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Convexity Properties of the Moment Mapping Re-examined

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Consider a Hamiltonian action of a compact Lie group on a compact symplectic manifold. A theorem of Kirwan's says that the image of the momentum mapping intersects the positive Weyl chamber in a convex polytope. I present a new proof

classes of noncompact or singular Hamiltonian spaces, such as cotangent bundles and complex affine varieties.

Key Words: Momentum mappings; geometric quantization; geometric invariant theory.

Contents.

1. INTRODUCTION

Let K be a compact Lie group acting smoothly on a compact symplectic manifold M and suppose there exists a moment(um) map for the action. This map has a host of interesting properties, one of the most important of which is the fact that the intersection of its image with any Weyl chamber is a convex polytope, referred to as the *momentum polytope* of M. This theorem, which is due to Kirwan, has a long history, which can be briefly

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summarized as follows. Kostant proved a convexity theorem for torus actions on conjugacy classes and flag manifolds in [16]. Atiyah in [2] and Guillemin and Sternberg in [5] dealt with the case of general Hamiltonian torus actions. In their paper, Guillemin and Sternberg further proved a convexity theorem for Hamiltonian actions of arbitrary compact Lie groups on integral Kähler manifolds (or projective manifolds), which was also proved by Mumford in [26]. Kirwan subsequently extended this result to Hamiltonian actions on arbitrary compact symplectic manifolds in [14]. Many useful refinements in the projective-algebraic case were made later by Brion in [3]. See [4], [11] and [22] for other results and more references. See [9] and [19] for some developments subsequent to the present paper.

A striking difference between Kirwan's general convexity theorem and the abelian convexity theorem of Atiyah–Guillemin–Sternberg lies in the fact that the latter offers far more quantitative information on the shape of the momentum polytope. For example, in the abelian case one knows that the vertices of the polytope are images of fixed points in M, and that the shape of the polytope near a vertex can be read off from the isotropy action on the tangent space at a corresponding fixed point. This follows from a combination of the equivariant Darboux Theorem and Morse theory applied to the components of the momentum map.

The goal of this paper is to obtain such information in the nonabelian case as well. The main result is Theorem 6.7, which is a sharpened version of Kirwan's convexity theorem. Given a point m in M mapping to a point μ in the momentum polytope, it provides a description of the shape of the polytope near μ in terms of the action of the stabilizer of m on polynomials on the tangent space at m. It also states a necessary criterion for μ to be a vertex, which generalizes the criterion for the abelian case referred to above. Other results include convexity theorems for actions on affine varieties, Theorem 4.9, and cotangent bundles, Theorem 7.6. Theorem 4.8 describes the relation between the momentum cone of an affine variety and the momentum polytopes of its projective closure and the divisor at infinity.

These results are inspired by Brion's treatment of Kirwan's theorem for projective varieties. It came as a surprise to me how well Brion's algebro-geometric techniques can be adapted to a C^∞ setting essentially without sacrificing any of their power. The main reason why this is possible is that every point in M possesses an invariant neighbourhood that is isomorphic as a Hamiltonian K-manifold to (a germ of) a complex quasi-projective variety.

In the language of the orbit method, the momentum polytope of M is the "classical" analogue of the set of highest weights of the unitary irreducible representations occurring in the "quantization" of M. There is also a classical analogue of the space of highest-weight vectors. This will be the subject of a forthcoming paper.

The paper is organized as follows. Section 2 is a review of some basic facts concerning representations and momentum maps. Section 3 is a review of the convexity theorem for complex projective varieties, where I have presented the argument in such a manner that it can be applied to non-compact varieties. In Sections 4 and 5 I prove convexity theorems for complex affine and Stein varieties. In Section 6 I apply these results to prove local convexity properties of arbitrary momentum maps, whence I derive the main result, Theorem 6.7. The local description of the momentum polytope given by this theorem, although explicit, is unwieldy in practice, and one often has to revert to ad hoc methods to calculate momentum polytopes. In Section 7 I illustrate this in a number of examples, such as actions on cotangent bundles and projective spaces.

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2. PRELIMINARIES

In this section I introduce notation and review basic material to be referred to later.

2.1. *Groups, Representations*

Throughout this paper K will be a compact connected Lie group with a fixed maximal torus T. The complexification of K is denoted by G and the complexification of T by H. Let us fix a Borel subgroup B of G containing H. Its unipotent radical $[B, B]$ is denoted by N, and the corresponding positive Weyl chamber in $\mathfrak{t}^*$ by $\mathfrak{t}^*_+$. The lattice $\ker(\exp|_{\mathfrak{t}})$ is denoted by Λ. Its dual lattice

$$\Lambda^* = \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^*$$

is the lattice of (real) weights and $\Lambda^*_+ = \Lambda^* \cap \mathfrak{t}^*_+$ is the monoid of dominant weights. To a real weight λ corresponds a character ζ_λ of T defined by $\zeta_\lambda(\exp \xi) = \exp(2\pi \sqrt{-1}\, \langle \lambda, \xi \rangle)$ for $\xi \in \mathfrak{t}$.

The complex reductive group $G = K^{\mathbb{C}}$ has a unique complex affine structure. Let $R = \mathbb{C}[G]^N$ be the algebra of polynomial functions on G which are N-invariant on the right, that is to say, $f \in R$ if $f(gn) = f(g)$ for all $g \in G$ and $n \in N$. Then G acts on R by left multiplication and, since H normalizes N, H acts on R by right multiplication. Under the right H-action, R has a weight space decomposition

$$R = \bigoplus_{\lambda \in \Lambda^*_+} R_\lambda, \tag{2.1}$$

and it follows from the Borel–Weil Theorem that R_λ is an irreducible G-module with highest weight λ. (See [17], Chapt. III.) This implies that the algebra R is of finite type and, hence, that the scheme $G//N = \operatorname{Spec} R$ "is" an affine variety. (I shall not distinguish between an affine variety and the scheme associated to it.)

Let $\mathfrak{W} = N(T)/T$ be the Weyl group of (K, T) and let w_0 be the longest Weyl group element. Define an involution $*: \mathfrak{t} \to \mathfrak{t}$ by $\mu^* = -w_0\mu$. The complex-linear extension of $*$ to $\mathfrak{t}^{\mathbb{C}}$ and the dual map on $(\mathfrak{t}^*)^{\mathbb{C}}$ will also be denoted by $*$. It is well-known that $*$ leaves the set of dominant weights invariant and that for all $\lambda \in \Lambda^*_+$ the representation R_{λ^*} is isomorphic to R^*_λ, the contragredient representation of R_λ.

2.2. *Cones, Polytopes*

Let E be a finite-dimensional vector space over $\mathbb{Q}$ and let S be a subset of E. The *$\mathbb{Q}$-convex hull* of S is the smallest convex subset of E containing S and is denoted by $\operatorname{hull}_{\mathbb{Q}} S$. The *convex hull* of S is the smallest convex subset of $E \otimes \mathbb{R}$ containing S and is denoted by $\operatorname{hull} S$. A subset of E (resp. $E \otimes \mathbb{R}$) is called a *cone* if it is invariant under multiplication by nonnegative rational (resp. real) scalars. The *convex $\mathbb{Q}$-cone* spanned by S is the smallest convex cone in E containing S and is denoted by $\operatorname{cone}_{\mathbb{Q}} S$. In other words, $\operatorname{cone}_{\mathbb{Q}} S = \mathbb{Q}_{\geqslant 0} \cdot \operatorname{hull}_{\mathbb{Q}} S$. The *convex cone* spanned by S is the smallest convex cone in $E \otimes \mathbb{R}$ containing S and is denoted by $\operatorname{cone} S$. That is, $\operatorname{cone} S = \mathbb{R}_{\geqslant 0} \cdot \operatorname{hull} S$. If S is finite, $\operatorname{hull} S$ and $\operatorname{cone} S$ are called a *rational convex polytope*, resp. a *rational convex polyhedral cone* in $E \otimes \mathbb{R}$. A cone is called *proper* if it does not contain any linear subspaces (apart from $\{0\}$).

2.3. *Hamiltonian Actions*

Let (M, ω) be a symplectic manifold with a K-action defined by a smooth map $\tau: K \times M \to M$. The action τ is called *Hamiltonian* if there exists a *momentum map*, that is, a map $\Phi: M \to \mathfrak{k}^*$ with the property that $d\Phi^\xi = \iota(\xi_M)\, \omega$ for all $\xi \in \mathfrak{k}$. Here ξ_M denotes the vector field on M induced by ξ, and Φ^ξ is the function defined by $\Phi^\xi(m) = (\Phi(m))(\xi)$. We may, and will, assume Φ to be K-equivariant with respect to the coadjoint action on $\mathfrak{k}^*$. The quadruple (M, ω, τ, Φ) is called a *Hamiltonian K-manifold.* (See e.g., [7].)

If Y is any subset of M, we denote the restriction of Φ to Y by Φ_Y. The *momentum set* $\Delta(Y)$ of Y is defined by

$$\Delta(Y) = \Phi(Y) \cap \mathfrak{t}^*_+.$$

In "good" cases, $\Delta(Y)$ is known to be a convex cone or polytope ([5], [14]), and is then called the *momentum cone*, resp. *polytope* of Y.

On every Hamiltonian K-manifold (M, ω, τ, Φ) there exists an almost-complex structure J that is *compatible* with the Hamiltonian action, that is to say, $J\colon TM \to TM$ is a symplectic map, the symmetric bilinear form $\omega(\cdot, J\cdot)$ is positive definite, and J is K-equivariant. If in addition J is integrable, then M is a Kähler manifold with K-invariant metric $ds^2 = \omega(\cdot, J\cdot) - \sqrt{-1}\,\omega(\cdot, \cdot)$, and K acts holomorphically.

Example 2.1 ([5]). Let (V, ω_V) be a symplectic vector space and assume K acts on V by linear symplectic transformations. This action is Hamiltonian; a momentum map is given by the quadratic map

$$\Phi_V^\xi(v) = \tfrac{1}{2}\omega_V(\xi v, v), \tag{2.2}$$

where ξv denotes the image of $v \in V$ under $\xi \in \mathfrak{k}$ (viewed as a linear operator on V). Choose a K-invariant ω_V-compatible complex structure J on V. Let $\langle \cdot, \cdot \rangle$ be the Hermitian inner product whose imaginary part is equal to $-\omega_V$. Then the momentum map can also be written as

$$\Phi_V^\xi(v) = \frac{\sqrt{-1}}{2}\langle \xi v, v\rangle. \tag{2.3}$$

Now suppose $K = T$ is a torus. Then V is an orthogonal direct sum of weight spaces $V = \bigoplus_{\nu \in \Lambda^*} V_\nu$. If $V_\nu \neq 0$, then ν is called a weight of the symplectic action of T on V. The weight space decomposition depends on the choice of the complex structure, but the weights do not. (This follows from the fact that any two K-invariant compatible complex structures on V are conjugate by a K-equivariant linear symplectic map.) If v is a vector of weight ν, then $\xi v = 2\pi\sqrt{-1}\,\nu(\xi)\, v$, so $\Phi(v) = -\pi\,\|v\|^2 \nu$ by (2.3), and therefore

$$\Delta(V) = -\operatorname{cone}\{\nu_1, ..., \nu_l\}, \tag{2.4}$$

where $\nu_1, ..., \nu_l$ are the (real) weights of V.

