Petri net languages revisited

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A R T I C L E   I N F O

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A B S T R A C T

In this paper, the behavior of place/transition Petri nets is discussed. As a formal tool of consideration a commutation homomorphism of monoids is applied, which gives rise to a comparison of the sequential behavior of nets with its commutative version. The sequential and commutative languages are discussed and compared by means of commutation homomorphism from the monoid of words to the monoid of multisets. First, atomic nets (nets with a single place only) are considered. It is proved that (1) the sequential behavior of atomic nets is a context free language, and (2) the commutative behavior, obtained as a homomorphic image of the sequential one, is regular. From here, via compositionality property of nets these results are lifted to the case of all place/transition nets. Namely, it turns out that the sequential behavior of any Petri net is the intersection of a finite number of context free languages, and next, that commutative behavior of any general net is regular. The substantial part in the presented approach plays reduced languages, as a “go between”: they are regular subsets of sequential languages of nets with the same commutative images as the original ones.

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1. Introduction

The paper aims to discuss, analyze, and compare some methods of concurrent system behavior representation. As a formal model of concurrent systems Petri nets are chosen, as offering a formal, sufficiently general, and widely known method of concurrency description. As a formal tool for comparing behavior of systems some algebraic constructs are used, such as monoids and morphisms between them. Compositionality of systems plays an essential part in this discussion: after having established properties of simple structures, the obtained results are lifted to more complex ones. Sets of executing sequences (sequences of actions), sets of multisets abstracting from irrelevant successions (giving some, but not full information about states), and sets of reachable states are of principal interest. To show how powerful are algebraically oriented methods in solving various problems of discrete processes theory, is another motivation for preparing the present paper. Petri nets discussed in the paper are nets without the so-called self loops, i.e. the set of input places of any transition is disjoint with its set of output places.

The paper is organized as follows. First, the so-called atomic nets, i.e. nets with one place only, are considered. For such nets three objects describing their behavior are introduced, as sequential behavior, containing well known execution sequences, next, by mapping these sequences via commutative homomorphism to commutative behavior, neglecting some ordering of actions, and finally, by mapping execution sequences via reachability homomorphism to the reachable set of states. It is proved that the set of all execution sequences of an atomic net is a context free language.

Second, the property of net composition such that the behavior of nets composition is the composition of their individual behavior, is used as a tool for general nets analysis. It is proved that the set of execution sequences of general nets is the intersection of a finite number of context free languages; it shows how close to undecidability problems the issues concerning sequential languages are. However, using the well known fact that intersection of regular languages results in
a regular language, it turns out that the commutative behavior of an arbitrary net is regular. It is helpful for proving the decidability of the reachability problem for general nets. Closing remarks end the paper.

The reader is assumed to be acquainted with some basic facts from the theory of formal languages and finite automata on one hand, and with a (very restricted) knowledge about Petri nets and their behavior on the other hand. Besides of it, some elementary notions on the abstract algebras will be helpful.

2. Preliminaries

Standard mathematical notions are used throughout the paper. Sets \{0, 1, 2, \ldots\} and \{\ldots, −2, −1, 0, 1, 2, \ldots\} will be denoted by \(\mathbb{N}\) and \(\mathbb{Z}\), respectively.

**Algebra of words.** Any finite set \(T\) of symbols will be called alphabet. Any finite sequence \(w\) of symbols from \(T\) is a word over \(T\), its length is denoted by \(|w|\); word of length 0 (the empty word) is denoted by \(\epsilon\). The set of all words over \(T\) is denoted by \(T^*\). Concatenation of word \(u = (a_1, a_2, \ldots, a_n)\) with word \(w = (b_1, b_2, \ldots, b_m)\), with \(n, m \geq 0\), is word \(u \cdot w = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)\) called the concatenation of \(u\) with \(w\). Commas separating symbols as well as the dot symbolizing concatenation operation are usually omitted. By \(T^*\) we understand the set of all words over \(T\). Clearly, \((T^*, \cdot, \epsilon)\) is a free monoid with \(T\) as the set of generators. Word \(u\) is a prefix of word \(uw\). Write \(u \leq w\) if \(u\) is a prefix of \(w\). Number \(|w|\) is the length of word \(w\). Subsets of \(T^*\) are called sequential languages, or briefly, languages over \(T\). Language \(L \subseteq T^*\) is said to be prefix-closed, if \(u \leq v \in L \Rightarrow u \in L\). For any languages \(L_1, L_2 \subseteq T^*\), write \(L_1 \subseteq L_2\) for the language \(\{w \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}\).

