{(n, m): n \geq k\}) \cup \{(n, m): n \geq k, m \geq f(m)\}$, where $k \in \omega$ and $f \in \omega^\omega$. There is no sequence in $\omega \times \omega$ converging to $\infty$. This space $S_2$ is sequential, but not Fréchet–Urysohn.

**Lemma 2.2.** If $\mathcal{F}[X]$ is a $k$-space, then it is Fréchet–Urysohn.

**Proof.** Assume that $\mathcal{F}[X]$ is a $k$-space. First we observe that $\mathcal{F}[X]$ is sequential. Let $K$ be a compact subset of $\mathcal{F}[X]$. Since every Tychonoff pseudocompact metacompact space is compact [21, Corollary 3] and every subset of $\mathcal{F}[X]$ is metacompact, every pseudocompact subset of $K$ is compact. Hence, by Zhou’s theorem [25] (if a compact Hausdorff space satisfies that every pseudocompact subset is closed, then it is Fréchet–Urysohn), $K$ is Fréchet–Urysohn. Therefore $\mathcal{F}[X]$ is sequential (in particular, it has countable tightness).

Next we observe that $X$ can be countable. Let $A \subseteq \mathcal{F}[X]$ and $A \subseteq \mathcal{F}[X]$. By countable tightness of $\mathcal{F}[X]$, there is a countable subset $B \subset A$ such that $A \subset B$. Let $Y = A \cup ([\bigcup B: B \in B])$. Then, in view of Lemma 1.1, the subspace $\{A \cup B \subset \mathcal{F}[X]\}$ can be embedded into the closed (hence, sequential) subspace $\mathcal{F}[Y]$. Therefore, to find a sequence in $B$ converging to $A$, we have only to show that the sequential space $\mathcal{F}[Y]$ is Fréchet–Urysohn. Hence we may assume that $X$ is Fréchet–Urysohn.

Now let $\mathcal{F}[X]$ be countable and sequential. We show that $\mathcal{F}[X]$ is Fréchet–Urysohn. Assume the contrary. Since $\mathcal{F}[X]$ is hereditarily normal, by Kannan [12, Corollary 2.3] (if a hereditarily normal sequential space is not Fréchet–Urysohn, then it contains a subspace which is homeomorphic to $S_2$), the Arens space $S_2$ can be embedded into $\mathcal{F}[X]$. Let $\varphi: S_2 \rightarrow \mathcal{F}[X]$ be an embedding, and let $\varphi(\infty) = F$, $\varphi(n) = F_n$ and $\varphi((n, m)) = F_{n, m}$. Considering the open neighborhoods $F, (F, X)$ of $F$ and $\{F_n, X\}$ of $F_n$, we may assume $F \subset F_n \subset F_{n, m}$ for all $n, m \in \omega$. Let $F = \cap \{G_k: k \in \omega\}$, where each $G_k$ is an open subset of $X$ containing $F$ and $G_{k+1} \subset G_k \ (k \in \omega)$ are satisfied. By a simple observation, for each $k \in \omega$ we can choose a strictly increasing sequence $\{m^k_l: l \in \omega\}$ in $\omega$ and a sequence $\{m^k_l: l \in \omega\}$ in $\omega$ such that:

1. $\bigcap_{m^k_l \in \omega} (m^k_l: l \in \omega) = \emptyset$ for all $k \neq k'$;
2. $\{F^k_m: m \geq m^k_l\} \subset \{F, G_{k+1}\}$ for all $l \in \omega$.

Since the space $\{F \cup \{F^k_m: k \in \omega\}\} \cup \{F^k_m, m \geq m^k_l\}$ is homeomorphic to $S_2$, we may assume $m^k_l = 0$ for all $k, l \in \omega$. Therefore, we may assume that for all $k, l \in \omega$, the following condition

$$\{F^k_m\} \cup \{F^k_{m+1}: m \in \omega\} \subset \{F, G_{k+1}\} \cdots$$

is satisfied. For each $k \in \omega, l \in \mathbb{N}$ and $m \geq n^k_l$, we put

$$A_{n^k_l, m} = F^k_{m+1} \cup \left(\bigcup \{F^k_{m+1}: n^k_j \leq j \leq m\}\right).$$
For each $k \in \omega$, $\iota \in \mathbb{N}$, let