Example 2.2 ([13], [26], [1]). Let V, J and $\langle \cdot, \cdot \rangle$ be as in the previous example, and let $\mathbb{P}V$ be the space of complex lines in V. The natural K-action on $\mathbb{P}V$ leaves the Fubini–Study symplectic form invariant, and with the volume of $\mathbb{P}V$ normalized to 1, a momentum map is given by

$$\Phi_{\mathbb{P}V}([v]) = \frac{\Phi_V(v)}{\pi\,\|v\|^2} = \frac{\sqrt{-1}}{2\pi}\frac{\langle \xi v, v\rangle}{\|v\|^2}, \tag{2.5}$$

where $[v]$ denotes the line through v. Consequently, if K is a torus and v is a weight vector in V with weight ν, then $\Phi_{\mathbb{P}V}([v]) = -\nu$. Hence,

$$\Delta(\mathbb{P}V) = -\text{hull}\{\nu_1, ..., \nu_l\}, \tag{2.6}$$

where $\nu_1, ..., \nu_l$ are the weights of V. See further Section 7.1.

2.4. *Coadjoint Orbits*

For every μ in $\mathfrak{t}^*_+$ the coadjoint orbit $K\mu$ with its Kirillov–Kostant symplectic form ω_μ is a Hamiltonian K-manifold. The momentum map is simply the inclusion $\iota_\mu \colon K\mu \to \mathfrak{k}^*$. (See [15], [7].) Let P_μ be the parabolic subgroup $(K_\mu)^{\mathbb{C}} N$ of G. The K-equivariant diffeomorphism $G/P_\mu \to K\mu$ sending the coset $1P_\mu$ to the vector μ provides $K\mu$ with a complex structure with respect to which ω_μ is Kähler.

Now let μ be a dominant weight. Then the cohomology class of the form ω_μ on $K\mu$ is integral and because $K\mu$ is compact, there exists a Hermitian holomorphic line bundle $\mathcal{O}_\mu$ on $K\mu$ with curvature $-2\pi\sqrt{-1}\,\omega_\mu$. The pullback of $\mathcal{O}_\mu$ to G/P_μ is just the homogeneous line bundle $G \times^{P_\mu} \mathbb{C}$, where P_μ acts on $\mathbb{C}$ by the character μ.

Let μ_1 and μ_2 be two points in $\mathfrak{t}^*_+$ and let $\mu = \mu_1 + \mu_2$. Then $K_\mu = K_{\mu_1} \cap K_{\mu_2}$ and $P_\mu = P_{\mu_1} \cap P_{\mu_2}$, so we have canonical holomorphically locally trivial fibrations $\pi_i \colon K\mu \to K\mu_i$. It is not hard to see that $\omega_\mu = \pi_1^* \omega_{\mu_1} + \pi_2^* \omega_{\mu_2}$. If μ, μ_1 and μ_2 are dominant, the holomorphic line bundle $\mathcal{O}_\mu$ is isomorphic to $\pi_1^* \mathcal{O}_{\mu_1} \otimes \pi_2^* \mathcal{O}_{\mu_2}$. Let me summarize this in a commutative diagram:

$$\begin{tikzcd}
 & \pi_1^*\mathcal{O}_{\mu_1} \oplus \pi_2^*\mathcal{O}_{\mu_2} \arrow[dl] \arrow[d, "\otimes"] \arrow[dr] & \\
\mathcal{O}_{\mu_1} \arrow[d] & \mathcal{O}_\mu \arrow[d] & \mathcal{O}_{\mu_2} \arrow[d] \\
K\mu_1 & K\mu \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & K\mu_2.
\end{tikzcd}$$

2.5. *Gradient Flows, Semistability*

Let X be a smooth connected Riemannian manifold and let f be a function on X with the property that at every point of X there exists a system of local coordinates in which f is real-analytic. Let $\mathfrak{F}(t, \cdot)$ be the gradient flow of $-f$. Assume that the path of steepest descent $\mathfrak{F}(t, x)$ through every point x is contained in a compact set. Then the flow is defined for all $t \geqslant 0$. Moreover, by results of Łojasiewicz [20] and Simon [28], the limit

$x_\infty = \lim_{t\to\infty} \mathfrak{F}(t, x)$ exists for all x. Let a be a critical level of f and let $S_a = \{x \in X: f(x_\infty) = a\}$ be the stable set of a. Then S_a is a locally closed subset of X and $x \mapsto x_\infty$ is a continuous retraction from S_a onto $f^{-1}(a)$. The decomposition $X = \coprod_a S_a$ is called the *Morse decomposition* of X with respect to f (even if f is not a Morse function).

(A note on the literature: Łojasiewicz' paper [20] does not contain a complete proof of these assertions. He explains part of the requisite estimates in [21]. Simon gives a fuller account of the retraction argument in [28], while also generalizing it to an infinite-dimensional situation. Apparently independently of both [20] and [28], Neeman [25] and Schwarz [27] rederive the retraction theorem for certain flows on vector spaces from the inequalities in [21]. Their arguments can easily be generalized to prove the above statements.)

As an example, let (X, ω, τ, Φ) be a connected Hamiltonian K-manifold equipped with a compatible almost-complex structure J. Let $(\cdot, \cdot)$ be a K-invariant inner product on $\mathfrak{k}$ and let $|\cdot|$ be the associated norm. I use the same symbols to denote the corresponding inner product and norm on the dual $\mathfrak{k}^*$. Put $f = |\Phi|^2$. It follows from the local model for Hamiltonian actions (see Section 6) that this function is real-analytic in suitable local coordinates. Note that since f is K-invariant, the flow $\mathfrak{F}(t, \cdot)$ is K-equivariant. Let us assume the momentum map to be *admissible* in the sense that for every $x \in X$ the path of steepest descent $\mathfrak{F}(t, x)$ is contained in a compact set. The set S_0 is called the set of (*analytically*) *semistable* points and is denoted by $X^{\mathrm{ss}}(\Phi)$. So $X^{\mathrm{ss}}(\Phi)$ is nonempty if and only if $0 \in \Phi(X)$. Kirwan has shown in [13] that S_a is a submanifold of even codimension for every critical level a (and if J is integrable, then S_a is a complex submanifold). Now assume that the Morse decomposition $X = \coprod_a S_a$ is locally finite. (This is for instance the case if for every a there are only finitely many critical levels below a). Then, if nonempty, $X^{\mathrm{ss}}(\Phi)$ is open, connected and dense.

3. SEMISTABILITY AND CONVEXITY

In this section I review the convexity theorem for Kähler manifolds due to Guillemin and Sternberg [5] and Mumford [26]. Guillemin and Sternberg have pointed out in [6] that semistability and convexity are closely related. This idea goes back to Heckman's paper [8], and can be formulated as follows.

PROPOSITION 3.1. *Let (Y_i, σ_i, Ψ_i) be compact Hamiltonian K-manifolds with compatible (integrable) complex structures J_i, where $i = 1, 2, \ldots, k$. Assume that the cohomology classes of the σ_i are integral. Let Y be a* *compact complex K-manifold and let* $p_i\colon Y \to Y_i$ *be K-equivariant surjective holomorphic maps. Let* $a_1, a_2, \ldots, a_k$ *be nonnegative numbers, and let* $\sigma = \sum_i a_i p_i^* \sigma_i$ *and* $\Psi = \sum_i a_i p_i^* \Psi_i$. *Assume* σ *is a Kähler form on* Y. *Then* $\bigcap_i p_i^{-1}(Y^{ss}(\Psi_i))$ *is contained in* $Y^{ss}(\Psi)$. *Hence, if* $0 \in \Psi_i(Y_i)$ *for every i, then* $0 \in \Psi(Y)$.

By the equivariant version of Kodaira's Embedding Theorem, the manifolds Y_i and Y are of course biholomorphically equivalent (but not necessarily isometric) to projective manifolds with linear G-actions. The proof is a straightforward adaptation of the techniques of [5] and [26]. With a view to later applications I supply an argument which can easily be made to work for noncompact manifolds.

Proof. Note that Ψ is a momentum map for the K-action on Y with respect to the symplectic form σ. Also, since Y is compact, Ψ is admissible in the sense of Section 2.5. For clarity I will first handle the case where $Y_i = Y$ and $p_i = \mathrm{id}_Y$. So we are given Kähler forms σ_i on Y and we wish to show $\bigcap_i Y^{ss}(\Psi_i) \subset Y^{ss}(\Psi)$.

For $i = 1, 2, \ldots, k$, let L_i be a Hermitian holomorphic line bundle on Y with curvature form $-2\pi\sqrt{-1}\,\sigma_i$. (These exist because Y is compact.) Let n_i be a positive integer and let $s_i \in \Gamma(Y, L_i^{n_i})^K$, where Γ stands for holomorphic sections. Let $\langle s_i, s_i\rangle$ denote the length squared of s_i with respect to the fibre metric on $L_i^{n_i}$. It follows from the invariance of the s_i that for every $\xi \in \mathfrak{k}$ we have $\mathscr{L}_{J\xi_Y}\langle s_i, s_i\rangle = -4\pi n_i \Psi_i^\xi \langle s_i, s_i\rangle$, and hence

$$\mathscr{L}_{J\xi_Y}\langle s_i, s_i\rangle^{a_i/n_i} = -4\pi a_i \Psi_i^\xi \langle s_i, s_i\rangle^{a_i/n_i}. \tag{3.1}$$

Here $\mathscr{L}$ stands for the Lie derivative. Let L be the line bundle $\bigotimes_i L_i$ with the product Hermitian metric and let $\mathfrak{F}(t, \cdot)$ be the flow of the vector field $-\mathrm{grad}\,|\Psi|^2$. From the elementary fact that

$$J\xi_Y = \mathrm{grad}\,\Psi^\xi \tag{3.2}$$

one easily deduces that

$$\mathrm{grad}\,|\Psi(y)|^2 = 2J\Psi(y)^\flat_{Y,y}, \tag{3.3}$$

where $\flat\colon \mathfrak{k}^* \to \mathfrak{k}$ is the linear isomorphism defined by the inner product, and $\Psi(y)^\flat_Y$ is the vector field on Y induced by $\Psi(y)^\flat$. (See [13].) Put $s = s_1^{n/n_1} \otimes s_2^{n/n_2} \otimes \cdots \otimes s_k^{n/n_k}$, where $n = n_1 n_2 \cdots n_k$. Then $s \in \Gamma(Y, L^n)^K$. Consider the function $u = \langle s_1, s_1\rangle^{a_1/n_1}\langle s_2, s_2\rangle^{a_2/n_2} \cdots \langle s_k, s_k\rangle^{a_k/n_k}$. By (3.1),

$$\mathscr{L}_{J\xi_Y} u = -4\pi(a_1\Psi_1^\xi + a_2\Psi_2^\xi + \cdots + a_k\Psi_k^\xi)\,u = -4\pi\Psi^\xi u.$$