**Formal grammars.** Context-free grammar is any quadruple \(G = (V, T, P, \sigma)\), where \(V\) is a finite set of symbols, \(T \subseteq V\) is a set of terminal symbols, \(P \subseteq (V - T) \times V^*\) is a finite relation with elements called productions, and \(\sigma \in \epsilon - T\) is the axiom of \(G\). Given context free grammar \(G = (V, T, P, \sigma)\), relation \(\rightarrow \subseteq V^* \times V^*\) is defined by the equivalence

\[ w' \rightarrow w'' \iff \exists u, u', v, w : w' = u'wv, w'' = u'w'u, (v, w) \in P \]

is referred to as the step relation in \(G\). Call the derivation relation in \(G\) and denote by \(\rightarrow^*\) the transitive and reflexive closure of \(\rightarrow\). Sequence of words \(w_0, w_1, \ldots, w_n\) with \(n \geq 0\) is called a derivation of \(w_n\) from \(w_0\), if \(w_{i-1} \rightarrow w_i\) for all \(i, 0 < i \leq n\), and \(w_0\) is said to be derived from \(w_0\). Number \(n\) is then called the length of derivation. If \(w''\) is derived from \(w'\), write \(w' \rightarrow^* w''\). We shall also write \((v \rightarrow w)\) for production \((v, w) \in P\). Language

\[ L(G) = \{w \in T^* \mid \sigma \rightarrow^* w\} \]

is a context free language defined by grammar \(G\). A particular case of context free grammars are regular grammars. Context free grammar \((V, T, P, \sigma)\) is regular, if \(P \subseteq (V - T) \times T^*(V \cup \{\epsilon\})\). Languages generated by regular grammars are said to be regular.

3. Atomic nets

**Definition 1.** Atomic net is any ordered triple \(A = (T, F, m^0)\) such that

- \(T\) is a finite set (of transitions),
- \(F : T \rightarrow \mathbb{Z}\) (valuation),
- \(m^0 \in \mathbb{N}\) (initial value).

Valuation \(F\) is extended to \(F^* : T^* \rightarrow \mathbb{Z}\) by setting \(F^*(\epsilon) = m^0, F^*(wt) = F^*(w) + F(t)\), and \(F^*(Z) = \{F^*(w) \mid w \in Z\}\) for any \(t \in T, w \in T^*, Z \subseteq T^*\). For each \(w \in T^*\) define condition \(P(w)\) as follows:

\[ P(w) \Leftrightarrow \forall u \leq w \in T^* : F^*(u) \geq 0. \]

Set of words

\[ L = \{w \in T^* \mid P(w)\} \]

is the sequential language of \(A\), or simply the language of \(A\). Set of integers \(F^*(L)\) is the reachability set of \(L\); integers in \(F^*(L)\) are reachable by \(L\). □

Some auxiliary notions and denotations related to \(A\) and used in the sequel for discussing atomic nets are

\[ T_p = \{t \in T \mid F(t) \geq 0\}, \quad \text{(set of producers)}, \]
\[ T_m = \{t \in T \mid F(t) < 0\}, \quad \text{(set of consumers)}, \]
\[ |A| = \sum_{t \in T} |F(t)| + m^0, \quad \text{the size of } A. \]

Observe that transitions \(t \in T\) with \(F(t) = 0\) are acceptable by the above definition and formally classified as producers; later they will prove to be useful in composition of a number of atomic nets, as transitions that participate in actions of some atomic nets but do not participate in the others.

**Example 1.** Triple \(X = ([a, b, c], \{(a, 2), (b, -3), (c, -1)\}, 3)\) is an atomic net, with \(|A| = 9, T_p = \{a\}, T_m = \{b, c\}\). □
In the rest of this section atomic net $A = (T, F, m^0)$ together with symbols $T_p, T_m, |A|, P$ will be fixed and defined as above; let $L$ be the sequential language of $A$.

3.1. Sequential behavior of atomic nets

Let start with some properties of the sequential language $L$ of $A$.

**Proposition 1.** The following properties hold for all $w \in L$:

1. $F_0(w) \geq |A|$, $t_1, t_2 \in T, t_1 \neq t_2 \Rightarrow w t_1 t_2 \in L$.
2. $w \in L, a \in T_p \Rightarrow wa \in L$.
3. $t_1 \in T, b \in T_m, w b t_1 \in L \Rightarrow wt_1 b \in L$.
4. $b_1, b_2 \in T_m \Rightarrow (w b_1 b_2 \in L \iff w b_2 b_1 \in L)$.

**Proof.** Clearly, $F_0(w) > |A|$ guarantees $P(w t_1 t_2)$ for arbitrary $t_1, t_2 \in T, t_1 \neq t_2$, since then $F_0(w)$ is sufficiently large to ensure $P(w t_1 t_2)$ as well as $P(w t_1 t_2)$. It proves 1. Implication 2 expresses the fact, that $P(w)$ implies $P(w a)$ for any $a \in T_p$. Implication 3 says that if $P(w b t_1)$ holds for $t_1 \in T, b \in T_m$, the more $P(w b t_1)$ is holding. Implication 4 is evident: changing $b_2 b_1$ to $b_2 b_1$ in case of $b_1, b_2 \in T_m$ preserves validity of $P(w b_1 b_2)$ and of $P(w b_2 b_1)$. □

**Proposition 2.** Let $w, w_1, w_2 \in L$. Then

1. $P(w t), t \in T \Rightarrow P(w)$.
2. $P(w), u \in T_m \Rightarrow 0 \leq F_0(u) \leq m^0$.
3. $P(w_1 w_2), a \in T_p \Rightarrow P(w_1 a w_2)$.
4. $P(w_1 b w_2), b \in T_m \Rightarrow P(w_1 w_2)$.

