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Linear Algebra and its Applications 387 (2004) 361–368

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Jordan maps of nest algebras

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Received 4 November 2003; accepted 23 February 2004

Submitted by C.-K. Li

Abstract

In this paper, we prove that a bijective map ϕ from $\mathscr{A}$, a standard subalgebra of a nest algebra on a Hilbert space, onto an algebra that satisfies

$$\phi(r(AB + BA)) = r(\phi(A)\phi(B) + \phi(B)\phi(A)) \quad (A, B \in \mathscr{A}),$$

where r is a fixed nonzero rational number, is additive.

AMS classification: 47L35; 47L20; 46H10

Keywords: Jordan maps; Standard subalgebras; Additivity; Standard arguments

Let $\mathscr{A}$ and $\mathscr{B}$ be associative algebras over the field $\mathbb{Q}$ of rational numbers. Let r be a rational number. If a bijective map $\phi : \mathscr{A} \to \mathscr{B}$ satisfies

$$\phi(r(AB + BA)) = r(\phi(A)\phi(B) + \phi(B)\phi(A))$$

for all $A, B \in \mathscr{A}$, we call it a r-Jordan map. In [3,4], Hakeda and Saitǒ proved that every $\frac{1}{2}$-Jordan map ϕ from a unital C*-algebra with a system of matrix units onto a *-algebra satisfying $\phi(a^*) = \phi(a)^*$ is additive. In [9], Molnár showed that every $\frac{1}{2}$-Jordan map between standard operator algebras is additive. In the theory of operator algebras the Jordan product of A and B is usually defined by $A \circ B = \frac{1}{2}(AB + BA)$ while in ring theory the definition is modified to $A \circ B = (AB + BA)$ because of obvious reasons. Therefore, from the ring theoretical point of view, it is more natural to consider 1-Jordan maps than $\frac{1}{2}$-Jordan maps. Molnár asked in [9]: "Is every $\frac{1}{2}$-Jordan map between standard operator algebras additive?" This question

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doi:10.1016/j.laa.2004.02.031

was affirmatively answered in [6]. Recently, it was shown in [7] that all results in [3,4,9] are also true for all r-Jordan maps.

Among all papers mentioned above algebras considered contain idempotents. It is our aim in this paper to find a class of operator algebras of which any Jordan map is additive but in which there are no idempotents.

Let $\mathscr{H}$ be a Hilbert space. Denote by $B(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$ and by I the identity operator on $\mathscr{H}$. A chain $\mathscr{N}$ of projections on $\mathscr{H}$ is called a nest if it contains 0 and I and it is closed in the strong operator topology. The nest algebra denoted by $\mathscr{T}(\mathscr{N})$ corresponding to the $\mathscr{N}$ is defined by

$$\mathscr{T}(\mathscr{N}) = \{T \in B(\mathscr{H}) : TE = ETE \text{ for all } E \in \mathscr{N}\}.$$

If $\mathscr{N}$ is trivial, i.e. $\mathscr{N} = \{0, I\}$, then $\mathscr{T}(\mathscr{N}) = B(\mathscr{H})$. The algebra of all finite rank operators in $\mathscr{T}(\mathscr{N})$ is denoted by $\mathscr{F}(\mathscr{N})$. The Erdos Density Theorem says that $\mathscr{F}(\mathscr{N})$ is dense in $\mathscr{T}(\mathscr{N})$ under the strong operator topology [2]. A subalgebra of $\mathscr{T}(\mathscr{N})$ is called a standard subalgebra if it contains $\mathscr{F}(\mathscr{N})$. In particular, $\mathscr{F}(\mathscr{N})$ is a standard subalgebra. If $x, y \in \mathscr{H}$, then the rank one operator $x \otimes y$ is defined by

$$(x \otimes y)z = (z, y)x, \ (z \in \mathscr{H}).$$

It is well known that $x \otimes y$ belongs to $\mathscr{T}(\mathscr{N})$ if and only if there is an element $E \in \mathscr{N}$ such that $x \in E\mathscr{H}$ and $y \in (I - E)\mathscr{H}$. For more information on nest algebras, we refer to [1].