Using this and (3.3) we find that the derivative of u along a trajectory $\mathfrak{F}(t, y)$ is equal to

$$\frac{d}{dt} u(\mathfrak{F}(t, y)) = -du(\operatorname{grad} |\Psi(\mathfrak{F}(t, y))|^2) = -2\, du(J\Psi(\mathfrak{F}(t, y))^\flat_{Y, \mathfrak{F}(t, y)})$$

$$= 8\pi \langle \Psi(\mathfrak{F}(t, y)), \Psi(\mathfrak{F}(t, y)) \rangle\, u(\mathfrak{F}(t, y))$$

$$= 8\pi\, |\Psi(\mathfrak{F}(t, y))|^2\, u(\mathfrak{F}(t, y)) \geqslant 0. \tag{3.4}$$

Now suppose $y \in Y$ is semistable with respect to all σ_i. Kirwan [13] and Ness [26] observed that for a projective manifold with the Fubini–Study metric analytic semistability is equivalent to semistability in Mumford's sense. This is true in general for a compact complex manifold with an integral Kähler metric; see [29]. This means we can find positive integers n_i and invariant global holomorphic sections s_i of $L_i^{n_i}$ such that $s_i(y) \neq 0$. Then $s(y) = s_1^{n/n_1}(y) \otimes \cdots \otimes s_k^{n/n_k}(y) \neq 0$, so $u(y) > 0$. Put $y_\infty = \lim_{t \to \infty} \mathfrak{F}(t, y)$. By (3.4), $u(\mathfrak{F}(t, y))$ is increasing along the path $\mathfrak{F}(t, y)$, so $u(y_\infty) > 0$. On the other hand, $du(\mathfrak{F}(t, y))/dt$ tends to zero as t tends to infinity, so from (3.4) we get $|\Psi(y_\infty)|^2\, u(y_\infty) = 0$, and therefore $|\Psi(y_\infty)|^2 = 0$. In other words, y is semistable for σ. We have shown $\bigcap_i Y^{\mathrm{ss}}(\Psi_i) \subset Y^{\mathrm{ss}}(\Psi)$.

Suppose now that $0 \in \Psi_i(Y_i)$ for all i. Then the sets $Y^{\mathrm{ss}}(\Psi_i)$ are nonempty for all i, and are therefore open and dense. It follows that their intersection is nonempty. If $y \in \bigcap_i Y^{\mathrm{ss}}(\Psi_i)$, then, by the first part of the proof, $\Psi(y_\infty) = 0$, that is, $0 \in \Psi(Y)$.

In the general case the argument is almost exactly the same. The difference is that one considers the line bundle $L = \bigotimes_i p_i^* L_i$ on Y and the section $s = p_1^* s_1^{n/n_1} \otimes \cdots \otimes p_k^* s_k^{n/n_k}$ of L^n. Further, the assumptions on the p_i guarantee that the pre-image $p_i^{-1}(S)$ of a complex-analytic subset $S \subset Y_i$ of positive codimension is a complex-analytic subset of positive codimension of Y. This implies that if $Y_i^{\mathrm{ss}} \neq \varnothing$ for all i, then $\bigcap_i p_i^{-1}(Y^{\mathrm{ss}}(\Psi_i)) \neq \varnothing$. ∎

For noncompact Y the flow of $-\operatorname{grad} |\Psi|^2$ may not be defined for all time or its trajectories may fail to converge, and the equivalence between analytic and algebraic semistability can break down. But the following qualified statement is still true. The proof is almost word for word the same.

PROPOSITION 3.2. *Let (Y_i, σ_i, Ψ_i) be Hamiltonian K-manifolds endowed with compatible complex structures J_i, where $i = 1, 2, ..., k$. Assume there exist K-equivariant Hermitian holomorphic line bundles L_i on Y_i with curvature forms $-2\pi \sqrt{-1}\, \sigma_i$ for all i. Let Y be a complex K-manifold and let $p_i \colon Y \to Y_i$ be K-equivariant surjective holomorphic maps. Let $a_1, a_2, ..., a_k$*

be nonnegative numbers, and let $\sigma = \sum_i a_i p_i^* \sigma_i$ *and* $\Psi = \sum_i a_i p_i^* \Psi_i$. *Assume* σ *is a Kähler form on* Y *and that the momentum map* Ψ *is admissible in the sense of Section 2.5. If for every* i *there exist a positive integer* n_i *and a nonzero* K*-invariant holomorphic section of* $L_i^{n_i}$, *then* $0 \in \Psi(Y)$.

Remark 3.3. Suppose that $Z \subset Y$ and $Z_i \subset Y_i$ are irreducible K-stable locally closed analytic subvarieties, and that for all i the restriction of the map p_i to Z is a surjective map $Z \to Z_i$. Assume that for every z in Z the path $\mathfrak{F}(t, z)$ and its limit z_∞ are contained in Z. Also assume that the sections s_i restrict to nonzero sections on Z_i. Then $0 \in \Psi_i(Z_i)$ for all i implies $0 \in \Psi(Z)$. Exactly the same proof works.

Here is an application of Proposition 3.2, where the notation is as in Section 2.4. Let (X, ω, Φ) be a Hamiltonian K-manifold, not necessarily compact, with a compatible complex structure J. Suppose that there exists a K-equivariant Hermitian holomorphic line bundle L on X with curvature form $-2\pi\sqrt{-1}\,\omega$. For $i = 1, 2, \ldots, k$, let μ_i be a dominant weight and let Y_i be the manifold $X \times K\mu_i^*$ with symplectic form $\sigma_i = \omega + \omega_{\mu_i^*}$. (Recall that $*$ is defined by $\nu^* = -w_0\nu$.) Then the K-action on (Y_i, σ_i) is Hamiltonian with momentum map $\Psi_i = \Phi + \iota_{\mu_i^*}$. Let L_i be the Hermitian line bundle $L \otimes \mathcal{O}_{\mu_i^*}$ on Y_i. Let a_i be arbitrary positive numbers, let $\mu = \sum_i a_i \mu_i$, and let Y be the K-manifold $X \times K\mu^*$. Consider the fibrations $p_i \colon Y \to Y_i$ induced by the fibrations of coadjoint orbits $K\mu^* \to K(a_i\mu_i) \xrightarrow{q_i} K\mu_i^*$, where q_i is the equivariant diffeomorphism sending $a_i\mu_i$ to μ_i. Since the a_i are positive, the form $\sigma = \sum_i a_i p_i^* \sigma_i$ is a Kähler form on Y, and the action on Y is Hamiltonian with momentum map $\Psi = \sum_i a_i p_i^* \Psi_i$. Let us assume that

$$\text{for all } i \text{ there exist } n_i > 0 \text{ and nonvanishing sections } s_i \in \Gamma(Y_i, L_i^{n_i})^K; \tag{3.5}$$

$$\text{for all } a_i \geqslant 0 \text{ the momentum map } \Psi \text{ is admissible.} \tag{3.6}$$

(Assumption 3.6 holds e.g., when Φ is proper.) Then by Proposition 3.2 there exists a point $(x, k\mu^*)$ in $Y = X \times K\mu^*$ with $\Psi(x, k\mu^*) = 0$, that is, $(a_1 + \cdots + a_k)\, \Phi(x) = kw_0(a_1\mu_1 + \cdots + a_k\mu_k)$. This shows that

$$\frac{a_1\mu_1 + a_2\mu_2 + \cdots + a_k\mu_k}{a_1 + a_2 + \cdots + a_k} \in \Phi(X) \cap \mathfrak{t}_+^* = \Delta(X)$$

for all $a_i > 0$. The same is obviously true if some of the a_i are 0 (just replace the λ_i by the subset consisting of those μ_i for which $a_i \neq 0$), so we have proved:

PROPOSITION 3.4. *Under the assumptions* (3.5) *and* (3.6) *the convex hull of* $\mu_1, \mu_2, \ldots, \mu_k$ *is contained in* $\Delta(X)$.

Remark 3.5. By Remark 3.3, a similar result holds for an irreducible K-stable locally closed analytic subvariety Z of X. Namely, assume that the gradient flow of the function $-|\Psi|^2$ preserves the subvariety $Z \times K\mu^* \subset X \times K\mu^*$ and that the forward trajectories converge to points in $Z \times K\mu^*$. (This assumption is satisfied e.g., if the K-action on X extends to a holomorphic G-action and if Z has the property that $\overline{Gz} \subset Z$ whenever $z \in Z$.) Assume further that for all i the sections s_i restrict to nonzero sections on $Z \times K\mu_i$. Then $\operatorname{hull}\{\mu_1, \mu_2, ..., \mu_k\} \subset \Delta(Z)$.

Now consider the graded G-algebra $\mathfrak{A} = \bigoplus_{n \geqslant 0} \Gamma(X, L^n)$.

Definition 3.6 (Brion). The *highest-weight set* of X is the subset $\mathscr{C}(X)$ of the $\mathbb{Q}$-vector space $\Lambda^* \otimes \mathbb{Q}$ consisting of all λ/n with the property that λ is a dominant weight and n a positive integer such that the irreducible representation R_λ occurs in the degree-n piece $\mathfrak{A}_n$. In other words, $\lambda/n \in \mathscr{C}(X)$ if and only if $n > 0$ and λ occurs as a weight of the T-action on the degree-n piece of the ring $\mathfrak{A}^N$.

Let me recall a few basic facts concerning highest-weight sets. See [3] for details. Note first that $\mathscr{C}(X)$ is contained in $\mathfrak{t}^*_+$. The fact that $\mathfrak{A}$ has no zero divisors implies that $\mathscr{C}(X)$ is a convex subset of the $\mathbb{Q}$-vector space $\Lambda^* \times \mathbb{Q}$. Now suppose X is compact. Then $\mathfrak{A}$ is of finite type. A result of Luna and Vust says that for every G-algebra $\mathfrak{A}$ of finite type (not necessarily graded)

$$\mathfrak{A}^N \cong (R \otimes \mathfrak{A})^G, \tag{3.7}$$

where R is as in (2.1). This implies that $\mathfrak{A}^N$ is of finite type. (See e.g., [17].) It follows from this that $\mathscr{C}(X)$ is the convex hull of a finite number of points in $\Lambda^* \otimes \mathbb{Q}$. Moreover, $\Delta(X)$ is closed. It is now easy to deduce the following result from Proposition 3.4.

Theorem 3.7 ([5], [26], [3]). *If X is a compact integral Hamiltonian K-manifold with a compatible Kähler structure, $\Delta(X)$ is equal to the closure of $\mathscr{C}(X)$ in $\mathfrak{t}^*_+$ and is therefore a rational convex polytope.*

By Remark 3.5, Theorem 3.7 also holds if we replace X with a G-stable irreducible closed analytic subvariety.