$$A_{k,\iota} = \{A_{n^k, m} : m \geq n^k\}.$$ 

We observe that each $A_{k,\iota}$ is closed and discrete in $\mathcal{F}[X]$. Assume the contrary. Since $\mathcal{F}[X]$ is sequential, there are a point $B \in \mathcal{F}[X]$ and a sequence $\{A_{n^k, m} : i \in \omega\} \subset A_{k,\iota}$ such that $A_{n^k, m} \rightarrow B$ ($i \rightarrow \infty$), where $n^k \leq m_0 < m_1 < \cdots$. By $F_{n^k, m} \rightarrow F_{n^k, m}$ ($m \rightarrow \infty$), the set $\bigcup \{F_{n^k, m} : m \geq n^k\}$ is infinite. Hence there is a point $p \in \bigcup \{F_{n^k, m} : m \geq n^k\} \setminus B$. Since $p \in A_{n^k, m}$ for all but finitely many $i \in \omega$, the neighborhood $[B, X \setminus \{p\}]$ of $B$ contains only finitely many $A_{n^k, m}$. This is a contradiction.

We observe that for each $k \in \omega$, $\bigcup \{A_{k,\iota} : \iota \in \mathbb{N}\}$ is closed and discrete in $\mathcal{F}[X]$. Assume the contrary. Then there is a point $B \in \mathcal{F}[X]$ and a sequence $\{A_{n^k, m} : i \in \omega\} \subset \bigcup \{A_{k,\iota} : \iota \in \mathbb{N}\}$ such that $A_{n^k, m} \rightarrow B$ ($i \rightarrow \infty$), where $1 \leq l_0 < l_1 < \cdots$ and $m_1 \geq n^k_{l_1}$ ($i \in \omega$). Since each $A_{n^k, m}$ contains $F_{n^k, m}$, $F_{n^k, m} \subset B$ holds. We show $B = F_{n^k, m}$. Assume that there is a point $p \in B \setminus F_{n^k, m}$. By virtue of $\bigcup \{F_{n^k, j} : n^k_{l_1} \leq j \leq m_1\} \rightarrow F_{n^k, m}$ ($i \rightarrow \infty$), considering the open neighborhood $[F_{n^k, m}, X \setminus \{p\}]$ of $F_{n^k, m}$, we have $p \notin \bigcup \{F_{n^k, j} : n^k_{l_1} \leq j \leq m_1\}$ for all but finitely many $i \in \omega$. Moreover, by the condition ($*$), $F_{n^k, m} \subset G_{k+\iota}$ for all $i \in \omega$. This implies $p \notin F_{n^k, m}$ for all but finitely many $i \in \omega$. Thus $p \notin A_{n^k, m}$ for all but finitely many $i \in \omega$. On the other hand, considering the neighborhood $[B, X]$ of $B$, we have $p \in B \subset A_{n^k, m}$ for all but finitely many $i \in \omega$. This is a contradiction. Thus $B = F_{n^k, m}$. Take some $j \geq 1$ with $(F_{n^k, m} \cap \mathcal{G}_{k+\iota} = \emptyset$). Take an open set $W \subset X$ such that $F \subset W \subset G_{k+\iota}$ and $F_{n^k, m} \notin [F, W]$ for all $i \in \omega$ (i.e., $F_{n^k, m} \setminus W \neq \emptyset$ for all $i \in \omega$). Since $A_{n^k, m} \rightarrow F_{n^k, m}$ ($i \rightarrow \infty$), considering the open neighborhood $[F_{n^k, m}, W \cup (X \setminus \mathcal{G}_{k+\iota})]$ of $F_{n^k, m}$, we have $F_{n^k, m} \subset W \cup (X \setminus \mathcal{G}_{k+\iota})$ for all but finitely many $i \in \omega$. Take some $i \in \omega$ such that $l_i \geq j$ and $F_{n^k, m} \subset W \cup (X \setminus \mathcal{G}_{k+\iota})$. By the condition ($*$), $F_{n^k, m} \subset G_{k+\iota}$. Therefore

$$F_{n^k, m} \subset G_{k+\iota} \cap (W \cup (X \setminus \mathcal{G}_{k+\iota})) \subset G_{k+\iota} \cap (W \cup (X \setminus \mathcal{G}_{k+\iota})) = W.$$ 

This is a contradiction.