**Proof.** Statement 1 expresses the prefix-closedness of set $\{ w \mid P(w) \}$, which is evident. Statement 2 follows directly from the definition of $P(w)$ which guarantees $F_0(u) \geq 0$ for all $u \leq w$. If $P(w_1 w_2)$ is valid, the more $P(w_1 a w_2)$ is valid; if $P(w_1 b w_2)$ is valid, the more $P(w_1 w_2)$ is valid, for all $w_1, w_2 \in L$, which proves 3 and 4. □

**Definition 2.** Let $A$ be atomic net defined as above, $\{ \langle k \rangle \mid -|A| \leq k \leq |A| \} \cap T = \emptyset$. Grammar $G_A = (V, T, P, \sigma)$, such that

$V = \{ \langle k \rangle \mid -|A| \leq k \leq |A| \} \cup T$,
$P = \{ (0) \rightarrow e \} \cup \{ (F(t)) \rightarrow t \mid t \in T \} \cup \{ \langle k \rangle \rightarrow \langle n \rangle \mid k \leq n \leq |A| \} \cup \{ \langle k+n \rangle \rightarrow \langle k \rangle \langle n \rangle \mid \langle k+n \rangle \in V, k \geq 0 \lor n \leq 0 \}$,

$\sigma = \langle (-m^0) \rangle$,

will be called the **sequential grammar** for atomic net $A$. □

In this section symbol $\rightarrow$ (of step) and symbol $\rightarrow^*$ (of derivation) will be used as related to grammar $G$ defined above.

**Lemma 1.** For all $w', w'' \in V^*, 0 \leq m \leq |A| : \sigma \rightarrow^* w' w'' \Rightarrow \sigma \rightarrow^* w' (m) w''$

**Proof.** Assume $\sigma \rightarrow^* w' w''$. Let $w' = u(k)$; then $w' w'' = u(k) w'' = u(0 + k) w''$; by splitting $u(k) w'' = w'(0) w'' = w'(0) w''$; by increasing $w'(0) w'' = w'(m) w''$; from here it follows $\sigma \rightarrow^* w'(m) w''$. Let now $w'' = \langle u \rangle k$; then $w' w'' = w'(k) u = u(0 + k) w''$; again by splitting $u(0 + k) w'' = w'(0) \langle k \rangle u = w'(m) w''$; from here by increasing $\sigma \rightarrow^* w'(m) w''$. In case of $w' w'' = \epsilon, \sigma = \langle (-m^0) \rangle \rightarrow \langle m \rangle = \epsilon$ by increasing. Therefore, in all cases $\sigma \rightarrow^* w'(m) w''$ □

**Proposition 3.** $L(G_A) = L$.

**Proof.** Let $G_A$ be the sequential grammar for atomic net $A = (T, F, m^0)$, and $\rightarrow^*$ be the derivation relation in $G_A$. Extend mapping $F$ to $V^*$ by setting $F(\langle k \rangle) = k, F^*(\epsilon) = m^0, F^*(\langle k \rangle) = F^*(\epsilon) + k$, and similarly extend condition $P$ to set $V^*$:

$P(w) \Leftrightarrow \forall v \leq w \in V^* : F^*(\langle v \rangle) \geq 0$.

In order to prove $L(G_A) = L$ we show that for all $w \in V^*$ equivalence:

$\sigma \rightarrow^* w \Leftrightarrow P(w)$

holds. First prove $\sigma \rightarrow^* w \Rightarrow P(w)$. The proof will be carried out by induction w.r. to the length of derivation of $w$. Let $w = \sigma$; then $P(\sigma)$ holds, since $\sigma = \langle (-m^0) \rangle$ and $F^*(\sigma) = m^0 + F^*(\langle (-m^0) \rangle) = 0$, proving $P(\langle (-m^0) \rangle)$. Assume $P(w')$ holds and let $w' \rightarrow^* w''$. Then either

$w' = w_1(F(t)) w_2, w'' = w_1 t w_2$, with $t \in T$, or
$w' = w_1(k) w_2, w'' = w_1(n) w_2$, with $k \leq n$, or
$w' = w_1(k+n) w_2, w'' = w_1(k) \langle n \rangle w_2$, with $(k, n) \geq 0 \lor n \leq 0$.