This result applies in particular to a G-stable irreducible closed subvariety X of $\mathbb{P}V$, the projective space of a G-module V, equipped with the Fubini–Study symplectic form. But now consider a subvariety X of $\mathbb{P}V$ that is not necessarily irreducible or even reduced. What is the connection between $\Delta(X)$ and $\mathscr{C}(X)$? Let X^{red} denote the reduced variety associated to X and let $X_1, X_2, ..., X_l$ be its irreducible components (each endowed with the reduced induced structure). Let Z and Z_i be the affine cones of X^{red}, resp. X_i, and let I, resp. I_i, be their homogeneous ideals. Then $I=\bigcap_i I_i$ and for each i we have an exact sequence

$$I_i/I \hookrightarrow \mathbb{C}[Z] \twoheadrightarrow \mathbb{C}[Z_i].$$

It follows easily from this that $\mathscr{C}(X^{\mathrm{red}})=\bigcup_i \mathscr{C}(X_i)$. It is evident that $\Delta(X^{\mathrm{red}})=\bigcup_i \Delta(X_i)$, so applying Theorem 3.7 to each of the X_i we obtain the following result.

ADDENDUM 3.8 ([18]). *If X is a (not necessarily irreducible or reduced) subvariety of $\mathbb{P}V$, then $\Delta(X^{\mathrm{red}})$ is equal to the closure of $\mathscr{C}(X^{\mathrm{red}})$ in $\mathfrak{t}^*_+$ and is therefore a union of finitely many rational convex polytopes.*

The following example shows that $\mathscr{C}(X)$ is in general not a union of finitely many $\mathbb{Q}$-convex sets and that it is not necessarily equal to $\mathscr{C}(X^{\mathrm{red}})$.

EXAMPLE 3.9. Fix a positive integer n. Let $\mathfrak{A}$ be the algebra $\mathbb{C}[x, y]/(x^n)$ and let $X=\operatorname{Proj}\mathfrak{A}$, considered as a closed subscheme of $\mathbb{P}^1$. For any weight λ of $K=S^1$ define an action of K on $\mathbb{C}^2$ by $g(x, y)=(\zeta_{-\lambda}(g)\, x, y)$, where $\zeta_{-\lambda}$ is the character defined by $\zeta_{-\lambda}(\exp \xi)=\exp(-2\pi\sqrt{-1}\,\langle\lambda, \xi\rangle)$. According to (2.6), $\Delta(\mathbb{P}^1)$ is equal to the interval between 0 and λ. Note that X^{red} is the point with homogeneous coordinates $[0, 1]$, so

$$\mathscr{C}(X^{\mathrm{red}})=\Delta(X^{\mathrm{red}})=\{0\}.$$

Write $\bar{f}$ for $f+(x^n)\in\mathfrak{A}$; then $\bar{x}^k\bar{y}^l$ has degree $k+l$ and weight $k\lambda$ for $k=0, 1, ..., n-1$ and $l\geqslant 0$, so that

$$\mathscr{C}(X)=\left\{\frac{k\lambda}{k+l} : k, l\in\mathbb{N}, k<n, kl\neq 0\right\},$$

which cannot be written as a union of finitely many $\mathbb{Q}$-convex sets.

Nevertheless, it is always true that the intersection of $\mathscr{C}(X)$ with a rational line in $\mathfrak{t}^*$ is a bounded set which contains both its endpoints.

ADDENDUM 3.10. *For every (not necessarily irreducible or reduced) subvariety X of $\mathbb{P}V$ and every v in $\mathscr{C}(X)$, let $I_v=\{q\in\mathbb{Q} : qv\in\mathscr{C}(X)\}$. Then* $\inf I_v$ *and* $\sup I_v$ *are in I_v.*

Proof. This is similar to the proof of Theorem 3.7. The assertion is trivial for $v=0$, so let me assume that $v\neq 0$. Let $\mathfrak{t}_1$ be the subalgebra of $\mathfrak{t}$ annihilated by v; then $T_1=\exp\mathfrak{t}_1$ is a subtorus of T of codimension one. Denote the quotient circle T/T_1 by T_2 and identify $\mathfrak{t}_2^*$, the kernel of the canonical projection $\mathfrak{t}^*\to\mathfrak{t}_1^*$, with the line $\mathbb{R}v\subset\mathfrak{t}^*$. Put $\mathscr{C}_v=\mathscr{C}(X)\cap\mathbb{R}v$.

Let Z be the affine cone on X and let $\mathfrak{A}$ be the graded algebra $\mathbb{C}[Z]^N$, which as noted above is of finite type. Since the maximal torus T normalizes N, it acts in a natural way on $\mathfrak{A}$. The algebra $\mathfrak{B}=\mathfrak{A}^{T_1}$ is likewise of finite type and it carries a representation of T_2. Let λ_0 be the (unique) primitive element of Λ^* such that $\lambda_0=pv$ for some positive rational p. Note that $f\in\mathfrak{A}$ is a weight vector of weight proportional to λ_0 if and only if f is in $\mathfrak{B}$ and is a weight vector for T_2. It follows that $\mathscr{C}_v$ is equal to the set of $m\lambda_0/n$ such that there exists $f\in\mathfrak{B}$ of weight $m\lambda^*$ and degree n. Choose (nonzero) generators $f_1, f_2, ..., f_l$ of $\mathfrak{B}$ with weights $m_1\lambda_0, m_2\lambda_0, ..., m_l\lambda_0$ and degrees $n_1, n_2, ..., n_l$. Then $m_j\lambda_0/n_j\in\mathscr{C}_v$, so $m_j/n_j\in I_v$ for $1\leqslant j\leqslant l$. Moreover, $qv\in\mathscr{C}_v$ if and only if there exist $a_1, a_2, ..., a_l$ such that the monomial $\prod_j f_j^{a_j}$ is nonzero and

$$qv=\frac{\sum_j a_j m_j}{\sum_j a_j n_j}\,\lambda_0.$$

In other words, every element of $\mathscr{C}_v$ is a convex combination of the $m_j\lambda_0/n_j$ (but not every such combination need be in $\mathscr{C}_v$, because $\mathfrak{B}$ may have zero divisors). Now let r be the minimum of the m_j/n_j and s their maximum. Then r and s are in I_v and $r=\inf I_v$ and $s=\sup I_v$. ∎

4. AFFINE VARIETIES

In this section X denotes an affine algebraic variety (not necessarily reduced or irreducible) on which G acts algebraically. The main results are theorems describing the momentum map image of X with respect to suitable K-invariant symplectic forms, Theorems 4.9 and 4.23. My main interest is in smooth varieties, but the proofs turn out to be no harder for general varieties. There are some examples at the end of Section 4.1.

4.1. *Highest Weights and the Momentum Cone*

The natural analogue of Definition 3.6 is the following.

DEFINITION 4.1 (Brion). The *set of highest weights* of X is the subset $\mathscr{C}(X)$ of Λ^*_+ consisting of all dominant weights λ such that the irreducible G-representation R_λ occurs in the coordinate ring $\mathbb{C}[X]$. In other words, $\lambda\in\mathscr{C}(X)$ if and only if λ is a weight of the T-action on the ring $\mathbb{C}[X]^N$. If G is a torus, we refer to $\mathscr{C}(X)$ as the *weight set* of X.

Note that if Y is another affine G-variety, then $\mathscr{C}(X)=\mathscr{C}(Y)$ if and only if the coordinate rings of X and Y contain the same irreducible G-representations (up to multiplicities). It is easy to see that if X is irreducible, then $\mathscr{C}(X)$ is a submonoid of Λ_+^* (that is, it contains 0 and is invariant under addition). It follows from (3.7) that $\mathscr{C}(X)$ is finitely generated as a monoid.

Let me first discuss a few simple tricks for computing highest-weight sets. Let $X//G = \operatorname{Spec} \mathbb{C}[X]^G$ denote the categorical quotient of X, that is, the variety of closed G-orbits in X, and let $\pi\colon X \to X//G$ be the quotient mapping. (See [17] or [23].) A subset U of X is called *saturated* (with respect to π) if $\pi^{-1}\pi(U) = U$, that is, $\overline{Gx} \subset U$ whenever $x \in U$. Saturated subsets are evidently G-stable. The following result says that $\mathscr{C}(X)$ is determined locally in the Zariski topology, or, more precisely, that it does not change when we remove from X a divisor defined by an invariant polynomial.

LEMMA 4.2. *Assume X is irreducible. Let Y be any saturated affine Zariski-open subvariety of X. Then $\mathscr{C}(X) = \mathscr{C}(Y)$.*

Proof. The coordinate ring of X embeds equivariantly into the coordinate ring of Y. This implies $\mathscr{C}(X) \subset \mathscr{C}(Y)$. For the reverse inclusion, note that the assumption that Y is saturated implies that the quotient $Y//G$ is an affine open subvariety of $X//G$. Let D be the complement of $Y//G$ in $X//G$, let $D_1, D_2, \ldots, D_k$ be the irreducible components of D, and let $f_i \in \mathbb{C}[X//G]$ be the polynomial defining D_i for $i = 1, 2, \ldots, k$. Put $f = f_1 f_2 \cdots f_k$; then the divisor $X - Y = \pi^{-1}(D)$ is the zero set of f (viewed as an element of $\mathbb{C}[X]$), and the coordinate ring of Y is just the localization of $\mathbb{C}[X]$ at f. Define for every p the linear map $\psi_p\colon \mathbb{C}[X]_p \to \mathbb{C}[Y]$ by $\psi_p(a) = a/f^p$. Since X is irreducible, ψ_p is injective. Since f is invariant, ψ_p is equivariant. The direct sum of the maps ψ_p is an equivariant map from $\mathbb{C}[X]$ to $\mathbb{C}[Y]$, which is surjective, because $\mathbb{C}[Y]$ is the localization of $\mathbb{C}[X]$ at f. It follows from this that if an irreducible G-representation occurs in $\mathbb{C}[Y]$, then it occurs in $\mathbb{C}[X]$. This proves $\mathscr{C}(Y) \subset \mathscr{C}(X)$. ∎

The problem of computing highest-weight sets can in principle be reduced to torus actions. The variety $\operatorname{Spec} \mathbb{C}[X]^N$ is a categorical quotient of X by N in the category of affine varieties, and will be denoted by $X//N$. (It is not always the same as the set-theoretical quotient of X by N and the natural map $X \to X//N$ is not always surjective. For instance, the homogeneous space G/N is a quasi-affine variety and $G//N$ is its affine closure.) Let $\mathscr{C}(X//N)$ be the weight set of the H-action on $X//N$. By the theorem of the highest weight, a weight occurs in $\mathbb{C}[X]^N$ if and only if it occurs as the highest weight of an irreducible component of $\mathbb{C}[X]$. This proves the following lemma.

LEMMA 4.3. $\mathscr{C}(X) = \mathscr{C}(X//N)$.

Furthermore, highest-weight sets are invariant up to denominators under finite morphisms.

