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Inequalities for Differentiable Mappings with Application to Special Means and Quadrature Formulæ

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Abstract—Improvements are obtained to some recent error estimates of Dragomir and Agarwal, based on convexity, for the trapezoidal formula. Corresponding estimates are established for the midpoint formula. A parallel development is made based on concavity.

Keywords—Hermite-Hadamard inequality, Convex functions, Integral inequalities, Trapezoidal formula, Midpoint formula, Special means, Numerical integration.

1. INTRODUCTION

One of the cornerstones of analysis is the Hadamard inequality, which states that if $[a, b]$ $(a < b)$ is a real interval and $f : [a, b] \to R$ a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Over the last decade this has been extended in a number of ways. An important question is estimating the difference between the middle and rightmost terms in (1.1). The following identity is a useful building block (see [1,2]).

LEMMA A. *Suppose (a) $a, b \in I \subseteq R$ with $a < b$ and $f : I^0 \to R$ is differentiable.*
If $f' \in L(a, b)$, then

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt.$$

Dragomir and Agarwal [1] used this lemma to prove the following results.

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THEOREM B. *Suppose (a) holds and $|f'|$ is convex on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)\,(|f'(a)|+|f'(b)|)}{8}.$$

THEOREM C. *Suppose (a) holds and let $p>1$. If $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right]^{(p-1)/p}.$$

In Section 2, we give an improvement and simplification of the constant in Theorem C and consolidate this result with Theorem B as Theorem 1 below. An analogous result, Theorem 2, is developed which relates in the same way to the first inequality in (1.1). The results of Dragomir and Agarwal are based on convexity. We develop analogous results based on concavity. In Sections 3 and 4, we note some consequent applications to special means and to estimates of the error term in the trapezoidal formula.

2. MAIN RESULT

THEOREM 1. *Suppose (a) holds and $q \ge 1$. If the mapping $|f'|^q$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}.$$

PROOF. From Lemma A

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \tag{2.1}$$

and by the power-mean inequality

$$\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \le \left(\int_0^1 |1-2t|\,dt\right)^{1-1/q}\left(\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{1/q}.$$

Because $|f'|^q$ is convex, we have

$$\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|^q\,dt \le \int_0^1 |1-2t|\left[t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\right]dt = \frac{|f'(a)|^q+|f'(b)|^q}{4}.$$

Since $\int_0^1 |1-2t|\,dt = 1/2$, we have from (2.1) and the displayed inequality following it that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{|f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q},$$

hence, the desired result. ∎

REMARK. For $q=1$ this reduces to Theorem B. For $q=p/(p-1)$ $(p>1)$ we have an improvement of the constant in Theorem C, since $2^p > p+1$ if $p>1$ and accordingly

$$\frac{1}{4} < \frac{1}{2(p+1)^{1/p}}.$$

We now proceed to an analogous result relating to the first inequality in (1.1).

THEOREM 2. *Suppose the assumptions of Theorem 1 are satisfied. Then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}.$$

PROOF. Our starting point is the identity

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b S(x)f'(x)\,dx, \tag{2.2}$$

where

$$S(x) = \begin{cases} x-a, & x \in \left[a, \dfrac{a+b}{2}\right), \\ x-b, & x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

An argument parallel to that of Theorem 1 but with (2.2) in place of Lemma A gives the desired result. ∎

We now derive comparable results to Theorems 1 and 2 with a concavity property instead of convexity.

THEOREM 3. *Suppose (a) holds and $|f'|^q$ ($q \ge 1$) is concave on $[a,b]$. Then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{4}\left| f'\left(\frac{a+b}{2}\right)\right| \tag{2.3}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{4}\left| f'\left(\frac{a+b}{2}\right)\right|. \tag{2.4}$$

PROOF. First, note that

$$\begin{aligned} |f'(\lambda x + (1-\lambda)y)|^q &\ge \lambda\,|f'(x)|^q + (1-\lambda)\,|f'(y)|^q \\ &\ge (\lambda\,|f'(x)| + (1-\lambda)\,|f'(y)|)^q, \end{aligned}$$

by the convexity of $|f'|^q$ and the power-mean inequality. Hence,

$$f'(\lambda x + (1-\lambda)y) \ge \lambda\,|f'(x)| + (1-\lambda)\,|f'(y)|,$$

so $|f'|$ is also concave.

Accordingly by the Jensen integral inequality we have

$$\begin{aligned} \int_0^1 |1-2t|f'(ta+(1-t)b)\,dt &\le \left(\int_0^1 |1-2t|\,dt\right)\left| f'\left(\frac{\int_0^1 |1-2t|(ta+(1-t)b)\,dt}{\int_0^1 |1-2t|\,dt}\right)\right| \\ &= \frac{1}{2}\left| f'\left(\frac{a+b}{2}\right)\right|. \end{aligned}$$

By (2.1) we have (2.3). Similarly using (2.2) we can prove (2.4). ∎

3. APPLICATIONS TO SPECIAL MEANS

As in [1] we shall consider extensions of arithmetic, logarithmic, and generalized logarithmic means from positive to real numbers. We take