for some $w_1, w_2 \in V^*$. Clearly, $P(w_1(F(t)) w_2) \Leftrightarrow P(w_1 w_2)$; next, also evidently, $P(w_1(k) w_2)$ and $k < n$ implies $P(w_1(n) w_2)$. Let now $P(w_1(k+n) w_2)$. Prefixes of $w_1(k) \langle n \rangle$ are $w_1, w_1(k), w_1(k) \langle n \rangle$. Since $P(w_1(k+n) w_2)$ holds, $F^*(w_1) \geq 0$ and $F^*(w_1) + k + n \geq 0$; in both cases, of $k \geq 0$ or $n \leq 0$, from $F^*(w_1) + k + n \geq 0$ it follows $F^*(w_1) + k \geq 0$. It
Proposition 4. Let atomic net $A$ be defined as in the preceding section, and let all notions concerning $A$ be denoted as above. As above, the language of $A$ is denoted by $L$. We start with a subset of the atomic net language called the reduced language of the net. The basic properties of reduced languages are: (1) the same reachability set as the original language; (2) regularity of the reduced language.

Definition 3. Let $A$ be atomic net defined as above. Reduced language of $A$ is the set of all words $w \in L$ such that for all $b \in T_m$, $vb \leq w$

$$F^*(v) \leq |A|.$$  \hspace{1cm} (1)

In the what follows let $R$ denotes the reduced language of atomic net $A$.

Proposition 4.

$$w \in R \iff \exists w' \in T^*, w'' \in T^*_p : w = w'w'' \in L, \quad \forall u \leq w' : F^*(u) \leq |A|.$$  \hspace{1cm} (2)
The above proposition offers an explicit form of Definition 3 characterizing words in $R_A$ as those composed of two words $w'w''$ such that $F^*(w') \leq |A|$ for any prefix $w'$ of the first, and the second containing exclusively producers (transitions with positive valuations). In the sequel it will help for discussing reachability issues for reduced languages.

**Proposition 5.** Let $w \in L$. The following statements are valid:

1. $\epsilon \in R$;
2. $w \in R, a \in T_p \Rightarrow wa \in R$;
3. $wa \in R, a \in T \Rightarrow w \in R$.
4. $u \leq w \in R \Rightarrow u \in R$.

**Proof.** First two statements say that $R \subseteq L$; none of them contradicts to (1). Statement 3 holds since prefix of any member of $R$ is in $R$ as well. Statement 4 follows from 3 by a simple induction. □

**Example 3.** Let $R_X$ be the reduced language of atomic net $X$ defined in Example 1. Then each word in $\{(ac)^n | n \geq 0\}$ is in $R_X$, while none of words in $\{a^n c^m | n > 6\}$ is in $R_X$. □

**Definition 4.** Let $A$ be the atomic net as defined above, $\{(k) | 0 \leq k \leq |A| \cap T = \emptyset; \}$ then grammar $E_A = (V, T, P, \sigma)$ such that

$$
V = \{(k) | 0 \leq k \leq |A|\} \cup T,
$$

$$
P = \{(k) \rightarrow \epsilon\} \cup
\{(k) \rightarrow t(k + F(t)) | k + F(t) \leq |A|\} \cup
\{(k) \rightarrow t|A| | k + F(t) > |A|\} \cup
\{|A| \rightarrow t|A| | F(t) > 0\},
$$

$$
\sigma = (m^0)
$$

for all $(k) \in V, t \in T,$ will be called the reduced grammar for atomic net $A$. □

Observe that $(|A| \rightarrow \epsilon) \in P$ by the first group of productions in $P$ (with $k = |A|$).

**Proposition 6.** $\mathcal{L}(E_A) = R$ for any atomic net $A$ with reduced language $R$.

**Proof.** Let $\rightarrow^*$ be the derivation relation in $E_A$, $R$ be the reduced language of $A$. First, prove that $\sigma \rightarrow^* w$ implies $w \in R$. Indeed, no production from $P$ can produce a word contradicting requirement (1), hence (1) holds for all derived words. It proves $\mathcal{L}(E_A) \subseteq R$. To prove $R \subseteq \mathcal{L}(E_A)$, let $w \in R$. If $w = \epsilon, \sigma \rightarrow^* w$ by a single termination production $(k) \rightarrow \epsilon$. Let $wa \in R$. Let then $w \in R$ and assume as induction hypothesis $\sigma \rightarrow^* w$. If $a \in T_p, then w \rightarrow wa$, hence $\sigma \rightarrow^* wa$. If $b \in T_m$ and $wb \in R$ proves $F^*(w) \leq |A|$. But then, by an appropriate production of $E_A, w \rightarrow wb$ which implies $\sigma \rightarrow^* wb$. It proves $\sigma \rightarrow^* w$ for all $w \in R$, hence $R \subseteq \mathcal{L}(E_A)$. It completes the proof. □

**Proposition 7.** Reduced grammars for atomic nets are regular.

**Proof.** It is obvious in view of the Definition 4. □

Observe that

$$
V - T = \{(k) | \exists w \in R_A : k = F^*(w) \leq |A|\}
$$

which means that values $k$ of variables $(k) \in V$ (states of the automaton corresponding to grammar $E_A$) in a course of derivation $\sigma \rightarrow^* w \in R_A$ are integers from $F^*(w)$.