LEMMA 4.4. *Let X and Y be affine G-varieties and let $\phi: X \to Y$ be a finite surjective G-morphism. Then $\mathscr{C}(Y)$ is contained in $\mathscr{C}(X)$ and $n!\,\mathscr{C}(X)$ is contained in $\mathscr{C}(Y)$, where n is the cardinality of the generic fibre of ϕ.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the coordinate rings of X, resp. Y. Then $\mathfrak{B}$ can be regarded as a subring of $\mathfrak{A}$ via the pull-back map ϕ^*. This implies $\mathscr{C}(Y) \subset \mathscr{C}(X)$. Now for the second inclusion. By Lemma 4.3, it suffices to show that $n!\,\mathscr{C}(X//N) \subset \mathscr{C}(Y//N)$. Recall that the finiteness of ϕ means that $\mathfrak{A}$ is a $\mathfrak{B}$-module of rank n. By Satz 1 on p. 192 of [17], $\mathfrak{A}^N$ is a $\mathfrak{B}^N$-module of rank $\leqslant n$. This implies that every element a of $\mathfrak{A}^N$ satisfies an equation $P(a)=0$, where $P(t)$ is a monic polynomial of degree n in $\mathfrak{B}^N[t]$. Let $a \in \mathfrak{A}^N$ be an element of weight $\lambda \in \Lambda^*$. I will show that $\mathfrak{B}^N$ contains an element of weight $k\lambda$ for some $k \leqslant n$. There exist $b_0, b_1, ..., b_{n-1}$ in $\mathfrak{B}^N$ such that $a^n + b_{n-1}a^{n-1} + \cdots + b_1 a + b_0 = 0$. Let k be the largest number l such that $b_{n-l} \neq 0$. Then

$$a^k + b_{n-1}a^{k-1} + \cdots + b_{n-k+1}a + b_{n-k} = 0.$$

Because the action of H on $\mathfrak{B}^N$ is completely reducible, we may assume all terms in this equation have the same weight. Then the weight of b_{n-k} is equal to the weight of a^k, which is $k\lambda$. ∎

EXAMPLE 4.5. Let Γ be a finite group acting on X and suppose that the actions of G and Γ commute. Let Y be the affine G-variety X/Γ. Then the lemma shows that $\mathscr{C}(Y) \subset \mathscr{C}(X)$ and $n!\,\mathscr{C}(X) \subset \mathscr{C}(Y)$, where n is the cardinality of Γ.

Remark 4.6. If $\phi: X \to Y$ is any surjective G-morphism of affine G-varieties, then $\mathscr{C}(Y)$ is a subset of $\mathscr{C}(X)$.

Remark 4.7. Suppose that G is the direct product of two reductive subgroups G_1 and G_2. Then the monoid of dominant weights of G is simply the product of the monoids of dominant weights of G_1 and G_2, and the positive Weyl chamber of G is the product of the positive Weyl chambers of G_1 and G_2. Moreover, every irreducible representation of G is a tensor product of an irreducible representation of G_1 and an irreducible representation of G_2. It follows that for every affine G_1-variety X_1 and for every affine G_2-variety X_2, $\mathscr{C}(X_1 \times X_2)$ is the product of $\mathscr{C}(X_1)$ and $\mathscr{C}(X_2)$.

Now take any equivariant algebraic closed embedding of X into some representation space V. Such embeddings always exist; see e.g., [17]. Let $\langle \cdot, \cdot \rangle$ be a K-invariant Hermitian inner product on V and let ω_V be the symplectic form $-\mathrm{Im}\langle\cdot,\cdot\rangle$. Denote by $\|\cdot\|$ the corresponding norm, and by Φ_V the momentum map given by (2.2). Now attach a copy of the one-dimensional trivial representation $\mathbb{C}$ to V. Then the projective space $\mathbb{P}(V\oplus\mathbb{C})$ carries a natural G-action, and the projective space $\mathbb{P}V$ can be identified equivariantly with the hyperplane at infinity in $\mathbb{P}(V\oplus\mathbb{C})$. By (2.5), the momentum map on $\mathbb{P}(V\oplus\mathbb{C})$ is given by

$$\Phi_{\mathbb{P}(V\oplus\mathbb{C})}([v,1])=\frac{\Phi_V(v)}{\pi(1+\|v\|^2)}=\frac{\sqrt{-1}}{2\pi}\frac{\langle\xi v,v\rangle}{1+\|v\|^2},\tag{4.1}$$

where $[v,1]$ denotes the line through $(v,1)$. Denote by $\bar{X}$ the closure of X in $\mathbb{P}(V\oplus\mathbb{C})$. Let X_∞ be the divisor at infinity $\bar{X}\cap\mathbb{P}V$ in $\bar{X}$. The highest-weight sets of X, $\bar{X}$ and X_∞, are closely related. If X is irreducible, then $\mathscr{C}(X)$ is a submonoid of Λ_+^*, so we have the equalities

$$\mathrm{cone}_{\mathbb{Q}}\,\mathscr{C}(X)=\mathrm{hull}_{\mathbb{Q}}\,\mathscr{C}(X)=\mathbb{Q}_{\geqslant 0}\cdot\mathscr{C}(X).$$

(See Section 2.2 for the notation.) Also, $\mathbb{Q}_{\geqslant 0}\cdot\mathscr{C}(\bar{X})=\mathrm{cone}_{\mathbb{Q}}\,\mathscr{C}(\bar{X})$, because $\mathscr{C}(\bar{X})$ is $\mathbb{Q}$-convex. As before, let X_∞^{red} denote the reduced variety associated to X_∞. By Addendum 3.8, $\Delta(X_\infty^{\mathrm{red}})$ is a union of convex polytopes, one for each irreducible component of X_∞^{red}.

THEOREM 4.8. *Let X be an irreducible affine G-variety.*

1. *The highest-weight set of $\bar{X}$ is the $\mathbb{Q}$-convex hull of the highest-weight set of X_∞^{red} and the origin in $\mathfrak{t}_+^*$* : $\mathscr{C}(\bar{X})=\mathrm{hull}_{\mathbb{Q}}(\mathscr{C}(X_\infty^{\mathrm{red}})\cup\{0\})$;

2. *the momentum polytope of $\bar{X}$ is the convex hull of the momentum set of X_∞^{red} and the origin in $\mathfrak{t}_+^*$* : $\Delta(\bar{X})=\mathrm{hull}(\Delta(X_\infty^{\mathrm{red}})\cup\{0\})$;

3. *the highest-weight sets of X and $\bar{X}$ span the same cone*: $\mathrm{cone}_{\mathbb{Q}}\,\mathscr{C}(X)=\mathrm{cone}_{\mathbb{Q}}\,\mathscr{C}(\bar{X})$.

Proof. 1. First I show that

$$\mathscr{C}(\bar{X})=\mathrm{hull}_{\mathbb{Q}}\,(\mathscr{C}(X_\infty)\cup\{0\}).\tag{4.2}$$

Let Z be the affine cone on $\bar{X}$ and let Y be the affine cone on X_∞. Let $z\colon V\times\mathbb{C}\to\mathbb{C}$ be the projection onto the second factor and let f be the restriction of z to Z. Then f can be regarded as an element of $\mathbb{C}[Z]_1$ and as such it is an invariant, because G acts trivially on the second factor. The coordinate ring of Y is equal to $\mathbb{C}[Z]/(f)$. From the G-equivariant exact sequence

$$(f)\hookrightarrow\mathbb{C}[Z]\twoheadrightarrow\mathbb{C}[Y]\tag{4.3}$$

it is clear that every irreducible representation occurring in $\mathbb{C}[Y]_n$ also occurs in $\mathbb{C}[Z]_n$. Therefore, $\mathscr{C}(X_\infty) \subset \mathscr{C}(\bar{X})$. Furthermore, $\mathbb{C}[Z]_1$ contains the copy $\mathbb{C}[Z]_0 f$ of the one-dimensional trivial representation, and so $0 \in \mathscr{C}(\bar{X})$. Consequently $\mathscr{C}(\bar{X}) \supset \operatorname{hull}_{\mathbb{Q}}(\mathscr{C}(X_\infty) \cup \{0\})$. Conversely, from (4.3) we see that if $R_{\lambda *}$ occurs in $\mathbb{C}[Z]_n$, then it occurs in either $\mathbb{C}[Z]_{n-1}$ or $\mathbb{C}[Y]_n$. In other words, if $\lambda/n \in \mathscr{C}(\bar{X})$, then $\lambda/(n-1) \in \mathscr{C}(\bar{X})$ or $\lambda/n \in \mathscr{C}(X_\infty)$. This implies that every element of $\mathscr{C}(\bar{X})$ lies on the segment joining an element of $\mathscr{C}(X_\infty)$ to the origin. Thus, $\mathscr{C}(\bar{X}) \subset \operatorname{hull}_{\mathbb{Q}}(\mathscr{C}(X_\infty) \cup \{0\})$. This proves (4.2).

To finish the proof of 1 it is enough to show that

$$\mathscr{C}(X_\infty) \subset \operatorname{hull}_{\mathbb{Q}}(\mathscr{C}(X_\infty^{\mathrm{red}}) \cup \{0\}).$$

I do this by showing that for every $v \in \mathscr{C}(X_\infty)$ there exists a rational $q \geqslant 1$ such that $qv \in \mathscr{C}(X_\infty^{\mathrm{red}})$. Let q be the largest rational number such that $\mu = qv \in \mathscr{C}(X_\infty)$. Such a q exists by Addendum 3.10 and is clearly $\geqslant 1$. There exist $g \in \mathbb{C}[Z]_n^N$ and $\lambda \in \Lambda_+^*$ such that $\mu = \lambda/n$, g transforms according to the weight λ^* under the action of T, and g is not in (f). I assert that g does not vanish identically on Y. For if it did, then by the Nullstellensatz there would exist l such that $g^l \in (f)$. Write $g^l = hf^m$ with $h \in \mathbb{C}[Z]$ and m as large as possible; then $h \notin (f)$, so $h + (f)$ is a nonzero element of $\mathbb{C}[Y] = \mathbb{C}[Z]/(f)$. Since f is an invariant of degree one, the weight of $h + (f)$ is equal to $l\lambda$ and its degree is $nl - m$. Therefore,

$$\frac{nlqv}{nl-m} = \frac{nl\mu}{nl-m} = \frac{l\lambda}{nl-m} \in \mathscr{C}(X_\infty),$$

which contradicts the maximality of q. We conclude that $g(y) \neq 0$ for some y in Y. This means that g represents a nonzero element of $\mathbb{C}[Y^{\mathrm{red}}]$, and hence $\mu \in \mathscr{C}(X_\infty^{\mathrm{red}})$.

2. This follows immediately from 1 and Theorem 3.7.

3. Note first that $\operatorname{cone} \mathscr{C}(\bar{X}) = \operatorname{cone} \mathscr{C}(Z)$, because Z is the affine cone on $\bar{X}$. Furthermore, the affine G-variety $Z - Y$ is saturated in Z, because Y is defined as the zero set of the invariant function f. Therefore, $\mathscr{C}(Z - Y) = \mathscr{C}(Z)$ by Lemma 4.2. Moreover, the G-equivariant map from $V \oplus \mathbb{C}$ to itself sending (x, t) to (tx, t) maps $X \times \mathbb{C}^\times$ isomorphically onto $Z - Y$. It follows that $\mathbb{C}[Z - Y]$ is isomorphic to $\mathbb{C}[X] \otimes \mathbb{C}[f, f^{-1}]$ as a G-algebra, where G acts trivially on $\mathbb{C}[f, f^{-1}]$. This implies $\mathscr{C}(Z) = \mathscr{C}(X)$. In sum, we have shown that $\operatorname{cone} \mathscr{C}(X) = \operatorname{cone} \mathscr{C}(\bar{X})$. ∎

The proof shows that $\Delta(\bar{X})$ is in fact equal to the *join* of $\Delta(X_\infty^{\mathrm{red}})$ with the origin in $\mathfrak{t}_+^*$, that is, the union of all intervals joining points in $\Delta(X_\infty^{\mathrm{red}})$ to the origin.