The main result in this paper is the following.

Theorem. *Let $\mathbb{F}$ be the real number field or the complex number field. Let $\mathscr{A}$ be a standard subalgebra of a nest algebra $\mathscr{T}(\mathscr{N})$ on a Hilbert space $\mathscr{H}$ over $\mathbb{F}$ of dimension greater than* 1 *and $\mathscr{R}$ be an algebra over the field of rational numbers. Let r be a nonzero rational number. Suppose $\phi : \mathscr{A} \to \mathscr{R}$ is a bijective map satisfying*

$$\phi(r(AB + BA)) = r(\phi(A)\phi(B) + \phi(B)\phi(A))$$

for all $A, B \in \mathscr{A}$. Then ϕ is additive.

The main technique we will use is the following argument which will be termed a "standard argument". Suppose $A, B, S \in \mathscr{A}$ are such that $\phi(S) = \phi(A) + \phi(B)$. Multiplying this equality by $\phi(T)(T \in \mathscr{A})$ from the left and the right respectively, we get $\phi(T)\phi(S) = \phi(T)\phi(A) + \phi(T)\phi(B)$ and $\phi(S)\phi(T) = \phi(A)\phi(T) + \phi(B)\phi(T)$. Summing them, we have that

$$\phi(T)\phi(S)+\phi(S)\phi(T)=\phi(T)\phi(A)+\phi(A)\phi(T)+\phi(T)\phi(B)+\phi(B)\phi(T).$$

Since ϕ is a r-Jordan map, it follows that

$$\phi(r(ST + TS)) = \phi(r(AT + TA)) + \phi(r(BT + TB)).$$

Moreover, if $\phi(r(AT + TA)) + \phi(r(BT + TB)) = \phi(r(AT + TA + BT + TB))$, then by injectivity of ϕ, we have that $ST + TS = AT + TA + BT + TB$.

The proof is purely algebraic and will be organized in a series of lemmas. We begin with the following trivial one.

Lemma 1. $\phi(0) = 0$.

Proof. Since ϕ is surjective, then we can find an $A \in \mathscr{A}$ such that $\phi(A) = 0$. Therefore $\phi(0) = \phi(r(0A + A0)) = r(\phi(0)\phi(A) + \phi(A)\phi(0)) = 0$. $\square$

In the next several lemmas we will assume that $\mathscr{N} \neq \{0, I\}$. Fix an element E in $\mathscr{N}$ with $0 < E < I$. For the sake of simplicity, we write $\mathscr{B} = F(\mathscr{N})$ and $\mathscr{T} = \mathscr{T}(\mathscr{N})$. We borrow the idea of [8]. Set $\mathscr{B}_{11} = E\mathscr{B}E$, $\mathscr{B}_{12} = E\mathscr{B}(I - E)$ and $\mathscr{B}_{22} = (I - E)\mathscr{B}(I - E)$. Then we can write $\mathscr{B} = \mathscr{B}_{11} \oplus \mathscr{B}_{12} \oplus \mathscr{B}_{22}$ since $\mathscr{B}$ is an ideal of $\mathscr{T}$. Similarly, we write $\mathscr{T} = \mathscr{T}_{11} \oplus \mathscr{T}_{12} \oplus \mathscr{T}_{22}$. The following lemma can be found in [5].

Lemma 2. *With the notation as above, we have that*

(i) *If $T_{11} \in \mathscr{T}_{11}$ is such that $T_{11}\mathscr{B}_{12} = 0$, then $T_{11} = 0$;*
(ii) *If $T_{22} \in \mathscr{T}_{22}$ is such that $\mathscr{B}_{12}T_{22} = 0$, then $T_{22} = 0$;*
(iii) *If $T_{12} \in \mathscr{T}_{12}$ is such that $\mathscr{B}_{11}T_{12} = 0$, then $T_{12} = 0$;*
(iv) *If $T_{12} \in \mathscr{T}_{12}$ is such that $T_{12}\mathscr{B}_{22} = 0$, then $T_{12} = 0$.*