**Theorem 2.** Reduced languages of atomic nets are sequential, regular, and prefix closed.

**Proof.** It follows from the Propositions 5 and 7. □
3.3. Commutative behavior of atomic nets

Algebra of multisets. Let $T$ be a finite set of symbols. Mapping $m : T \rightarrow \mathbb{N}$ is a multiset (or a linear form) over $T$. For multiset $m$ over $T$ and $a \in T$ write $m_a$ for $m(a)$. Symbol $a$ with $m_a > 0$ is an element of multiset $m$. Symbol $T^0$ denotes the set of all multisets over $T$, including the empty multiset $\emptyset$. $T^+$ with addition $m + m'$ of $m', m'' \in Z^T$ defined by $m_a = m'_a + m''_a$ and neutral element $0 \in T^+$ defined by $0_a = 0$ for all $a \in T$ forms a free commutative monoid. This monoid is generated by unit multisets $a$ defined for all $a \in T$ by $a_a = 1$ and $a_b = 0$ for all $a \neq b \in T$ (unit multiset $a$ is usually identified with its single element $a$, similarly as words with one symbol are identified with that symbol). Number $\sum_{a \in T} m(a)$ (called the length of $m$) is denoted by $|m|$.

Example 4. Let $T = \{a, b, c\}$; then multiset $m : T \rightarrow \mathbb{N}$ such that $t_a = 2, t_b = 0, t_c = 1$ is denoted by $2a + c$, as sum $a + a + c$ of unit multisets $a, a, c$; length of $t$ is 3. □

Algebra $(T^+, +, 0)$ of multisets is free in the family of all commutative monoids over $T$ and $T$ is its set of generators. Subsets of $T^+$ will be called commutative languages over $T$, or simply languages, if its commutativity follows from the context.

Language $M \subseteq T^+$ is connected, if

$\forall m \in M \Rightarrow m = 0 \lor \exists a \in T, \ m' \in M : m = m' + a.$

Example 5. Language $\{k \cdot a + n \cdot b | k \geq n \geq 0\}$ is connected, while $\{k \cdot a + n \cdot b | k = n \geq 0\}$ is not. □

For $M', M'' \subseteq T^+$ the sum of $M', M''$ is defined as $M' + M'' = \{m' + m'' | m' \in M', m'' \in M''\}$. For any commutative language $M \subseteq T^+$, language $M \cdot 0 = \{0\}$, and $M \cdot (n + 1) = (M \cdot n) + M$, for all $n \in \mathbb{N}$; then the closure of $M$ is defined as $M^0 = \bigcup_{n=0}^{\infty} M \cdot n$. Mapping $\gamma : T^* \rightarrow T^+$ is the commutation homomorphism, if

$\gamma(\epsilon) = 0,$

$\gamma(a) = a, \text{ for all } a \in T,$

$\gamma(w'w'') = \gamma(w') + \gamma(w''), \text{ for all } w', w'' \in T^*.$

Commutative language $M \subseteq T^+$ is regular, if there is regular language $L \subseteq T^*$ s.t. $\gamma(L) = M$. Extended now valuation $F : T \rightarrow \mathbb{Z}$ defined for atomic nets to $F : T^* \rightarrow \mathbb{Z}$ by setting $F(0) = 0, F(w_1 + w_2) = F(w_1) + F(w_2)$ for all $w_1, w_2 \in T^+$.

Call the equivalence induced by commutation homomorphism $\gamma$, i.e. the equivalence $\equiv$ such that

$w' \equiv w'' \Leftrightarrow \gamma(w') = \gamma(w'')$

the commutation equivalence. This equivalence is known sometimes as Parikh equivalence. From the above definition it follows that equivalence classes of the commutation equivalence are multisets over $T$. We are going to show that commutative languages of atomic nets are regular, by proving that commutative language for $A$ is the image of regular language $\bar{A}$ under the commutation homomorphism.

Proposition 8. Let $\gamma : T^* \rightarrow T^+$ be commutation homomorphism, $L \subseteq T^*$. Then

1. $L$ is prefix closed $\Rightarrow \gamma(L)$ is connected;
2. $|w| = |\gamma(w)|$ for all $w \in T^*$, (\gamma length preserving);
3. $F^+(w) = F^+(\gamma(w))$ (\gamma valuation preserving).

Proof. Statement 1 follows directly from the definition of connectedness, since from the implication $wa \in L \Rightarrow w \in L$ it follows implication $\gamma(w) + a \in \gamma(L) \Rightarrow \gamma(w) \in \gamma(L)$, which is a condition for connectedness of $\gamma(L)$. Statements 2 and 3 follow from the definition of $\gamma$ and of $F$. □

Theorem 3. Let $A$ be an atomic net, $L$ be the sequential language of $A$, $R$ be the reduced language of $A$. Then $L \equiv R$, i.e. $\gamma(L) = \gamma(R)$.