We now state the main result of this section.

THEOREM 4.9. *For every G-stable irreducible closed affine subvariety X of V the set $\Delta(X)$ is equal to* cone $\mathscr{C}(X)$. *In particular, it is a rational convex polyhedral cone.*

Proof. Because the monoid $\mathscr{C}(X)$ is finitely generated, it spans a rational convex polyhedral cone in $\mathfrak{t}^*$. To prove that $\Delta(X) = \operatorname{cone} \mathscr{C}(X)$ it suffices to prove that

$$\Delta(X) \subset \operatorname{cone} \mathscr{C}(X); \tag{4.4}$$

$$\operatorname{hull}\{\lambda_1, ..., \lambda_k\} \subset b\,\Delta(X) \qquad \text{for all} \quad b > 0 \quad \text{and} \quad \lambda_1, ..., \lambda_k \quad \text{in } \mathscr{C}(X). \tag{4.5}$$

Proof of (4.4). By 4.1, $\Delta(X)$ is a subset of the cone on $\Delta(\bar{X})$. By Theorem 3.7, $\Delta(\bar{X})$ is equal to the closure of $\mathscr{C}(\bar{X})$ in $\mathfrak{t}^*_+$, so by 3 of Theorem 4.8, the cone on $\Delta(\bar{X})$ is equal to the convex hull of $\mathscr{C}(X)$. Consequently, $\Delta(X)$ is a subset of the convex hull of $\mathscr{C}(X)$.

Proof of (4.5). Let b be any positive number, let $\mu \in \mathfrak{t}^*_+$ and let Y be the product $V \times K\mu$ with symplectic form $\sigma = b\omega_V + \omega_\mu$ and momentum map $\Psi = b\Phi_V + \iota_\mu$. I assert that

$$\text{the momentum map } \Psi \text{ is admissible for all } b > 0 \text{ and } \mu \in \mathfrak{t}^*_+. \tag{4.6}$$

Assuming this for the moment, let us consider the trivial line bundle $\mathcal{O}_V = V \times \mathbb{C}$ on V with the Hermitian metric defined by the Gaussian $h(v) = \exp(-\pi b\,\|v\|^2)$. The curvature form of $(\mathcal{O}_V, h)$ is $-2\pi\sqrt{-1}\,b\omega_V$. Lift the K-action on V to $\mathcal{O}_V$ by letting K act trivially on the fibre $\mathbb{C}$. Then the fibre metric is K-invariant, the K-invariant (or G-invariant) holomorphic sections of $\mathcal{O}_V$ are just the G-invariant holomorphic functions on V, and the associated momentum map is $b\Phi_V$.

Let us apply this to the special case $\mu = \sum_i a_i \mu_i$, where the a_i are nonnegative numbers and the μ_i are in $\mathscr{C}(X)$. Consider the varieties $Y_i = V \times K\mu_i^*$, on which we have the line bundles $L_i = \mathcal{O}_V \otimes \mathcal{O}_{\mu_i^*}$, symplectic forms $\sigma_i = b\omega_V + \omega_{\mu_i^*}$ and momentum maps $\Psi_i = b\Phi_V + \iota_{\mu_i^*}$. By (4.6), the momentum map $\Psi = \sum_i a_i p_i^* \Psi_i$ on $Y = V \times K\mu^*$ is admissible for all $b > 0$. Moreover, by the definition of $\mathscr{C}(X)$,

$$\begin{aligned} &\mu_i \in \mathscr{C}(X) \\ &\quad \Leftrightarrow \text{there exists a } G\text{-equivariant linear surjection } \mathbb{C}[X] \to R_{\mu_i} \\ &\quad \Leftrightarrow \text{there exists a nonzero } G\text{-invariant vector in } \mathbb{C}[X] \otimes R_{\mu_i^*} \\ &\quad \Leftrightarrow \text{there exists a nonzero } G\text{-invariant algebraic section of } \mathcal{O}_X \otimes \mathcal{O}_{\mu_i^*}. \end{aligned}$$

Remark 3.5 now implies that for all $b>0$ the polytope $\operatorname{hull}\{\mu_1, \mu_2, \ldots, \mu_k\}$ is contained in $b\Psi(X)$.

Proof of (4.6). After rescaling Ψ we may assume that $b=1$. Let $\mathfrak{F}(t, \cdot)$ be the flow of $-\operatorname{grad}|\Psi|^2$. We have to show that for every y in Y the trajectory $\mathfrak{F}(t, y)$ is contained in a compact subset of $Y=V\times K\mu$. Since $K\mu$ is compact, we need only show that the projection of $\mathfrak{F}(t, y)$ onto V is contained in a compact subset of V. For any function u on Y, let $\operatorname{grad}_V u$ denote the component of $\operatorname{grad} u$ along V. Then using (3.2) we find for every pair (v, β) in Y

$$\begin{aligned}\operatorname{grad}_V|\Psi(v,\beta)|^2 &= \operatorname{grad}_V|\Phi_V(v)+\beta|^2\\ &= \operatorname{grad}_V(|\Phi_V(v)|^2+2(\Phi_V(v),\beta)+|\mu|^2)\\ &= \operatorname{grad}_V|\Phi_V(v)|^2+2J\beta^\flat_{V,v}.\end{aligned}$$

This implies

$$\begin{aligned}\langle \operatorname{grad}_V|\Psi(v,\beta)|^2, v\rangle &= 4\,|\Phi_V(v)|^2+2\langle v, J\beta^\flat_{V,v}\rangle\\ &= 4\,|\Phi_V(v)|^2+2\langle v, \operatorname{grad}\Phi_V^{\beta^\flat}(v)\rangle\\ &= 4\,|\Phi_V(v)|^2+4\Phi_V^{\beta^\flat}(v)\\ &= 4\,|\Phi_V(v)|^2+4(\Phi_V(v),\beta)\\ &= 4\left|\Phi_V(v)+\frac{\beta}{2}\right|^2-|\mu|^2, \end{aligned} \tag{4.7}$$

where I have used the fact that Φ^ξ is homogeneous of degree two for all ξ in $\mathfrak{k}$, and that $|\Phi_V|^2$ is homogeneous of degree four. Now suppose (v, β) is a point where $\langle \operatorname{grad}_V|\Psi(v,\beta)|^2, v\rangle\leqslant 0$. Then it follows from (4.7) that $\Phi_V(v)$ is contained in the ball of radius $|\mu|/2$ about the point $-\beta/2\in\mathfrak{k}^*$. Therefore, $\Phi_V(v)$ is contained in the ball of radius $|\mu|$ about the point $-\beta$. In other words, $|\Phi_V(v)+\beta|^2\leqslant|\mu|^2$, that is, $|\Psi(v,\beta)|^2\leqslant|\mu|^2$. In short,

$$\langle \operatorname{grad}_V|\Psi(v,\beta)|^2, v\rangle\leqslant 0 \Rightarrow |\Psi(v,\beta)|^2\leqslant|\mu|^2. \tag{4.8}$$

Now let $\gamma(t)$ be the projection onto V of the trajectory $\mathfrak{F}(t,(v,\beta))$ through any point $(v,\beta)\in V\times K\mu$. It follows from (4.8) that we have the following two (non-exclusive) possibilities: $\langle \operatorname{grad}_V|\Psi(\mathfrak{F}(t,(v,\beta)))|^2, v\rangle>0$ for all $t>0$, or $|\Psi(\mathfrak{F}(s,(v,\beta)))|^2\leqslant|\mu|^2$ for some $s>0$. In the first case, the curve $\gamma(t)$ is trapped inside the ball of radius $\|v\|$ about the origin in V. In the second case, $|\Psi(\mathfrak{F}(t,(v,\beta)))|^2\leqslant|\mu|^2$ for all $t\geqslant s$, because $|\Psi|^2$ is decreasing along $\mathfrak{F}(t,(v,\beta))$. This implies that $|\Phi_V|^2(\gamma(t))\leqslant 4\,|\mu|^2$ for all $t\geqslant s$.

Moreover, $\gamma(t)$ is contained in the G-orbit through v. It now follows from Lemma 4.10 below that $\gamma(t)$ is contained in a compact subset of V. ∎

The following lemma implies that for every point v in V the restriction of Φ_V to the affine variety $\overline{Gv}$ is a proper map.

LEMMA 4.10. *For every bounded subset D of $\mathfrak{k}^*$ and for every bounded subset B of V the intersection $\Phi_V^{-1}(D) \cap GB$ is a bounded subset of V.*

Proof. Let B be any bounded subset of V. Let $\{g(n)\}_{n \geqslant 0}$ be a sequence of elements of G, let $\{v(n)\}_{n \geqslant 0}$ be a sequence of vectors in B, and put $f(n) = \|g(n)\, v(n)\|^2$. Suppose that $\lim_{n \to \infty} f(n) = \infty$. We need to show that the sequence $\{\Phi_V(g(n)\, v(n))\}_{n \geqslant 0}$ is unbounded. By the Cartan decomposition, $G = K \exp(\sqrt{-1}\, \mathfrak{t})\, K$, so we can write $g(n) = k(n) \exp(\sqrt{-1}\, \xi(n))\, h(n)$, where $k(n)$, $h(n) \in K$ and $\xi(n) \in \mathfrak{t}$. Choose an orthonormal basis $\{e_i\}$ of V with respect to which the T-action is diagonal. Then there exist $\beta_i \in \mathfrak{t}^*$ such that $\xi e_i = \sqrt{-1}\, \beta_i(\xi)\, e_i$ for all $\xi \in \mathfrak{t}$. Write $h(n)\, v(n) = \sum_i v_i(n)\, e_i$ and $\rho = \sup\{\|v\| : v \in B\}$; then

$$|v_i(n)| \leqslant \rho \qquad \text{for all } i \text{ and } n,$$

$$g(n)\, v = k(n) \sum_i \mathrm{e}^{\beta_i(\xi(n))} v_i(n)\, e_i,$$

$$f(n) = \sum_i \mathrm{e}^{2\beta_i(\xi(n))}\, |v_i(n)|^2.$$

Consider the set I consisting of all i such that the sequence $\exp(2\beta_i(\xi(n)))\, |v_i(n)|^2$ is unbounded. Then I is nonempty, because $\lim_{n \to \infty} f(n) = \infty$. After replacing $\{g(n)\}$ and $\{v(n)\}$ by suitable subsequences we may assume that

$$\lim_{n \to \infty} \mathrm{e}^{2\beta_i(\xi(n))}\, |v_i(n)|^2 = \infty$$

for all $i \in I$. Then $\lim_{n \to \infty} \beta_i(\xi(n)) = \infty$ for all $i \in I$, because $|v_i(n)|$ is bounded. This implies there exists an $\eta \in \mathfrak{t}$ with $\beta_i(\eta) > 0$ for all $i \in I$. We may assume η has length 1. Then $k(n)^{-1}\, \eta$ has length 1, so

$$|\Phi_V(g(n)\, v)|^2 \geqslant |\Phi^{k(n)^{-1} \eta}(g(n)\, v)|^2 = |\Phi^{\eta}(\exp(\sqrt{-1}\, \xi(n))\, h(n)\, v)|^2. \tag{4.9}$$