We observe that $\bigcup \{A_{k,\iota} : k \in \omega, \iota \in \mathbb{N}\}$ is closed and discrete in $\mathcal{F}[X]$. Assume the contrary. Then there is a point $B \in \mathcal{F}[X]$ and a sequence $\{A_{n^k, m} : i \in \omega\} \subset \bigcup \{A_{k,\iota} : k \in \omega, \iota \in \mathbb{N}\}$ such that $A_{n^k, m} \rightarrow B$ ($i \rightarrow \infty$), where $k_0 < k_1 < \cdots$ and $l_i \geq 1$ ($i \in \omega$) and $m_1 \geq n^k_{l_1}$ ($i \in \omega$). Since each $A_{n^k, m}$ contains $F$, $F \subset B$ holds. We show $B = F$. By the condition ($*$), $F_{n^k, m} \in [F, G_{k_1}]$ for all $m \in \omega$, hence $\bigcup \{F_{n^k, j} : n^k_{l_1} \leq j \leq m_1\} \subset G_{k_1}$ for all $i \in \omega$. By the condition ($*$) again, $F_{n^k, m} \in [F, G_{k_1+i_1}]$ for all $i \in \omega$, hence $F_{n^k, m} \subset G_{k_1+i_1} \subset G_{k_1}$ for all $i \in \omega$. Therefore $A_{n^k, m} \subset G_{k_1}$ for all $i \in \omega$. Considering the open neighborhood $[B, X]$ of $B$, we have $B \subset \bigcap \{G_{k_1} : i \in \omega\} = F$. Thus $B = F$. Hence $A_{n^k, m} \rightarrow F$ ($i \rightarrow \infty$), in particular we have $F_{n^k, m} \rightarrow F$ ($i \rightarrow \infty$). This is a contradiction.

Let $A = \bigcup \{A_{k,\iota} : k \in \omega, \iota \in \mathbb{N}\}$. In spite of the observation above, we can see $F \in \overline{A} \setminus A$. Obviously $F \neq A$. Let $[F, U]$ be an open neighborhood of $F$. Since $F_{n^k, m} \rightarrow F$ ($k \rightarrow \infty$), there is some $k \in \omega$ such that $[F_{n^k, m}] \cap \{\iota \in \mathbb{N} \subset [F, U]$). By $F_{n^k, m} \in [F, U]$), there is some $i \in \mathbb{N}$ such that $F_{n^k, m} \in [F, U]$ for all $m \geq n^k_i$. By $F_{n^k, m} \in [F, U]$, there is some $m \geq n^k_i$ such that $F_{n^k, m} \in [F, U]$. Then we have $A_{n^k, m} \in [F, U]$.

Consequently we conclude that $\mathcal{F}[X]$ must be Fréchet–Urysohn. $\square$

In this paragraph, let $X = \omega \cup \{\infty\}$ be a space with a unique non-isolated point $\infty$. Let $G = (\omega ^{<\omega}, \Delta )$ be the group with the symmetric difference operation $\Delta$, where $\omega ^{<\omega}$ is the set of all finite subsets of $\omega$. For each open neighborhood $U$ of $\infty$, let $V_U = \{F \in G : F \subset U\}$. This defines a neighborhood base at $\emptyset \in G$ making $G$ a topological group. Reznichenko and Sipacheva [19] proved that $X$ is $FU_{fin}$ if and only if $G$ is Fréchet–Urysohn. On the other hand, consider the subspace $[\{\infty\}, X]$ of $\mathcal{F}[X]$ (i.e., it is the space of all non-isolated points in $\mathcal{F}[X]$). For $A, B \in [\{\infty\}, X]$, define $A + B = [\{\infty\} \cup (A \Delta B$. Then it is easy to check that $[\{\infty\}, X, +)$ is a topological group and naturally isomorphic to the topological group $G$. Therefore we can have that $\mathcal{F}[X]$ is Fréchet–Urysohn if and only if $X$ is $FU_{fin}$. However this equivalence follows from the following general characterization.

**Lemma 2.3.** For a space $X$, the following are equivalent:

1. $\mathcal{F}[X]$ is Fréchet–Urysohn;
2. Every finite power of $X$ is $FU_{fin}$;
3. Every finite power of $\mathcal{F}[X]$ is $FU_{fin}$.
Proof. The implication (3) → (1) is trivial. First we show the equivalence (1) ↔ (2).