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \qquad \alpha,\beta \in R,$$
$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}, \qquad |\alpha| \neq |\beta|, \quad \alpha\beta \neq 0,$$
$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n}, \qquad n \in Z\setminus\{-1,0\}, \quad \alpha,\beta \in R, \quad \alpha \neq \beta.$$

PROPOSITION 1. *Let $a,b \in R$, $a < b$, $0 \notin [a,b]$, and $n \in Z$, $|n| \geq 2$. Then for all $q \geq 1$,*

$$|A(a^n,b^n) - L_n(a,b)^n| \leq \frac{|n|(b-a)}{4}\left[A\left(|a|^{(n-1)q},|b|^{(n-1)q}\right)\right]^{1/q} \tag{3.1}$$

and

$$|A(a,b)^n - L_n(a,b)^n| \leq \frac{|n|(b-a)}{4}\left[A\left(|a|^{(n-1)q},|b|^{(n-1)q}\right)\right]^{1/q}. \tag{3.2}$$

PROOF. The proof is immediate from Theorem 1 and Theorem 2 with $f(x) = x^n$, $x \in R$, $n \in Z$, $n \geq 2$. ∎

From (3.1), for $q = 1$, $n \geq 2$, we have [1, Proposition 3.1]. For $q = p/(p-1)$ $(p > 1, n \geq 2)$ we have an improvement of [1, Proposition 3.2].

PROPOSITION 2. *Suppose $a,b \in R$, $a < b$, and $0 \notin [a,b]$. Then for $q \geq 1$,*

$$\left|A\left(a^{-1},b^{-1}\right) - L^{-1}(a,b)\right| \leq \frac{b-a}{4}\left[A\left(|a|^{-2q},|b|^{-2q}\right)\right]^{1/q} \tag{3.3}$$

and

$$\left|A(a,b)^{-1} - L^{-1}(a,b)\right| \leq \frac{(b-a)}{4}\left[A\left(|a|^{-2q},|b|^{-2q}\right)\right]^{1/q}. \tag{3.4}$$

PROOF. The result follows from Theorem 1 and Theorem 2 with $f(x) = 1/x$. ∎

From (3.3), for $q = 1$, we have [1, Proposition 3.3], while for $q = p/(p-1)$ $(p > 1)$ we have an improvement of [1, Proposition 3.4].

4. THE TRAPEZOIDAL AND MIDPOINT FORMULÆ

Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a,b]$ and consider the quadrature formula

$$\int_a^b f(x)\,dx = T(f,d) + E(f,d),$$

where

$$T(f,d) = \sum_{i=0}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}\,(x_{i+1}-x_i),$$

for the trapezoidal version and

$$T(f,d) = \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i),$$

for the midpoint version and $E(f,d)$ denotes the associated approximation error. We derive an improvement of the error estimate derived for the former case in [1], as well as a corresponding result for the latter.

PROPOSITION 3. *Suppose (a) holds and $|f'|^q$ is convex on $[a,b]$, where $q \geq 1$. Then for every division d of $[a,b]$, the quadrature and midpoint errors satisfy*

$$|E(f,d)| \leq \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2}\right]^{1/q}$$
$$\leq \frac{1}{4} \max\{|f'(a)|, |f'(b)|\} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2.$$

PROOF. On applying Theorem 1 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, \dots, n-1)$ of the division d, we get

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) - \int_{x_i}^{x_{i+1}} f(x)\,dx\right|$$
$$\leq \frac{(x_{i+1} - x_i)^2}{4} \left[\frac{|f'(x_i)|^q + \left|f'(x_{i+1})\right|^q}{2}\right]^{1/q}.$$

Hence, in the trapezoidal case

$$\left|T(f,d) - \int_a^b f(x)\,dx\right| = \left|\sum_{i=0}^{n-1} \left\{\frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) - \int_{x_1}^{x_{i+1}} f(x)\,dx\right\}\right|$$
$$\leq \sum_{i=0}^{n-1} \left|\frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) - \int_{x_1}^{x_{i+1}} f(x)\,dx\right|$$
$$\leq \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[\frac{|f'(x_i)|^q + \left|f'(x_{i+1})\right|^q}{2}\right]^{1/q}$$
$$\leq \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \max\{|f'(x_i)|, |f'(x_{i+1})|\}$$
$$\leq \frac{1}{4} \max\{|f'(a)|, |f'(b)|\} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2.$$

The last inequality follows from the fact that $|f'(x)|^q$ is convex. A parallel application of Theorem 2 gives the result for the midpoint formula. ∎

Similarly using Theorem 3 we can prove the following.

PROPOSITION 4. *Suppose (a) holds and $|f'|^q$ $(q \geq 1)$ is concave on $[a,b]$. Then for every division d of $[a,b]$, the trapezoidal and midpoint errors satisfy*

$$|E(f,d)| \leq \frac{1}{4} \sum_{i=o}^{n-1} (x_{i+1} - x_i)^2 \left|f'\left(\frac{x_{i+1} + x_i}{2}\right)\right|.$$

REFERENCES

1. S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.* **11** (5), 91–95, (1998).
2. S.S. Dragomir and C.E.M. Pearce, On some inequalities for differentiable convex functions and applications, (preprint).