A straightforward computation using (2.3) shows that for all $\eta \in \mathfrak{t}$

$$\Phi^{\eta}(\exp(\sqrt{-1}\, \xi(n))\, h(n)\, v) = \tfrac{1}{2} \sum_i \beta_i(\eta)\, \mathrm{e}^{2\beta_i(\xi(n))}\, |v_i(n)|^2. \tag{4.10}$$

The vector η was chosen in such a way that all unbounded terms in the right-hand side of (4.10) tend to ∞ for $n \to \infty$. It now follows from (4.9) that $|\Phi_V(g(n)\, v)|^2$ tends to ∞ for $n \to \infty$. ∎

COROLLARY 4.11. *The momentum cone of X is the cone over the momentum polytope of the projectivization of X: $\Delta(X) = \operatorname{cone} \Delta(\bar{X}) = \mathbb{Q}_{\geqslant 0} \cdot \Delta(\bar{X})$.*

Proof. Combine Theorem 4.8.3 with Theorem 4.9. ∎

COROLLARY 4.12. *The momentum cone of X is a closed subset of $\mathfrak{t}^*_+$, and it does not depend on the embedding of X into the unitary K-module V.*

COROLLARY 4.13. *If X is normal, all fibres of the momentum map Φ_X are connected.*

Proof. The function $|\Psi|^2 = |\Phi_V + \iota_\mu|^2$ is real-algebraic on $Y = V \times K\mu$ and has therefore only finitely many critical levels. This implies that the Morse decomposition of Y with respect to $|\Psi|^2$ is finite. By the proof of Theorem 4.9, the momentum map Ψ is admissible. It now follows from the results quoted in Section 2.5 that its zero level, which is $K\Phi_V^{-1}(-\mu)$, is a deformation retract of an open subset of V the complement of which is a finite union of complex-analytic subsets of positive codimension. The same holds with V replaced by X, because the flow of $|\Psi|^2$ leaves $X \times K\mu$ invariant. Since X is normal, the complement of a finite number of analytic subsets is always connected. This implies that $K\Phi_X^{-1}(-\mu)$, and hence $\Phi_X^{-1}(-\mu)$, are connected. ∎

Remark 4.14. The zero fibre $\Phi_X^{-1}(0)$ is connected regardless of whether X is normal. The reason is that the function $|\Phi_V|^2$ has only one critical level, namely 0.

COROLLARY 4.15. *Let Y be a saturated Zariski-open subvariety of X. Then $\Delta(Y) = \Delta(X)$.*

Proof. Evidently, $\Delta(Y)$ is contained in $\Delta(X)$. For the reverse inclusion it suffices to show that $\operatorname{hull}\{\lambda_1, ..., \lambda_k\} \subset b\, \Delta(Y)$ for all $b > 0$ and $\lambda_1, ..., \lambda_k$ in $\mathscr{C}(X)$. The proof of this fact is a straightforward generalization of the proof of (4.5). (Cf. Remark 3.5.) ∎

EXAMPLE 4.16. Suppose the affine G-variety X is defined over the real numbers in the sense that the complex G-algebra $\mathbb{C}[X]$ is the complexification of a real K-algebra of finite type. Then complex conjugation defines an antilinear involution on the G-module $\mathbb{C}[X]$, so whenever an irreducible representation R_λ occurs in $\mathbb{C}[X]$, its contragredient representation R_λ^* also occurs. It follows from this that the monoid $\mathscr{C}(X)$ is invariant under the involution $*: \Lambda_+^* \to \Lambda_+^*$. Therefore, by Theorem 4.9, the cone $\Delta(X)$ is invariant under the involution $*: \mathfrak{t}_+^* \to \mathfrak{t}_+^*$. This can also be shown directly as follows. We may assume the embedding of X into the G-module V to be defined over the real numbers (in the sense that both V and the G-morphism $X \to V$ are defined over the reals). Then X is invariant under complex conjugation on V. From (2.2) one deduces immediately that $\Phi_V^\xi(\bar{v}) = -\Phi_V^\xi(v)$. Hence $\Phi_V(X) = -\Phi_V(X)$, and therefore $\Delta(X)^* = -w_0\,\Delta(X) = \Delta(X)$.

Example 4.17 (Peter-Weyl). The group G is an affine variety in its own right, and it acts on itself by left multiplication: $\mathfrak{L}_g h = gh$, and by right multiplication: $\mathfrak{R}_g h = hg^{-1}$. Consider the $\mathfrak{L} \times \mathfrak{R}$-action of $G \times G$ on G. Let us denote the highest-weight set of G for this action by $\mathscr{C}(G, \mathfrak{L} \times \mathfrak{R})$ and the momentum cone (with respect to any algebraic $G \times G$-equivariant embedding of G into a unitary $K \times K$-module) by $\Delta(G, \mathfrak{L} \times \mathfrak{R})$. The monoid of dominant weights of $G \times G$ is simply the product $\Lambda_+^* \times \Lambda_+^*$ and its positive Weyl chamber is $\mathfrak{t}_+^* \times \mathfrak{t}_+^*$. By the Peter-Weyl Theorem the coordinate ring of G is a direct sum of irreducible $G \times G$-modules: $\mathbb{C}[G] = \bigoplus_{\lambda \in \Lambda_+^*} R_\lambda \otimes R_\lambda^*$. This implies that the highest-weight monoid $\mathscr{C}(G, \mathfrak{L} \times \mathfrak{R})$ is equal to the subset $\{(\lambda, \lambda^*) : \lambda \in \Lambda_+^*\}$ of $\Lambda_+^* \times \Lambda_+^*$. By Theorem 4.9, the momentum cone $\Delta(G, \mathfrak{L} \times \mathfrak{R})$ is therefore the "anti-diagonal" $\{(\mu, \mu^*) : \mu \in \mathfrak{t}_+^*\}$ inside $\mathfrak{t}_+^* \times \mathfrak{t}_+^*$. Notice that this set is $*$-invariant as it should be, because G is defined over the real numbers.

Example 4.18. We use the notation of the previous example. There are three different embeddings of G into $G \times G$: the maps $i_1(g) = (g, 1)$, $i_2(g) = (1, g)$ and $d(g) = (g, g)$. Pulling back the $\mathfrak{L} \times \mathfrak{R}$-action via these three embeddings yields three actions of G on itself: the actions $\mathfrak{L}$ (left multiplication), $\mathfrak{R}$ (right multiplication) and $\mathfrak{C}$ (conjugation). The momentum maps for these actions are obtained by composing the $\mathfrak{L} \times \mathfrak{R}$-momentum map with the maps i_1^*, i_2^* and d^*, respectively. Thus we find that $\Delta(G, \mathfrak{L})$ and $\Delta(G, \mathfrak{R})$ are equal to the positive Weyl chamber, $\mathfrak{t}_+^*$, and $\Delta(G, \mathfrak{C})$ is the positive Weyl chamber of the semisimple part of K: $\Delta(G, \mathfrak{C}) = \mathfrak{t}_+^* \cap [\mathfrak{k}, \mathfrak{k}]$.

Example 4.19 (Gelfand's variety $G//N$). The action $\mathfrak{L}$ of G on itself descends to an action of G on $G//N$. Every irreducible G-module occurs exactly once in the coordinate ring $R = \mathbb{C}[G]^N$, so, once again, $\mathscr{C}(G//N) = \Lambda_+^*$ and $\Delta(G//N) = \mathfrak{t}_+^*$.

We can embed $G//N$ into affine space and compute the momentum polytope of its projectivization. First assume G is semisimple and simply connected. Then the algebra R is generated by the subspace $E = \bigoplus_{i=1}^r R_{\pi_i}$, where $\pi_1, \pi_2, \ldots, \pi_r$ are the fundamental weights of G and r is the rank of G.

Choose a highest-weight vector v_i in each of the R_{π_i}. Consider the left-G-equivariant map from G to E defined by sending the identity of G to the vector $v_1 \oplus v_2 \oplus \cdots \oplus v_r$. This map is right-$N$-equivariant, so it descends to a map from $G//N$ to E, which is by construction an embedding. Let us identify $G//N$ with its image in E. It is not hard to show that the subvariety $G//N$ is invariant under the standard $\mathbb{C}^\times$-action on E, and the divisor at infinity $(G//N)_\infty$ is therefore the quotient $(G//N - \{0\})/\mathbb{C}^\times$. In other words, $G//N$ is the affine cone on $(G//N)_\infty$. It now follows immediately from Theorem 3.7 that the momentum polytope of $(G//N)_\infty$ (with respect to any K-invariant inner product on E) is the $r-1$-dimensional simplex spanned by the fundamental weights. By Theorem 4.8, the momentum polytope of the projective closure of $G//N$ is therefore the r-dimensional simplex spanned by the fundamental weights and the origin in $\mathfrak{t}^*_+$.

Now assume G is a torus of dimension k. Then the subgroup N is trivial and so $R = \mathbb{C}[G]$ and $G//N = G$. Let $\zeta_1, \zeta_2, ..., \zeta_k$ be a basis over $\mathbb{Z}$ of the weight lattice Λ^*, and identify $\mathfrak{t}^*$ with $\mathbb{R}^k$ by sending this basis to the standard basis in $\mathbb{R}^k$. This choice of basis gives an identification of G with the product $(\mathbb{C}^\times)^k = \{(t_1, t_2, ..., t_k) : t_i \in \mathbb{C}^\times\}$. A *closed* affine embedding of G is given by sending $(t_1, t_2, ..., t_k)$ to $(t_1, t_1^{-1}, t_2, t_2^{-1}, ..., t_k, t_k^{-1}) \in \mathbb{C}^{2k}$. The projective closure of G in $\mathbb{P}^{2k}$ is a product of k copies of $\mathbb{P}^1$. The divisor at infinity G_∞ contains 2^k fixed points for the action of G, whose images under the momentum map are the points $\pm\zeta_1 \pm \zeta_2 \pm \cdots \pm \zeta_k$. Theorem 4.8 now implies that the momentum polytope of the projective closure of G is the parallelepiped spanned by these 2^k points.

For an arbitrary connected reductive group G, the variety $G//N$ can be embedded into affine space in a similar way, by choosing a basis of the monoid of highest weights. One can show that the momentum polytope of the projective closure of $G//N$ under such an embedding is the product of the simplex spanned by the origin and the fundamental weights of $[\mathfrak{k}, \mathfrak{k}]$, and the parallelepiped spanned by the points $\pm\zeta_1 \pm \zeta_2 \pm ...$, where the ζ_i are a basis of the weight lattice of $\mathfrak{z}(\mathfrak{k})$, the centre of $\mathfrak{k}$.