(1) → (2): Fix $k \in \mathbb{N}$ and a point $x = (x_1, \ldots, x_k) \in X^k$. Let $\mathcal{P}$ be a $\pi$-network at $x$ consisting of finite subsets of $X^k$. We take an open neighborhood $U_i$ of $x_i$ such that $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Let $A = \{x_1, \ldots, x_k\}$ and $U = U_1 \cup \cdots \cup U_k$. Let

$$ D = \{ F \in \{ A, U \} : \text{there is a member } P \in \mathcal{P} \text{ with } P \subseteq (U_1 \times \cdots \times U_k) \cap F^k \}. $$

We observe $A \in D$. Take any basic open neighborhood $[A, V]$ of $A$. Since $(U_1 \cap V) \times \cdots \times (U_k \cap V)$ is an open neighborhood of $x$, there is a member $P \in \mathcal{P}$ with $P \subseteq (U_1 \cap V) \times \cdots \times (U_k \cap V)$. Let $F = A \cup p_1(P) \cup \cdots \cup p_k(P)$, where $p_i$ is the projection of $X^k$ to the $i$-th coordinate. Obviously $F \in \{ A, V \} \cap \mathcal{F}(X)$. Since $F^k$ contains $P$, $P \subseteq (U_1 \times \cdots \times U_k) \cap F^k$, thus $F \in \{ A, V \} \cap D$. Since $\mathcal{F}(X)$ is Fréchet–Urysohn, there is a sequence $\{ F_n : n \in \omega \} \subseteq D$ converging to $A$. For each $F_n$, take a member $P_n \in \mathcal{P}$ such that $P_n \subseteq (U_1 \times \cdots \times U_k) \cap (F_n)^k$. We observe that $\{ P_n : n \in \omega \}$ converges to $x$. Let $W_1 \times \cdots \times W_k$ be an open neighborhood of $x$, where $W_i$ is an open neighborhood of $x_i$ such that $W_i \subset U_i$, and $W_i = W_j$ if $x_i = x_j$. Take $n_0 \in \omega$ such that $F_n \in \{ A, W_1 \cup \cdots \cup W_k \}$ for all $n \geq n_0$. Then, for all $n \geq n_0$ we have

$$ P_n \subseteq (U_1 \times \cdots \times U_k) \cap (F_n)^k \subseteq (U_1 \times \cdots \times U_k) \cap (W_1 \cup \cdots \cup W_k)^k = W_1 \times \cdots \times W_k. $$

(2) → (1): Let $A = \{ x_1, \ldots, x_k \} \subseteq \mathcal{F}(X)$ and assume $A \not\subseteq \mathcal{A} \setminus \{ A \} \subseteq \mathcal{F}(X)$. Take an open neighborhood $U_i$ of $x_i$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Since $\mathcal{A} \cup \{ A \}$ is an open neighborhood of $A$, we may assume that every $B \subseteq A$ satisfies $A \subseteq B \subseteq U_1 \cup \cdots \cup U_k$. Let

$$ \mathcal{P} = \{(U_1 \cap B) \times \cdots \times (U_k \cap B) : B \subseteq A \}. $$

Obviously each member of $\mathcal{P}$ is nonempty and finite. We observe that $\mathcal{P}$ is a $\pi$-network at the point $x = (x_1, \ldots, x_k) \in X^k$. Let $W_1 \times \cdots \times W_k$ be an open neighborhood of the point $x$, where $W_i \subset U_i$ $(i \leq k)$. Take a point $B \in \{ A, W_1 \cup \cdots \cup W_k \} \cap \mathcal{A}$. Then

$$ (U_1 \cap B) \times \cdots \times (U_k \cap B) = (W_1 \cap B) \times \cdots \times (W_k \cap B) \subseteq W_1 \times \cdots \times W_k. $$

Thus $\mathcal{P}$ is a $\pi$-network at $x$. Since $X^k$ is FU$_{fin}$, there is a sequence $\{ P_n : n \in \omega \} \subseteq \mathcal{P}$ converging to $x$. Let $P_n = (U_1 \cap B_n) \times \cdots \times (U_k \cap B_n)$, where $B_n \subseteq A$. We observe that the sequence $\{ B_n : n \in \omega \}$ converges to $A$. Take a basic open neighborhood $[A, V]$ of $A$. Since $(U_1 \cap V) \times \cdots \times (U_k \cap V)$ is an open neighborhood of $x$, there is an $n_0 \in \omega$ such that for all $n \geq n_0$

$$ P_n = (U_1 \cap B_n) \times \cdots \times (U_k \cap B_n) \subseteq (U_1 \cap V) \times \cdots \times (U_k \cap V). $$

Since $B_n$ is contained in $U_1 \cup \cdots \cup U_k$,

$$ B_n = (U_1 \cap B_n) \cup \cdots \cup (U_k \cap B_n) \subseteq (U_1 \cap V) \cup \cdots \cup (U_k \cap V) \subset V. $$

Hence $B_n \in \{ A, V \}$ for all $n \geq n_0$.