Lemma 3. *Let $i \in \{1, 2\}$ and $T_{ii} \in \mathscr{T}_{ii}$. If $T_{ii}\mathscr{B}_{ii} = 0$ or $\mathscr{B}_{ii}T_{ii} = 0$, then $T_{ii} = 0$.*

Proof. By the Erdos Density Theorem, there exists a net $\{F_\alpha\}$ of finite rank operators in $\mathscr{T}$ such that SOT-$\lim_\alpha F_\alpha = I$. Set $E_1 = E$ and $E_2 = I - E$. Then we have that $T_{ii}E_iF_\alpha E_i = 0$ or $E_iF_\alpha E_iT_{ii} = 0$. Note that $T_{ii} = E_iT_{ii} = T_{ii}E_i$. Taking the limit, we get that $T_{ii}E_i = 0$ or $E_iT_{ii} = 0$, that is $T_{ii} = 0$. $\square$

Lemma 4. *Suppose that $S = S_{11} + S_{12} + S_{22} \in \mathscr{T}$.*

(i) *If $ST_{11} + T_{11}S = 0$ for every $T_{11} \in \mathscr{B}_{11}$, then $S_{11} = S_{12} = 0$.*
(ii) *If $ST_{22} + T_{22}S = 0$ for every $T_{22} \in \mathscr{B}_{22}$, then $S_{12} = S_{22} = 0$.*

Proof. (i) Arguing as in the proof of Lemma 3, we get that $SE + ES = 0$. It follows that $2S_{11} = E(SE + ES)E = 0$ and $S_{12} = (SE + ES)(I - E) = 0$. (ii) can be proved similarly. $\square$

Lemma 5. *Let $B_{ij} \in \mathscr{B}_{ij}$, $1 \leqslant i \leqslant j \leqslant 2$. Then $\phi(B_{11} + B_{12}) = \phi(B_{11}) + \phi(B_{12})$ and $\phi(B_{22} + B_{12}) = \phi(B_{22}) + \phi(B_{12})$.*

Proof. Since ϕ is surjective, we may find an element $S = S_{11} + S_{12} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(B_{11}) + \phi(B_{12}). \tag{1}$$

For $T_{22} \in \mathscr{B}_{22}$, applying a standard argument to (1), we have that

$$\begin{aligned}\phi(r(T_{22}S + ST_{22})) &= \phi(r(T_{22}B_{11} + B_{11}T_{22})) + \phi(r(T_{22}B_{12} + B_{12}T_{22})) \\ &= \phi(0) + \phi(rB_{12}T_{22}) = \phi(rB_{12}T_{22}).\end{aligned}$$

It follows that $T_{22}S + ST_{22} = B_{12}T_{22}$. Hence $T_{22}(S - B_{12}) + (S - B_{12})T_{22} = 0$ for every $T_{22} \in \mathscr{B}_{22}$. Thus by Lemma 4(ii), we get $S_{22} = 0$, $S_{12} = B_{12}$.

For $T_{12} \in \mathscr{B}_{12}$, applying a standard argument to (1), we get

$$\begin{aligned}\phi(r(T_{12}S + ST_{12})) &= \phi(r(T_{12}B_{11} + B_{11}T_{12})) + \phi(r(T_{12}B_{12} + B_{12}T_{12})) \\ &= \phi(rB_{11}T_{12}) + \phi(0) \\ &= \phi(rB_{11}T_{12}).\end{aligned}$$

Therefore, $T_{12}S + ST_{12} = B_{11}T_{12}$ for every $T_{12} \in \mathscr{B}_{12}$. Equivalently, $T_{12}S_{22} + S_{11}T_{12} = B_{11}T_{12}$ for all $T_{12} \in \mathscr{B}_{12}$. But we have shown that $S_{22} = 0$, it follows that $S_{11}T_{12} = B_{11}T_{12}$ for all $T_{12} \in \mathscr{B}_{12}$. Thus by Lemma 2(i), $S_{11} = B_{11}$. Consequently, $S = B_{11} + B_{12}$. This proves the first equality. The second can be proved similarly. □

The next goal is to prove that ϕ is additive on $\mathscr{B}_{12}$. If $E \in \mathscr{B}$, this can be shown easily. But E does not necessarily belongs to $\mathscr{B}$. The following lemma bridges this gap.

