The Nonnegativity of Solutions of Delay Differential Equations

M. BODNAR
Institute of Applied Mathematics and Mechanics
Warsaw University, Banacha 2, 02-097 Warsaw, Poland
mbodnar@hydra.mimuw.edu.pl

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Abstract—General conditions guaranteeing the nonnegativity of solutions of delay differential equations are proposed. Some examples when the nonnegativity is not preserved in time are given. The nonnegativity of solutions of the logistic equation with time delay is considered. Behaviour like in discrete time case is observed. © 2000 Elsevier Science Ltd. All rights reserved.

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1. GENERAL THEORY

Various models of biological or physical systems in terms of systems of delay differential equations DDE have been studied in different contexts, e.g., [1-7]. It is very important to show nonnegativity of the solutions of biological or physical models. It is obvious, if a certain variable represents an amount of something (i.e., concentration of wolves), then we expect that this variable is nonnegative. Moreover, although several specific results are available in the literature, which looks true for specific cases, still general results for abstract systems need to be developed in order to provide a general framework for specific class of models. The case of linear equations was studied in [8]. The general conditions guaranteeing the nonnegativity of solutions as well as some examples when the nonnegativity is not preserved in time are proposed in this paper.

Consider the following equation:

\[ \dot{x}(t) = F(x(t)) + G(x(t - \tau)), \]

where \( t \in \mathbb{R} \), \( x(t) \in \mathbb{R}^n \); \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a Lipschitz continuous function, and \( G: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function, \( F = (F_1, \ldots, F_n), G = (G_1, \ldots, G_n) \) and is a constant \( \tau > 0 \). Let \( \Phi: [-\tau, 0] \rightarrow \mathbb{R}^n \) be a continuous function and

\[ x(t) = \Phi(t), \quad \text{for } t \in [-\tau, 0], \]

be an initial data of equation (1.1). Consider the auxiliary problem

\[ \dot{x}(t) = F(x(t)), \quad \text{for } t \geq 0, \]
\[ x(0) = \Phi(0). \]

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THEOREM 1.1. Assume that equation (1.3), for each \( \Phi(0) \geq 0 \), has a nonnegative solution and the following inequality holds:

\[
\forall s \in (\mathbb{R}^+)^n, \quad i = 1, \ldots, n, \quad G_i(s) \geq 0. \tag{1.4}
\]

Then the solution of problem (1.1),(1.2) with \( \Phi(t) \geq 0 \) for \( t \in [-\tau, 0] \) is nonnegative on the interval on which it exists.

PROOF. Let \( x \) be a solution of problem (1.1),(1.2). Let \( t \in [0, \tau] \), then we have \( t - \tau \in [-\tau, 0] \).

Inequality (1.4) implies

\[
\forall i \in \{1, \ldots, n\}, \quad \forall t \geq 0, \quad \dot{x}_i(t) \geq F_i(x(t)). \tag{1.5}
\]

Assume that \( x_i \) can be negative for some \( i \in \{1, \ldots, n\} \). Then there exists \( t_0 \geq 0 \) such that

\[
\dot{x}(t_0) \leq 0 \quad \text{and} \quad x(t_0) = 0. \tag{1.6}
\]

But the autonomous equation \( \dot{y} = F(y) \) has nonnegative solutions, and solving this equation with initial data \( (t_0, x(t_0)) \), we obtain \( F_i(x(t_0)) \geq 0 \). For \( G(x(t_0)) > 0 \), we have \( \dot{x}_i(t_0) > F_i(x(t_0)) \geq 0 \), which contradicts (1.6). It is easy to see for \( G(x(t_0)) = 0 \), we also have \( x(t) \geq 0 \). By induction on \( k \), the inequality \( x(t) \geq 0 \) holds for all \( k \) and for \( t \in [k\tau, (k + 1)\tau] \). Hence,

\[
x_i(t) \geq 0, \quad \forall t \geq 0, \quad \forall i \in \{1, \ldots, n\}. \tag{1.7}
\]

THEOREM 1.2. Let

\[
\exists s \in (\mathbb{R}^+)^n, \quad \exists i \in \{1, \ldots, n\}, \quad \exists \epsilon > 0 : G_i(s) < -(F_i(0) + \epsilon). \tag{1.8}
\]

Then there exists a function \( \Phi(t) \geq 0 \) such that the corresponding solution of problem (1.1),(1.2) becomes negative in a finite interval of time.