(1) → (3): First we show that $\mathcal{F}(X)$ is FU$_{fin}$. Let $A \in \mathcal{F}(X)$ and let $\mathcal{P} = \{ A_\alpha : \alpha < \kappa \}$ be a $\pi$-network at $A$ consisting of finite subsets of $\mathcal{F}(X)$. Without loss of generality, we may assume $A_\alpha \subseteq [A, X]$ for all $\alpha < \kappa$ (i.e., every member of $\bigcup_{\alpha < \kappa} A_\alpha$ contains $A$). For each $\alpha < \kappa$, let $A_\alpha = \bigcup A_\alpha$. Note that for a basic open neighborhood $[A, U]$ of $A$, $A_\alpha \subseteq [A, U]$ if and only if $A_\alpha \subseteq [A, U]$. We observe $A \subseteq [A_\alpha] = \alpha < \kappa$. Take any basic open neighborhood $[A, U]$ of $A$. Then there is an $\alpha < \kappa$ with $A_\alpha \subseteq [A, U]$, hence $A_\alpha \subseteq [A, U]$. Since $\mathcal{F}(X)$ is Fréchet–Urysohn, there is a sequence $\{ A_\alpha n : n \in \omega \}$ converging to $A$. Then $\{ A_\alpha n : n \in \omega \}$ converges to $A$. Thus $\mathcal{F}(X)$ is FU$_{fin}$. For each $n \in \mathbb{N}$, by Lemma 1.2 the $n$-times product $\mathcal{F}(X)^n$ can be embedded into $\mathcal{F}(X^n)$. Since every finite power of $X^n$ is FU$_{fin}$, $\mathcal{F}(X^n)$ is Fréchet–Urysohn, so it is FU$_{fin}$. Hence the subspace $\mathcal{F}(X^n)$ of $\mathcal{F}(X^n)$ is also FU$_{fin}$. □

Let $S_\omega = [\omega]^{\omega} \cup \{(n, m) : n, m \in \omega\}$ be the sequential fan, where every $(n, m)$ is isolated in $S_\omega$ and a basic open neighborhood of $\infty$ is of the form $N(\infty) = [\infty] \cup \{(n, m) : n \in \omega, m \geq f(n)\}$ for a function $f \in \omega^\omega$. Obviously $S_\omega$ is Fréchet–Urysohn, but not strongly Fréchet–Urysohn.

Since every FU$_{fin}$ space is strongly Fréchet–Urysohn, we have the following.

Corollary 2.4. Neither $\mathcal{F}(S_2)$ nor $\mathcal{F}(S_\omega)$ is a k-space.

For a space $X$, let $\mathcal{F}^1(X) = \mathcal{F}(X)$, and let $\mathcal{F}^{m+1}(X) = \mathcal{F}(\mathcal{F}^{m-1}(X))$ for $n \geq 2$. By Lemma 2.3, we have immediately the following.

Corollary 2.5. If $\mathcal{F}(X)$ is Fréchet–Urysohn, then $\mathcal{F}^m(X)$ is FU$_{fin}$ for all $n, m \in \mathbb{N}$. 

Concerning Theorem 2.1, Gerlits and Nagy [8,7], and independently Pytkeev [18] proved that in the function space $C_p(X)$ the k-space property, sequentiality and the Fréchet–Urysohn property coincide, where $C_p(X)$ is the function space over a Tychonoff space $X$ with the topology of pointwise convergence. Concerning the duality between topological properties of a space $X$ and the space $\mathcal{F}(X)$, Daniels [2] and Scheepers [20] investigated the duality between covering properties of a set $X \subseteq \mathbb{R}$ and the space $\mathcal{F}(X)$ and proved that some of those covering properties coincide in $\mathcal{F}(X)$. 

3. Applications to metrizability of $\mathcal{F}[X]$

A space is said to be Lašnev if it is the closed image of a metric space, and a space $X$ is said to be a quasi-$k$-space if it satisfies that a subset $A \subset X$ is closed in $X$ if and only if $A \cap K$ is closed in $K$ for every countably compact subset $K \subset X$. Since every countably compact metacompact space is compact, every quasi-$k$-space $\mathcal{F}[X]$ is a $k$-space.

Tanaka claimed that the following theorem holds.

**Theorem 3.1.** ([24, Theorem 1.1]) For a space $X$, the following are equivalent:

1. $\mathcal{F}[X]$ is metrizable;
2. $\mathcal{F}[X]$ is Lašnev;
3. $\mathcal{F}[X]$ is a paracompact perfectly normal quasi-$k$-space.