Lemma 6. *Let $T_{11} \in \mathscr{B}_{11},\ P_{12}, Q_{12} \in \mathscr{B}_{12}, S_{22} \in \mathscr{B}_{22}$. Then*

$$\phi(T_{11}P_{12} + Q_{12}S_{22}) = \phi(T_{11}P_{12}) + \phi(Q_{12}S_{22}).$$

Proof. Compute

$$\begin{aligned}T_{11}P_{12} + Q_{12}S_{22} &= (T_{11} + Q_{12})(P_{12} + S_{22}) \\ &= (T_{11} + Q_{12})(P_{12} + S_{22}) + (P_{12} + S_{22})(T_{11} + Q_{12}).\end{aligned}$$

Thus, we have that

$$\begin{aligned}&\phi(T_{11}P_{12} + Q_{12}S_{22}) \\ &\quad = \phi\left(r\left(\frac{1}{r}(T_{11}P_{12} + Q_{12}S_{22})\right)\right) \\ &\quad = \phi\left(r\left(\left(\frac{1}{r}T_{11} + \frac{1}{r}Q_{12}\right)(P_{12} + S_{22}) + (P_{12} + S_{22})\left(\frac{1}{r}T_{11} + \frac{1}{r}Q_{12}\right)\right)\right) \\ &\quad = r\phi\left(\frac{1}{r}T_{11} + \frac{1}{r}Q_{12}\right)\phi(P_{12} + S_{22}) + r\phi(P_{12} + S_{22})\phi\left(\frac{1}{r}T_{11} + \frac{1}{r}Q_{12}\right) \\ &\quad = r\left(\phi\left(\frac{1}{r}T_{11}\right) + \phi\left(\frac{1}{r}Q_{12}\right)\right)(\phi(P_{12}) + \phi(S_{22})) \\ &\qquad + r(\phi(P_{12}) + \phi(S_{22}))\left(\phi\left(\frac{1}{r}T_{11}\right) + \phi\left(\frac{1}{r}Q_{12}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= r\left(\phi\left(\frac{1}{r}T_{11}\right)\phi(P_{12}) + \phi(P_{12})\phi\left(\frac{1}{r}T_{11}\right)\right)\\
&\quad + r\left(\phi\left(\frac{1}{r}Q_{12}\right)\phi(S_{22}) + \phi(S_{22})\phi\left(\frac{1}{r}Q_{12}\right)\right)\\
&\quad + r\left(\phi\left(\frac{1}{r}T_{11}\right)\phi(S_{22}) + \phi(S_{22})\phi\left(\frac{1}{r}T_{11}\right)\right)\\
&\quad + r\left(\phi\left(\frac{1}{r}Q_{12}\right)\phi(P_{12}) + \phi(P_{12})\phi\left(\frac{1}{r}Q_{12}\right)\right)\\
&= \phi\left(r\left(\left(\frac{1}{r}T_{11}\right)P_{12} + P_{12}\left(\frac{1}{r}T_{11}\right)\right)\right)\\
&\quad + \phi\left(r\left(\left(\frac{1}{r}Q_{12}\right)S_{22} + S_{22}\left(\frac{1}{r}Q_{12}\right)\right)\right)\\
&\quad + \phi\left(r\left(\left(\frac{1}{r}T_{11}\right)S_{22} + S_{22}\left(\frac{1}{r}T_{11}\right)\right)\right)\\
&\quad + \phi\left(r\left(\left(\frac{1}{r}Q_{12}\right)P_{12} + P_{12}\left(\frac{1}{r}Q_{12}\right)\right)\right)\\
&= \phi(T_{11}P_{12}) + \phi(Q_{12}S_{22}). \qquad \square
\end{aligned}
$$