PROOF. Let \( \Phi = (\Phi_1, \ldots, \Phi_n) \) be given:

\[
\Phi_j(t) = \begin{cases} 
\dot{s}_j, & \text{for } t \in \left[-\tau, -\frac{\tau}{2}\right], \\
\frac{2s_j}{\tau}t, & \text{for } t \in \left(-\frac{\tau}{2}, 0\right),
\end{cases} \quad \text{for } j = 1, \ldots, n. \tag{1.9}
\]

Hence,

\[
x_i(0) = 0 \quad \text{and} \quad \dot{x}_i(0) = F_i(0) + G_i(s) < 0. \tag{1.10}
\]

This shows that there exists \( t > 0 \) such that \( x_i(t) < 0 \).

REMARK 1.1. If \( F(0) = 0 \), it is enough that \( G_i(s) < 0 \), for some \( s, i \), to have negative values of the solution of problem (1.1),(1.2).

REMARK 1.2. Let the solutions of problem (1.1),(1.2), with \( \tau = 0 \), be nonnegative for each nonnegative initial condition. It does not imply that the solutions of problem (1.1),(1.2) with some \( \tau > 0 \) are nonnegative for nonnegative initial data.

PROOF. Consider the following equation:

\[
\dot{x}(t) = x(t) - \frac{x(t - \tau)}{2}. \tag{1.11}
\]

It is easy to see that all solutions of equation (1.11) with nonnegative initial data have nonnegative solutions with \( \tau = 0 \). On the other hand, Theorem 1.2 implies that for the nonnegative initial data, equation (1.11) can have solutions with negative values.
It is easy to generalise Theorems 1.1 and 1.2 to the following equation:

$$\dot{x}(t) = F(x(t)) + \sum_{j=1}^{k} G^{(j)}(x(t - \tau_j)).$$  \hfill (1.12)

In this case, condition (1.4) becomes

$$\forall (x^1, \ldots, x^k) : x^i \in (\mathbb{R}^+)^n, \quad \forall j \in \{1, \ldots, n\}, \quad \sum_{i=1}^{k} G_j^{(i)}(x^i) \geq 0.$$  \hfill (1.13)

and condition (1.8) becomes

$$\exists (s^1, \ldots, s^k) : s^i \in (\mathbb{R}^+)^n, \quad \exists j \in \{1, \ldots, n\}, \quad \exists \epsilon > 0 : \sum_{i=1}^{k} G_j^{(i)}(s^i) < -(F_j(0) + \epsilon).$$  \hfill (1.14)

## 2. LOGISTIC EQUATION WITH TIME DELAY

The logistic equation is applied frequently to biological systems [1,6–9]. We show that the solutions of the logistic equation with time delay can become negative in an finite interval of time. Then some special cases when the property of nonnegative solution is fulfilled will be discussed.

Consider the following problem:

$$\dot{x}(t) = ax(t - \tau)\left(1 - \frac{x(t - \tau)}{K}\right), \quad \text{for } t > 0,$$
$$x(t) = \phi(t), \quad \text{for } t \in [-\tau, 0].$$  \hfill (2.1)

where \(a\) is the growth rate, \(K\) the environment capacity, and \(\tau\) are positive constants. This model was studied in [9], but not the nonnegativity of solutions. Scaling variable \(x\) and time (see [1]), we obtain

$$\dot{x}(t) = a\tau x(t - 1)(1 - x(t - 1)), \quad \text{for } t > 0,$$
$$x(t) = \phi(t), \quad \text{for } t \in [-1, 0].$$  \hfill (2.2)

As a consequence of Theorem 1.2, we have the following.

**Corollary 1.2.** For the nonnegative initial condition, the solution of problem (2.2) can have negative values.

Since \(K = 1\) is the environment capacity, there are biological reasons to consider initial conditions \(0 \leq \phi(t) \leq 1\). Consider the following polynomials:

$$W_1(x) = -\frac{1}{48}x^3 - \frac{1}{8}x^2 + \frac{1}{4}x + 1$$  \hfill (2.3)

and

$$W_2(x) = -\frac{1}{16}x^3 - \frac{1}{4}x^2 + 1.$$  \hfill (2.4)

Let \(p_1\) and \(p_2\) be the greatest roots of \(W_1\) and \(W_2\), respectively, i.e.,

$$p_1 = \max\{x \in R : W_1(x) = 0\}, \quad p_2 = \max\{x \in R : W_2(x) = 0\}.$$  \hfill (2.5)
Theorem 2.1. Assume

\[ 0 \leq \phi(t) \leq 1, \quad \text{for } t \in [-1,0]. \tag{2.6} \]

Then

(i) if \( aT > P_1 \), then there exists a function \( \phi \), which satisfies condition (2.6) and the corresponding solution to problem (2.2) has negative values;

(ii) if \( aT < P_2 \) and the function \( \phi \) satisfies (2.6), then the solution to problem (2.2) is nonnegative.