In this section, we note that the implication $(3) \rightarrow (1)$ is incorrect, and improve the implication $(2) \rightarrow (1)$.

**Lemma 3.2.** ([3,13]) For a space $X$, the following are equivalent:

1. $X$ is first-countable;
2. $\mathcal{F}_2[X]$ is first-countable;
3. $\mathcal{F}[X]$ is a Moore space.

**Example 3.3.** Under CH (the continuum hypothesis), there is a countable Fréchet–Urysohn space $\mathcal{F}[X]$ which is not metrizable.

**Proof.** Under CH, there is an uncountable $\gamma$-set $F$ [6]. For this $\gamma$-set $F$, consider the countable space $X_F$ with a unique non-isolated point constructed in [10, Example 1]. This space $X_F$ is FU$_{fin}$ (every finite power of it is also FU$_{fin}$) and not first-countable [10, Theorem 6]. Therefore, by Theorem 2.1 and Lemma 3.2, $\mathcal{F}[X_F]$ is a countable Fréchet–Urysohn space which is not metrizable.

It is open [10, Question 1] whether in ZFC there is a countable FU$_{fin}$ space which is not first-countable. If there were a model of set theory satisfying that every countable FU$_{fin}$ space is first-countable, then every countable Fréchet–Urysohn $\mathcal{F}[X]$ would be metrizable in the model.

Let $\mathcal{P}$ be a family of subsets of a space $X$. According to [15], $\mathcal{P}$ is said to be a $k$-network for $X$ if for each compact subset $K \subset X$ and an open set $U \subset X$ containing $K$, there is a finite subfamily $Q \subset \mathcal{P}$ such that $K \subset \bigcup Q \subset U$.

We show the following metrization theorem.

**Theorem 3.4.** For a space $X$, $\mathcal{F}[X]$ is a Moore space if and only if it is a $k$-space and $\mathcal{F}_2[X]$ has a point-countable $k$-network. Hence, $\mathcal{F}[X]$ is metrizable if and only if it is a paracompact $k$-space and $\mathcal{F}_2[X]$ has a point-countable $k$-network.

**Proof.** Assume that $\mathcal{F}[X]$ is a Moore space. Then, since $\mathcal{F}[X]$ is a metacompact Moore space, it has a point-countable base. Therefore $\mathcal{F}[X]$ is a $k$-space and $\mathcal{F}_2[X]$ has a point-countable $k$-network. Conversely assume that $\mathcal{F}[X]$ is a $k$-space and $\mathcal{F}_2[X]$ has a point-countable $k$-network. By Theorem 2.1, $\mathcal{F}_2[X]$ is strongly Fréchet–Urysohn. Gruenhage, Michael and Tanaka [9, Corollary 3.6] showed that every strongly Fréchet–Urysohn space with a point-countable $k$-network has a point-countable base. Therefore $\mathcal{F}_2[X]$ has a point-countable base. By Lemma 3.2, $\mathcal{F}[X]$ is a Moore space. The latter part on metrizability follows from Bing’s metrization criterion [4, 5.4.1, p. 329].

**Corollary 3.5.** If $\mathcal{F}[X]$ is a paracompact $k$-space with a point-countable $k$-network, then it is metrizable.

Every Lašnev space is paracompact and a $k$-space. Moreover every Lašnev space has a point-countable $k$-network: for example see [5]. Hence we have:

**Corollary 3.6.** If $\mathcal{F}[X]$ is Lašnev, then it is metrizable.

Alternatively, to show the preceding corollary, we may use that every strongly Fréchet–Urysohn Lašnev space is metrizable.

**Example 3.7.** (1) Let $\mathbb{R}$ be the real line. Since $\mathcal{F}[\mathbb{R}]$ is a metacompact Moore space, it has a point-countable base. But $\mathcal{F}[\mathbb{R}]$ is not metrizable [16].
(2) We observe that $F[S_n]$ has a countable $k$-network. Let $T_n = [\omega \setminus \{0\}] \cup \{(i, j) : i \leq n, j \in \omega\}$ for each $n \in \omega$. Let $X_n = \{A \in F[S_n] : A \subset T_n\}$. Then $F[S_n] = \bigcup X_n$ and it is easy to check that every compact subset of $F[S_n]$ is contained in some $X_n$. Since each $X_n$ is homeomorphic to $F[T_n]$, it is metrizable. Therefore, if $B_n$ is a countable base for $X_n$, $B = \bigcup B_n$ is a countable $k$-network. Thus $F[S_n]$ is countable and has a countable $k$-network, but not first-countable.