Lemma 7. *ϕ is additive on $\mathscr{B}_{12}$.*

Proof. Let $A_{12}, B_{12} \in \mathscr{B}_{12}$ and choose $S = S_{11} + S_{12} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{12}) + \phi(B_{12}). \tag{2}$$

For $T_{22} \in \mathscr{B}_{22}$, applying a standard argument to (2) we get that

$$\phi(r(T_{22}S + ST_{22})) = \phi(rA_{12}T_{22}) + \phi(rB_{12}T_{22}). \tag{3}$$

For $T_{11} \in \mathscr{B}_{11}$, applying a standard argument to the equation (3), we have that

$$
\begin{aligned}
&\phi(r^2((T_{22}S + ST_{22})T_{11} + T_{11}(T_{22}S + ST_{22})))\\
&\quad = \phi((r^2T_{11}A_{12})T_{22}) + \phi(T_{11}(r^2B_{12}T_{22}))\\
&\quad = \phi((r^2T_{11}A_{12})T_{22} + T_{11}(r^2B_{12}T_{22}))
\end{aligned}
$$

making use of Lemma 6, from which we have that

$$(T_{22}S + ST_{22})T_{11} + T_{11}(T_{22}S + ST_{22}) = T_{11}(A_{12} + B_{12})T_{22}$$

and hence

$$T_{11}S_{12}T_{22} = T_{11}(A_{12} + B_{12})T_{22}.$$

Thus by Lemma 2, we conclude that $S_{12} = A_{12} + B_{12}$.

For $T_{12} \in \mathscr{B}_{12}$, applying a standard argument to (3), we get that

$$\phi(r^2(T_{12}(T_{22}S + ST_{22}) + (T_{22}S + ST_{22})T_{12})) = 0.$$

It follows that $T_{12}(T_{22}S + ST_{22}) + (T_{22}S + ST_{22})T_{12} = 0$. Hence, $T_{12}(T_{22}S_{22} + S_{22}T_{22}) = 0$ for all $T_{12} \in \mathscr{B}_{12}$ and $T_{22} \in \mathscr{B}_{22}$. Thus by Lemma 2(ii) and Lemma 4(ii), $S_{22} = 0$.

For $T_{12} \in \mathscr{B}_{12}$, applying a standard argument to (2), we get that $\phi(r(T_{12}S + ST_{12})) = 0$, and so $T_{12}S + ST_{12} = 0$, from which we get that $T_{12}S_{22} + S_{11}T_{12} = 0$ for every $T_{12} \in \mathscr{B}_{12}$. It follows from the fact $S_{22} = 0$ and Lemma 2(i) that $S_{11} = 0$.

Consequently, $S = A_{12} + B_{12}$. □

Lemma 8. *ϕ is additive on $\mathscr{B}_{11}$.*

Proof. Let $A_{11}, B_{11} \in \mathscr{B}_{11}$ and choose $S = S_{11} + S_{12} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{11}) + \phi(B_{11}). \tag{4}$$

For $T_{22} \in \mathscr{B}_{22}$, by a standard argument to (4), we have that $T_{22}S + ST_{22} = 0$. It follows from Lemma 4(ii) that $S_{22} = S_{12} = 0$.

Now there remains to prove that $S_{11} = A_{11} + B_{11}$. For $T_{12} \in \mathscr{B}_{12}$, applying a standard argument to (4) again, we get

$$\phi(r(T_{12}S + ST_{12})) = \phi(rA_{11}T_{12}) + \phi(rB_{11}T_{12}).$$

Hence by Lemma 7, we have that

$$T_{12}S + ST_{12} = (A_{11} + B_{11})T_{12}$$

for every $T_{12} \in \mathscr{B}_{12}$. Since $S_{22} = S_{12} = 0$, it follows that $S_{11}T_{12} = (A_{11} + B_{11})T_{12}$ for every $T_{12} \in \mathscr{B}_{12}$. Therefore by Lemma 2(i) we have that $S_{11} = A_{11} + B_{11}$. □

Similarly, we can prove the following.