Proof. Let

\[ \phi(t) = \begin{cases} 
1, & \text{for } t \in [-1,0), \\
2, & \text{for } t = 0, \\
1, & \text{for } t > 0,
\end{cases} \tag{2.7} \]

and \( x_n \) be a solution of problem (2.2) on the interval \([n-1,n]\). It is easy to see that

\[ x_1(t) = \phi(0) + \int_0^t aT\phi(s)(1-\phi(s)) \, ds. \tag{2.8} \]

Hence,

\[ x_1(t) = 1 + \int_0^t \frac{aT}{4} \, ds = 1 + \frac{aT}{4}. \tag{2.9} \]

Therefore,

\[ x_2(t) = x_1(t) + \int_1^t aTx_1(s)(1-x_1(s)) \, ds = 1 + \frac{aT}{4} - \int_1^t \left(1 + \frac{aTs}{4}\right) \frac{aTs}{4} \, ds. \tag{2.10} \]

Consequently, we have

\[ x(2) = x_2(2) = 1 + \frac{aT}{4} - \frac{(aT)^2}{8} - \frac{(aT)^3}{48} = W_1(aT). \tag{2.11} \]

(i) If \( aT > P_1 \), then \( W_1(aT) = x(2) < 0 \).

(ii) First we show that if condition (2.6) is fulfilled then the following inequality holds:

\[ \forall \, t > 0, \quad x(t) \leq 1 + \frac{aT}{4}. \tag{2.12} \]

If \( x(t) > 1 \), then there must exist a point \( t_0 < t \) such that \( x(t_0) = 1 \). Since

\[ x(t) = x(t_0) + \int_{t_0-1}^{t-1} aTx(s)(1-x(s)) \, ds, \tag{2.13} \]

we have

\[ x(t) \leq 1 + \int_{t_0-1}^{t_0} \frac{aT}{4} \, ds = 1 + \frac{aT}{4}. \tag{2.14} \]

On the other hand, we have

\[ x(t) \geq 1 + \int_{t_0-1}^{t_0} \left(1 + \frac{aT}{4}\right) \left(\frac{aT}{4}\right) \, ds = W_2(s) \, ds. \tag{2.15} \]

Thus, \( aT < P_2 \) implies \( W(aT) > 0 \) and consequently \( x(t) \geq 0 \).

Remark 2.1. Computing values of \( P_1 \) and \( P_2 \), one gets

\[ P_1 \approx 3.0578, \quad P_2 \approx 1.6786. \tag{2.16} \]
3. FINAL REMARKS

Many researchers use models with delay without checking its basic properties and believing that the behaviour for small delay is similar to the case without delay. It should be pointed out that it may be false.

We have shown that the nonnegativity of solutions of delay differential systems need not be preserved. Some general conditions guaranteeing the nonnegativity have been proposed, but some problems are still open. For example, consider equation (1.1) with the function $G = G(x(t), x(t - \tau))$, i.e.,

$$\dot{x}_i(t) = x_i(t)(F_i(x) + G_i(x(t), x(t - \tau))).$$

(3.1)

Then we note the following.

REMARK 3.1. For each initial data $\Phi$ such that $\Phi_i(0) > 0$ for $i = 1, \ldots, n$, the solutions of problem (3.1) are nonnegative.

Theorem 1.2 shows that there exists an initial data $\Phi$, such that the solution of problem (1.1),(1.2) becomes negative. But the nonnegativity may depend on the way in which delay is introduced to the equation. If we consider the logistic delayed equation of the form (2.1), then the solution is negative for nonnegative initial data. Even if the solution can be negative sometimes, there exists a set of parameters and initial data that the nonnegativity is guaranteed. The logistic equation with time delay is a good example. The behaviour of its solutions, for small values of $a\tau$, is similar to the logistic discrete time model.

REFERENCES