Proof. Since $R \subseteq L$, it suffices to prove that for any $w \in L$ there exists $u \in R$ such that $w \equiv u$; if so, then clearly $\gamma(w) = \gamma(u)$ for each $w \in L$, $u \in R$. The proof of this claim will be carried out by induction. Let $w \in L$. If $w = \epsilon$, $w \equiv \epsilon \in R$. If $w$ is a one symbol word in $L$, it is clearly $w \in R$. Let now $w = w't_1t_2 \in L$ and assume, as induction hypothesis, that there exists $u \in R$ with $\equiv w't_1 \in R$, i.e. such that $\gamma(u) = \gamma(w't_1)$. It implies $F^+(u) = F^+(w't_1)$. If $F^+(u) < |A|$ or $t_2 \equiv w't_2 \in R$. Then $F^+(w't_1) > |A|$ and $t_2 = b \in T_m$, in such a situation it must be $t_1 = a \in T_p$, since otherwise it would be $F^+(w') > |A|$ and $w't_1 \equiv u \in R$ would be contradicting to (1). Thus, $w'ab \in L, F(a) > 0, F(b) < 0$, and $F^+(w'a) > |A|$. From this it follows that $w'ba \in L$, since then the value of $F^+(w') > |A| - F(a)$ is sufficiently large to guarantee $F^+(w') + F(b) \geq 0$. Thus, $w'ba \in L$ and $w'ba \equiv w'ab \in L$. By induction hypothesis and (1) we have $w'ba \equiv w'ab \equiv ub \in R$, hence $w'ba$, as equivalent to $ub \in R$, is in $R$ either. By induction, we infer $I \equiv R$. □

Theorem 4. Commutative languages of atomic nets are regular and connected.

Proof. Let $A$ be atomic net and $L$ be the language of $A$. By Proposition 3 there is reduced language $R \subseteq L$ such that $R \equiv L$. By Proposition 7 grammar $E_R$ is regular, hence set $R$ is regular. Therefore $\gamma(L)$ is regular too. Connectedness of $\gamma(L)$ follows from prefix closedness of $L$. □
Example 6. The commutative language for atomic net $X$ defined in Example 1 is:

$$M = \{ m \in \{ a, b, c \}^\mathbb{N} \mid 2m_a - 3m_b - m_c + 3 \geq 0 \}$$

and e.g. $\gamma(aabacaabb) = 5a + 4b + c \in M$. □

Theorem 5. In atomic nets, reachability by sequential languages is equivalent to reachability by reduced languages.

Proof. Let $L$ be the sequential language of atomic net $A = (T, F, m^0)$, $R$ be its reduced language; $F^*(L) = F^*(\gamma(L))$ by Proposition 8; $F^*(\gamma(L)) = F^*(\gamma(R))$ by Theorem 3; again by Proposition 8 $F^*(\gamma(R)) = F^*(R)$ which proves $F^*(L) = F^*(R)$. □

The following diagram represents basic relationships of languages discussed so far:

<table>
<thead>
<tr>
<th>Reduced language</th>
<th>Execution language</th>
</tr>
</thead>
<tbody>
<tr>
<td>regular, sequential</td>
<td>context free, sequential</td>
</tr>
<tr>
<td>$\gamma \uparrow$</td>
<td>$\gamma \downarrow$</td>
</tr>
</tbody>
</table>

Commutative language, regular, non-sequential

3.4. Reachability in atomic nets

Let $A = (T, F, m^0)$ be atomic net fixed for the rest of this section, and $L$ be the sequential language of $A$. Say that $k \in \mathbb{N}$ is reachable by $A$ in $m$ steps, if

$$\exists w \in L : |w| \leq m, \quad k = F^*(w).$$

Say that $w \in T^*$ is univalent, if for any prefixes $u, v \leq w$

$$F^*(u) = F^*(v) \Rightarrow u = v.$$

Lemma 2. For any word $w \in L$ there is a univalent word $u \in L$ such that $F^*(w) = F^*(u)$ and $|u| \leq |w|$. 

Proof. Let $F^*(w_1uw_2) = k, F^*(w_1) = F^*(w_1u)$. Then $F(w_1w_2) = k$ and $|w_1w_2| \leq |w_1uw_2|$. Thus, word $u$ can be removed from $w_1uw_2$ keeping the value of $F^*(w_1uw_2)$ unchanged. Clearly, $w_1w_2 \in L$. The proof is completed by repeating this procedure. □

Lemma 3. Let $w \in T^*$ be a univalent word, $F^*(u) \leq k$ for all $u \leq w$. Then $|w| \leq k$.

Proof. Assume $F^*(w) \leq k, \forall w \in T^*$ with univalent $w$. If it were $|w| > k$, then by Dirichlet principle (known also as pigeon holes principle) there would be at least two different prefixes $u \leq w, v \leq w$ with $F^*(u) = F^*(v)$, contradicting univalency of $w$. □

Proposition 9. For any $w \in L$ with $F^*(w) = k \geq 0$ there is $u \in L$ with $F^*(u) = k$ and $|u| \leq |A| + k$.