(3) Let $A(\omega)$ be the one-point compactification of the discrete space of cardinality $\omega_1$. Then $F[A(\omega)]$ is a paracompact Fréchet–Urysohn space which is not first-countable. Paracompactness of $F[A(\omega)]$ follows from Przymusinski [17, Theorem 4.2]. The Fréchet–Urysohn property of $F[A(\omega)]$ follows from Theorem 2.1. In fact, it is easy to see that $A(\omega)$ is $\text{FU}_{\text{fin}}$.

Supplementally we show that each $F_n[S_n]$ is sequential. Therefore, in Theorem 3.4 we cannot replace the condition “$F[X]$ is a $k$-space” by “each $F_n[X]$ is sequential”.

**Lemma 3.8.** Fix $n \in \mathbb{N}$, and assume that $F_n[S_n]$ is sequential. If $A \subset F_{n+1}[S_n] \setminus F_n[S_n]$ and $\overline{A} = \{[\omega] \cup \{n\} \cup A$ are satisfied, then $A$ contains a sequence converging to the point $[\omega]$.

**Proof.** If $n = 1$, our statement is true, in fact note that $F_1[S_1]$ (resp., $F_2[S_2]$) is homeomorphic to $\omega$ (resp., $S_0$). Assume $n \geq 2$. Considering the neighborhood $[[\omega] \cup \{n\} \cup A$ of $[\omega]$, we may assume $[\omega] \in A$ for all $A \in A$. Let $A = \{A_k : k \in \omega\}$. For each $k \in \omega$, take a point $(n, m_k) \in A_k \setminus [\omega]$ such that $(n, m_k) \in A_k$ implies $n_k \leq n$. Let $B_k = A_k \setminus \{(n, m_k)\}$ and let $B = \{B_k : k \in \omega\}$. Then $[\omega] \in B \setminus B$. Since $F_n[S_n]$ is sequential by our assumption, there is a sequence $\{B_k : j \in \omega\} \subset B$ converging to some point $B \in F_n-1[S_n]$. Since the set $\{B_k : j \in \omega\}$ is compact, there is an $l \in \omega$ such that $\bigcup B_k \cap \{B_k : j \in \omega\} \cap \{(n, m) : n \leq l, m \in \omega\}$. Then automatically $\bigcup B_k \cap \{(n, m) : n \leq l, m \in \omega\}$. Assume that $\{m_k, m_k : j \in \omega\}$ is finite. Then there is a $(p, q) \in \{(n, m) : n \leq l, m \in \omega\}$ such that $j \in [\omega] : (p, q) = (m_k, m_k)$ of finite. If $j \in J$, then

$$A_k = B_k \cup \{(m_k, m_k)\} = B_k \cup \{(p, q)\}$$

and these converge to $B \cup \{(p, q)\} \in F_n[S_n]$. Thus $\overline{A} \supseteq B \cup \{(p, q)\} \neq [\omega]$, this is a contradiction. Hence $\{m_k, m_k : j \in \omega\}$ is infinite. In this case, there are an $l \leq l$ and an infinite subset $J \subset \omega$ such that for each $j \in J$, $(m_k, m_k) = (l', m_k)$ and $\{m_k, m_k : j \in \omega\}$ is strictly increasing. If $j \in J$, then

$$A_k = B_k \cup \{(m_k, m_k)\} = B_k \cup \{(l', m_k)\}$$

and these converge to $B$, because $(l', m_k) : j \in J$ converges to $[\omega]$ and $\omega \in B$. By $\overline{A} = \{[\omega] \cup A$, we have $B = [\omega]$. $\square$

**Theorem 3.9.** For each $n \in \mathbb{N}$, $F_n[S_n]$ is sequential.

**Proof.** Assume that $F_n[S_n]$ is sequential. We show that $F_{n+1}[S_n]$ is sequential. Let $A$ be a non-closed subset of $F_{n+1}[S_n]$. We have to find a sequence in $A$ converging to a point in $F_{n+1}[S_n] \setminus A$. Since $F_n[S_n]$ is sequential, we may assume that $A \cap F_n[S_n]$ is closed in $F_{n+1}[S_n]$. Let

$$k = \max \{m : 1 \leq m \leq n, (\overline{A} \setminus A) \cap \{F_m[S_n] \cup F_{m-1}[S_n]\} \neq \emptyset\},$$