Lemma 9. *ϕ is additive on $\mathscr{B}_{22}$.*

Lemma 10. *Let $A_{ij} \in \mathscr{B}_{ij}$, $1 \leqslant i \leqslant j \leqslant 2$. Then $\phi(A_{11} + A_{12} + A_{22}) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{22})$.*

Proof. Choose $S = S_{11} + S_{12} + S_{22} \in \mathscr{A}$ such that $\phi(S) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{22})$. Then for $T_{11} \in \mathscr{B}_{11}$ we have that

$$\begin{aligned}
\phi(r(T_{11}S + ST_{11})) &= r\phi(T_{11})\phi(S) + r\phi(S)\phi(T_{11}) \\
&= r\phi(T_{11})(\phi(A_{11}) + \phi(A_{12}) + \phi(A_{22})) \\
&\quad + r(\phi(A_{11}) + \phi(A_{12}) + \phi(A_{22}))\phi(T_{11}) \\
&= \phi(r(T_{11}A_{11} + A_{11}T_{11})) + \phi(r(T_{11}A_{12} + A_{12}T_{11})) \\
&\quad + \phi(r(T_{11}A_{22} + A_{22}T_{11})) \\
&= \phi(r(T_{11}A_{11} + A_{11}T_{11})) + \phi(r(T_{11}A_{12})) \\
&= \phi(r(T_{11}A_{11} + A_{11}T_{11} + T_{11}A_{12}))
\end{aligned}$$

making use of Lemma 5 in the last equality. It follows that $ST_{11} + T_{11}S = A_{11}T_{11} + T_{11}A_{11} + T_{11}A_{12}$. Equivalently, $(S - A_{11} - A_{12})T_{11} + T_{11}(S - A_{11} - A_{12}) = 0$ for every $T_{11} \in \mathscr{B}_{11}$. Hence by Lemma 4(i), we get that $S_{11} = A_{11}$ and $S_{12} = A_{12}$. Similarly, we can prove that $S_{22} = A_{22}$. □

Lemma 11. *ϕ is additive on $\mathscr{B}$.*

Proof. Let $A = A_{11} + A_{12} + A_{22}$ and $B = B_{11} + B_{12} + B_{22}$ be in $\mathscr{A}$. Then Lemmas 7–10 are all used in seeing the equalities

$$\begin{aligned}\phi(A + B) &= \phi((A_{11} + B_{11}) + (A_{12} + B_{12}) + (A_{22} + B_{22}))\\ &= \phi(A_{11} + B_{11}) + \phi(A_{12} + B_{12}) + \phi(A_{22} + B_{22})\\ &= \phi(A_{11}) + \phi(B_{11}) + \phi(A_{12}) + \phi(B_{12}) + \phi(A_{22}) + \phi(B_{22})\\ &= \phi(A_{11} + A_{12} + A_{22}) + \phi(B_{11} + B_{12} + B_{22})\\ &= \phi(A) + \phi(B)\end{aligned}$$

hold true. That is, ϕ is additive on $\mathscr{B}$. □

We are now in a position to prove our main theorem.

Proof of Theorem. If $\mathscr{N} = \{0, I\}$, then $\mathscr{T}(\mathscr{N}) = B(\mathscr{H})$. Hence the result follows from [7, Theorem 1.6].

We now assume that $\mathscr{N} \neq \{0, I\}$. Then ϕ is additive on $\mathscr{B}$. Let $A, B, S \in \mathscr{A}$ such that $\phi(S) = \phi(A) + \phi(B)$. Let $T \in \mathscr{B}$ be arbitrary. Noting that $AT + TA$ and $BT + TB$ are both in $\mathscr{B}$, we get that $\phi(r(ST + TS)) = \phi(r(AT + TA)) + \phi(r(BT + TB)) = \phi(r(AT + TA + BT + TB))$. It follows that $(S - (A + B))T + T(S - (A + B)) = 0$ for all $T \in \mathscr{B}$. Hence by the Erdos Density Theorem, $2(S - (A + B)) = (S - (A + B))I + I(S - (A + B)) = 0$. Thus $S = A + B$. We are done. □

Acknowledgements

The authors would like to thank the referee for very thorough reading of the manuscript and many helpful comments.

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