Proof. Assume $w \in L, F^*(w) = k \geq 0$. Then there is $w' \in R$ such that $F^*(w') = k \geq 0$. As it follows from Lemma 2 there is a univalent word $u \in R$ such that $F^*(u) = k$. By the definition of reduced language $R$ the length of univalent word $u'$ such that $F^*(u') \leq |A|$ is not greater than $|A|$, and the length of univalent word $u'' \in T^*_p$ such that $0 \leq F^*(u'') \leq k$ is not greater than $k$. Since $|u| \leq |u'| + |u''|$, we get $|u| \leq |A| + k$. Since $u \in R \subseteq L$, the proof is completed. □

Theorem 6. Reachability in atomic nets is decidable.

Proof. Let $A$ be atomic net, $w \in L$, and $F^*(w) = k \geq 0$. Then by Theorem 3 there is $u \in R$ (hence $u \in L$, since $R \subseteq L$) such that $F^*(u) = k$. By Lemma 2 there is univalent word $u' \in L$ such that $F^*(u') = k$. By Proposition 9 $|u'| \leq |A| + k$. It means that in order to check whether $k$ is reachable by $A$ one has to check a finite number of words $w$ (namely, all words $w \in L$ with length not greater than $|A| + k$). If for some $w$ out of them $F^*(w) = k$, $k$ is reachable; if not, $F^*(w) \neq k$ for all $w \in L$, since otherwise $F^*(w) = k$ would contradicts to the result of Proposition 9. Thus, reachability of $k$ by an atomic net $A$ can be proved by checking a finite number of words. It completes the proof. □
4. General Petri nets

In this section we lift the results achieved for atomic nets to the general ones, known widely as Place/Transition nets, or P/T nets. First, some necessary definitions are given. Next, the compositionality principle will be formulated that makes possible to discuss properties of complex nets by means of their atoms. Finally, by combined properties of projection and commutation homomorphisms, we prove basic facts concerning general nets, namely language types and reachability property.

Definition 5. By a general net we shall understand any quadruple \( N = (P, T, F, m^0) \), where

\[
P, T, \text{ finite non-empty sets,} \\
F : P \times T \rightarrow \mathbb{Z}, \\
m^0 : P \rightarrow \mathbb{N};
\]

\( P, T \) are referred to as sets of places and transitions of \( N \), respectively, any vector \( m : P \rightarrow \mathbb{N} \) is a marking of \( N \), and \( m^0 \) is the initial marking of \( N \). It is assumed that for each \( t \in T \) there is \( p \in P, k \in \mathbb{Z} \) with \( F(p, t) = k \) (there is no isolated transitions).

Let \( M = \{ m \mid m : P \rightarrow \mathbb{N} \} \). Relation \( \rightarrow \subseteq M \times M \) is defined for all \( t \in T \) and \( m', m'' \in M \) by the equivalence:

\[
m' \rightarrow m'' \iff \forall p \in P : m''(p) = m'(p) + F(p, t)
\]

called the one step relation of net \( N \). Observe that \( m' \rightarrow m'' \) implies \( m'(p) + F(p, t) \geq 0 \) for any \( p \in P \). Extend relation \( \rightarrow \) with \( t \in T \) to \( \rightarrow^* \) with \( w \in T^* \) in the standard way:

\[
m' \rightarrow^* m'' \iff \begin{cases} \mbox{if } w = \epsilon, \\ \mbox{if } w = t \in T, \\ \exists m : m' \xrightarrow{w}^* m \xrightarrow{t} m'' \end{cases} \mbox{, if } w = uv, u, v \in T^*. 
\]

Then

\[ L_N = \{ w \mid \exists m : m^0 \xrightarrow{w}^* m \}, \quad F^*(L_N) = \{ m \mid \exists w : m^0 \xrightarrow{w}^* m \} \]

are called respectively the sequential language and reachability set of net \( N \). Elements of \( L(N) \) are called, traditionally, firing sequences of \( N \).  

Composition of nets. Let \( N = (P, T, F, m^0) \) be a general net and let for each \( p \in P \) atomic net \( A_p = (T, F_p, m^0_p) \) be such that

\[
F_p(t) = F(p, t), \quad m^0_p = m^0(p);
\]

then say that \( N \) is the composition of atomic nets \( A_p \) for \( p \in P \). From now on let general net \( N = (P, T, F, m^0) \) be the composition of atomic nets \( A_p = (T, F_p, m^0_p) \) defined as above and be fixed for the rest of the paper. Note that all atomic nets that are composed into a single general net have a common set of transitions; such a decomposition is always possible since transitions with zero valuations are accepted in the definition of atomic nets.

The following proposition, given in [2], offers a useful tool for proving properties of general nets, first decomposing them into atoms, proving their individual properties, and next composing the results to get global properties of general nets in question.

Proposition 10. Let \( L, L_p \) be the sequential languages of \( N, A_p \), respectively. Then

\[
L = \bigcap_{p \in P} L_p.
\]

(language of the composition of atoms is the intersection of their languages).