where $F_0[S_n] = \emptyset$. Fix a point $A \in (\overline{A} \setminus A) \cap (\overline{F_k[S_n] \cup F_{k-1}[S_n]}).$ Take an open subset $U \subset S_n$ such that $A \subset U$ and $A \cup A \cap F_{n}[S_n] = \emptyset$. Let $B = A \cap A$. Then obviously $B = B \cup \{A\}$. If $k = 1$, then $A = [\omega]$. By Lemma 3.8, we can take a sequence in $B$ converging to $A = [\omega]$. Thus we assume $2 \leq k \leq n$. Let $A = [\omega] \cup A'$, where $A' \subset S_n \setminus [\omega]$ and $|A'| = k$. We consider the map $D_{A'} : [A, U] \to F[S_n]$ defined by $D_{A'}(F) = F \setminus A'$. It is a routine to check that $D_{A'}$ is an embedding of $[A, U]$ into $F[S_n]$, $D_{A'}(A) = [\omega]$ and $D_{A'}([A, U]) = [\omega] \cup A'$. Hence $D_{A'}(B) = D_{A'}(B) \cup \{[\omega] \cup F_{n}[S_n]$. Since $F_n[S_n]$ is sequential, $D_{A'}(B)$ contains a sequence converging to $[\omega]$. In other words, $B$ contains a sequence converging to $A$. $\square$

We note that $F_3[S_3]$ is not Fréchet–Urysohn. For each $n \in \omega$, let $A_n = \{[\omega], (0, n), \{n + 1, m\}) : m \in \omega\$, and let $\mathcal{A} = \bigcup \{A_n : n \in \omega\}. Then \{[\omega] \in \overline{A} \setminus A$, but there is no sequence in $A$ converging to $[\omega]$.

**4. Remarks on weak-bases for $F[X]$**

Concerning metrizability of $F[X]$, we investigate weak-bases for $F[X]$. Let $X$ be a space. For each point $x \in X$, let $B_X$ be a family of subsets of $X$. According to [1], $B = \bigcup \{B_x : x \in X\} \subset X$ is said to be a weak-base for $X$ if it satisfies (1) every member of $B_X$ contains $x$, (2) for $B_0, B_1 \subset B_X$, there is a $B \in B_X$ such that $B \subset B_0 \cap B_1$ and (3) $G \subset X$ is open if and only if for each $x \in G$ there is a $B \in B_X$ with $B \subset G$. A space $X$ is said to be g-first-countable (resp., g-metrizable) [23] if it has a weak-base $B = \bigcup \{B_x : x \in X\}$ such that each $B_x$ is countable (resp., $B$ is $\sigma$-locally finite). Every g-metrizable space is g-first-countable.

**Lemma 4.1.** If $F_2[X]$ is sequential, then it is Fréchet–Urysohn.
Proof. We have only to see that each point in \( F_2[X] \) has a Fréchet–Urysohn neighborhood. If \( A \in F_2[X] \) is non-isolated in \( F_2[X] \), then \( A = \{x\} \) for some non-isolated point \( x \in X \) and \( [A, X] \cap F_2[X] \) is an open-and-closed neighborhood of \( A \) in \( F_2[X] \). Since \( [A, X] \cap F_2[X] \) is sequential and all points in it but \( A \) is isolated, \( [A, X] \cap F_2[X] \) is a Fréchet–Urysohn neighborhood of \( A \). \( \square \)

**Proposition 4.2.** For a space \( X \), the following are equivalent:

1. \( F[X] \) is g-first-countable;
2. \( F_2[X] \) is g-first-countable;
3. \( F[X] \) is a Moore space.

**Proof.** The implications (1) \( \rightarrow \) (2) and (3) \( \rightarrow \) (1) are trivial.

(2) \( \rightarrow \) (3): Assume that \( F_2[X] \) is g-first-countable. Since every g-first-countable space is sequential [1], \( F_2[X] \) is sequential. By Lemma 4.1, \( F_2[X] \) is Fréchet–Urysohn. Since every g-first-countable Fréchet–Urysohn space is first-countable [1], \( F_2[X] \) is first-countable. By Lemma 3.2, \( F[X] \) is a Moore space. \( \square \)

Since every Fréchet–Urysohn g-metrizable space is metrizable [23, Theorem 1.10], we have:

**Corollary 4.3.** For a space \( X \), \( F[X] \) is g-metrizable if and only if it is metrizable.

**References**