Proof. Detailed proof is given in [2], and also in the book [1]. Here, only a sketch of the proof is presented. By definitions of execution sequences in atomic nets and of execution sequences in general nets, we have for all \( m', m'' \in M \) and any \( t \in T \):

\[
\forall p \in P : m'(p) + F_p(t) = m''(p) \iff m' \xrightarrow{t}_N m''
\]

where \( F_p(t) = F(p, t) \). From this, by induction, we infer that for any \( w \in T^* \) and any \( m \) we have

\[
\forall p \in P : m^0_p(w) = m(p) \iff m^0 \xrightarrow{w}_N m,
\]

which means

\[
w \in L \iff \forall p \in P : w \in L_p.
\]

It proves the validity of the Proposition.  

From Proposition 10 we get the following property of general nets.

**Theorem 7.** Sequential languages of general nets are intersections of a finite number of context free languages.

**Proof.** It is a direct consequence of Theorem 1 and Proposition 10. □

**Commutative behavior of general nets.** As in the case of atomic nets, commutation homomorphism will play a substantial role in this part of the paper. Let \( N = (P,T,F,m^0) \) be a general net, composed of atomic nets \( A_p = (T_p,F_p,m^0_p) \) for all \( p \in P \). Let \( \gamma : T^* \rightarrow T^\oplus \) be the commutation homomorphism and \( L \) be the sequential language of general net \( N \), \( L_p \) be the sequential language of \( A_p \) for all \( p \in P \).

**Definition 6.** \( \gamma(L) \) is the commutative language of general net \( N \). □

Observe that the commutative language of any net, as the homomorphic image of a prefix closed language, is a connected set of multisets.

**Proposition 11.**

\[ \gamma(L) = \bigcap_{p \in P} \gamma(L_p). \]

**Proof.** Recall that \( L, L_p \) are prefix closed and consequently, \( \gamma(L), \gamma(L_p) \) are connected; it enables proof of equivalence \( \gamma(u) \in \gamma(L) \iff \gamma(u) \in \bigcap_{p \in P} \gamma(L_p) \) by induction w.r. to \( |u| \). The above equivalence is obvious for \( u = \varepsilon \). Let \( u = wt \in L, w \in L, t \in T \). By induction hypothesis \( \gamma(w) \in \gamma(L) \iff \gamma(w) \in \bigcap_{p \in P} \gamma(L_p) \). Then \( \gamma(wt) \in \gamma(L) \iff \gamma(wt) \in \bigcap_{p \in P} \gamma(L_p) \), which proves the required equivalence. By induction, equivalence \( \gamma(u) \in \gamma(L) \iff \gamma(u) \in \bigcap_{p \in P} \gamma(L_p) \) holds for all \( u \in L \). It implies \( \gamma(L) = \bigcap_{p \in P} \gamma(L_p) \). □

**Theorem 8.** Commutative languages of general nets are regular.

**Proof.** Let \( R_p \) be the reduced language of \( A_p \), for all \( p \in P \). By Theorem 2 \( R_p \) is regular for each \( p \in P \). Then \( \gamma(R_p) \) is regular for each \( p \in P \), as the image of a regular language. Since by Theorem 3 \( \gamma(R_p) = \gamma(L_p) \), language \( \gamma(L_p) \) is regular for each \( p \in P \). Since intersection of regular languages is regular, by Proposition 11 \( \gamma(L) \) is regular as intersection of \( \gamma(L_p) \) for \( p \in P \). □

To sum up the results of the above discussion on general nets, we proved that languages of general nets are intersections of context free languages (hence not context free), sequential, and prefix closed. Nevertheless, their images under commutation homomorphism are clearly non-sequential, connected, and regular. Here we call regular any subset of any monoid which is the homomorphic image of a regular subset of a free monoid (of words over an alphabet).

5. Conclusions

In the paper Petri nets behavior has been discussed. At the beginning atomic nets, i.e. nets with a single place only, were considered. It has been shown that the firing (execution) sequences of such nets form a context free languages. Next, the so-called commutative languages, arising from sequential ones by applying the commutation homomorphism from monoid of words to monoid of multisets (linear forms), both defined on the same alphabets. It gives rise to an alternative way of description of nets, in which some of words with irrelevant order of symbol occurrences are identified, but reachability properties are preserved. It turns out that sequential behavior of nets is equivalent w.r. to the commutation homomorphism to its regular subset. This subset, called the reduced language of the net, is then used for proving the decidability of reachability in atomic nets. Thus, three types of behavior description of atomic nets are discussed: sequential one, commutative one, and reduced one. All of them serve as tools for proving context sensitiveness of the sequential behavior, regularity of commutative behavior, and decidability of reachability in atomic nets. Then these results are lifted to the general case of place/transition nets. It has been proved that the sequential behavior of such nets is the intersection of finite number of context free languages, and that the commutative behavior of general nets is regular.

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