EXAMPLE 4.20 (associated bundles). Let F be a reductive subgroup of G and let Y be an affine F-variety. Consider the bundle $X = G \times^F Y$ associated to the principal fibration $F \to G \to G/F$. The action $\mathfrak{L}$ of G on itself induces a G-action on X. Also, X is an affine variety with coordinate ring $\mathbb{C}[X] = (\mathbb{C}[G] \otimes \mathbb{C}[Y])^F$. Note that if F^0 is the identity component of F, there is a finite map $G \times^{F^0} Y \to X$, so $\Delta(X) = \Delta(G \times^{F^0} Y)$ by Lemma 4.4. This means we may assume F to be connected. The categorical quotient of X by N has coordinate ring $\mathbb{C}[X]^N = (\mathbb{C}[G] \otimes \mathbb{C}[Y])^{F \times N}$, which is isomorphic to $(R \otimes \mathbb{C}[Y])^F$, where R is the ring $\mathbb{C}[G]^N$, on which F acts by *left* multiplication. By Lemma 4.3 and Theorem 4.9, the momentum cone of X is therefore the convex cone spanned by the weights of the action of the maximal torus H on the algebra $(R \otimes \mathbb{C}[Y])^F$ defined by right multiplication on R. In general, this is hard to calculate explicitly.

EXAMPLE 4.21 (tori). In the setting of the previous example, let us assume that $G = H$ is a torus, and let us write $F = H_1$. As noted above, we may assume H_1 to be connected. Then H_1 is the complexification of a subtorus T_1 of T. We let T_2 be the quotient T/T_1 and identify it with a complement of T_1 in T, so that $T \cong T_1 \times T_2$. Put $H_2 = (T_2)^{\mathbb{C}}$. Then $H \cong H_1 \times H_2$ and

$$X = H \times^{H_1} Y \cong (H_1 \times H_2) \times^{H_1} Y = H_2 \times (H_1 \times^{H_1} Y) = H_2 \times Y.$$

Therefore, by Remark 4.7,

$$\mathscr{C}(X) \cong \mathscr{C}(H_2) \times \mathscr{C}(Y) = \Lambda_2^* \times \mathscr{C}(Y),$$

where Λ_2^* is the weight lattice of H_2. Consequently, $\Delta(X) \cong \mathfrak{t}_2^* \times \Delta(Y)$. (These identifications depend on the splitting $H \cong H_1 \times H_2$. An invariant way of stating these facts is: X is a trivial principal H_2-bundle over Y; $\mathscr{C}(X)$ is equal to the preimage of $\mathscr{C}(Y)$ under the canonical projection $\Lambda^* \to \Lambda_1^*$; and $\Delta(X)$ is equal to the preimage of $\Delta(Y)$ under the canonical projection $\mathfrak{t}^* \to \mathfrak{t}_1^*$.) If Y is a vector space, then by (2.4), $\Delta(X) \cong \mathfrak{t}_2^* \times -\mathrm{cone}\{v_1, ..., v_l\}$, where $v_1, ..., v_l$ are the weights of the H_1-action on Y.

4.2. *The Momentum Cone and Étale Slices*

Remarkably, the momentum cone $\Delta(X)$ of an affine G-variety X turns out to be entirely determined by infinitesimal data at any point on a closed G-orbit. I shall deduce this from Luna's étale slice theorem. First I discuss a variation on Lemma 4.4.

PROPOSITION 4.22. *Let X and Y be affine G-varieties, let $\phi: X \to Y$ be a G-morphism, let x be a point in X, and let $y = \phi(x)$. Suppose that ϕ has finite fibres, that the image of ϕ is open in Y, and that the orbits Gx and Gy are closed. Then $\mathscr{C}(Y)$ is contained in $\mathscr{C}(X)$, and $\mathscr{C}(X)$ is contained in the cone on $\mathscr{C}(Y)$.*

Proof. The orbit Gy and the complement of $\phi(X)$ are G-stable Zariski-closed subsets of Y. Because G-invariant polynomial functions separate G-stable Zariski-closed subsets, there exists an $f \in \mathbb{C}[Y]^G$ that vanishes outside $\phi(X)$ and satisfies $f(y) = 1$. Let $Y' = Y_f$ and $X' = \phi^{-1}(Y)$. Then X' and Y' are saturated affine open subsets of X, resp. Y, containing the orbits Gx, resp. Gy, and the restriction of ϕ to X' is surjective onto Y'. Then $\mathscr{C}(X') = \mathscr{C}(X)$ and $\mathscr{C}(Y') = \mathscr{C}(Y)$ by Lemma 4.2. We are therefore reduced to proving that $\mathscr{C}(Y') \subset \mathscr{C}(X')$ and $\mathscr{C}(X') \subset \mathrm{cone}\, \mathscr{C}(Y')$.

Let $\mathfrak{C}$ be the integral closure of $\mathbb{C}[Y']$ in $\mathbb{C}[X']$, and let $Z = \operatorname{Spec} \mathfrak{C}$. By Luna's equivariant version of Zariski's Main Theorem ([23], part I), the natural maps $\iota: X' \to Z$ and $\psi: Z \to Y'$ have the following properties: ι is an open immersion, ψ is a finite morphism, and $\phi = \psi \circ \iota$. Also, ψ is surjective, because $\phi: X' \to Y'$ is. Consequently, $\mathscr{C}(Y') \subset \mathscr{C}(Z)$ and $\mathscr{C}(Z) \subset$ cone $\mathscr{C}(Y')$ by Lemma 4.4. So if we can show that $\mathscr{C}(X') = \mathscr{C}(Z)$, we are done. Let us identify X' with its image $\iota(X')$ in Z. The orbit $Gx \subset X'$ is closed in Z (cf. [23], p. 94): since Gy is closed in Y' and ψ is finite, $\psi^{-1}(Gy)$ is closed in Z and consists of a finite number of orbits, one of which is Gx. The conclusion is that Gx and the complement of X' in Z are G-stable closed subsets of Z. This implies the existence of a G-invariant $h \in \mathbb{C}[Z]$ that vanishes outside X' and satisfies $h(x) = 1$. Then X'_h is a G-stable affine open subset of X', and it is saturated as a subset of both X' and Z. Hence, by Lemma 4.2, $\mathscr{C}(X') = \mathscr{C}(X'_h) = \mathscr{C}(Z)$. ∎

THEOREM 4.23. *Let X be an affine G-variety, let x be a point on a closed G-orbit, and let S_x be an étale slice at x. Then the momentum cone of X is equal to the momentum cone of $G \times^{G_x} S_x$. If x is a smooth point of X, then $\Delta(X) = \Delta(G \times^{G_x} V_x)$, where V_x is the tangent space to S_x at x.*

Proof. By Luna's Etale Slice Theorem the natural map from the bundle $G \times^{G_x} S_x$ into X is étale and its image is Zariski-open. Furthermore the G-orbits through the point $[1, x]$ in $G \times^{G_x} S_x$ and the point x in X are closed. It now follows from Proposition 4.22 that $\mathscr{C}(X)$ and $\mathscr{C}(G \times^{G_x} S_x)$ span the same cone. Hence $\Delta(X) = \Delta(G \times^{G_x} S_x)$ by Theorem 4.9.

If x is a smooth point of X, we may assume the étale slice S_x to be smooth, and there exists a G_x-equivariant étale morphism $\psi: S_x \to V_x$ with Zariski-open image. The map ψ extends to a G-equivariant map $G \times^{G_x} S_x \to G \times^{G_x} V_x$, which is étale and has Zariski-open image as well. Again by Proposition 4.22, $\mathscr{C}(G \times^{G_x} S_x)$ and $\mathscr{C}(G \times^{G_x} S_x)$ span the same cone. We conclude that $\Delta(X) = \Delta(G \times^{G_x} V_x)$. ∎

COROLLARY 4.24. *The cone on $\mathscr{C}(G \times^{G_x} S_x)$ is independent of the point x. Here x ranges over the set of all points in X whose G-orbit are closed.*

The following result is a necessary condition for the origin to be an extreme point of $\Delta(X)$. Here $[G, G]$ denotes the commutator subgroup of G. Note that for every subgroup F of G, $[G, G]\, F$ is a subgroup of G, because $[G, G]$ is normal. If F is a closed reductive subgroup, then so is $[G, G]\, F$.

THEOREM 4.25. *Assume that $\Delta(X)$ is a proper cone. Then for every point x such that Gx is closed the following condition holds: $G = [G, G]\, G_x$.*

Proof. Let x be any point such that Gx is closed. Let Y denote the homogeneous space G/G_x and Z the homogeneous space $G/[G, G]\, G_x$. Consider the maps

$$X \xleftarrow{\iota} Y \xrightarrow{\tau} Z,$$

where ι is the G-map sending the coset $1G_x$ to x and τ is the canonical projection. Clearly, $\Delta(Y)$ is a subset of $\Delta(X)$ and, by Remark 4.6 and Theorem 4.9, $\Delta(Z)$ is a subset of $\Delta(Y)$. Therefore, since $\Delta(X)$ is a proper cone, so is $\Delta(Z)$. On the other hand, the torus $G/[G, G]$ acts transitively on Z, so $\Delta(Z)$ is a vector space. (Cf. Example 4.21.) We conclude that Z is a point, in other words, $G = [G, G]\, G_x$. ▮

Note that if G is semisimple, the condition $G = [G, G]\, G_x$ is void. This is as it should be, because in this case the positive Weyl chamber $\mathfrak{t}^*_+$ is a proper cone, so every cone contained in it is a proper cone.

5. STEIN VARIETIES

In this section I prove a convexity theorem for certain Stein K-varieties, Theorem 5.4. It can be regarded as a local version of Theorem 4.9. The results are far from optimal, but will be sufficient for our purposes. Let me start with a number of elementary observations on Kähler potentials and momentum maps.

LEMMA 5.1. *Suppose Y is a connected complex manifold and ρ a strictly plurisubharmonic function on Y. Let σ be the Kähler form $\sqrt{-1}\,\partial\,\bar{\partial}\rho$ with associated Riemannian metric $\langle\cdot,\cdot\rangle$, and let ϑ be the Hamiltonian vector field of ρ. Then the vector field $J\vartheta = \operatorname{grad}\rho$ is expanding: $\mathscr{L}_{J\vartheta}\sigma = 2\sigma$.*

Proof. Let $J\colon TY \to TY$ denote the complex structure on Y and also the transpose operator $T^*Y \to T^*Y$. For all functions f,

$$\bar{\partial} f = \tfrac{1}{2}(d + \sqrt{-1}\, Jd)\, f$$

and (5.1)

$$dJ\, df = -2\sqrt{-1}\, d\bar{\partial} f = -2\sqrt{-1}\, \partial\bar{\partial} f.$$

Moreover, for all functions f and all tangent vectors η,

$$\sigma(\operatorname{grad} f, \eta) = -\langle \operatorname{grad} f, J\eta\rangle = -df(J\eta) = -J\, df(\eta),$$

and therefore $J\,df = -\iota_{\operatorname{grad} f}\sigma$. Together with (5.1) this implies that $\mathscr{L}_{\operatorname{grad}\rho}\sigma = d\iota_{\operatorname{grad}\rho}\sigma = -dJ\,d\rho = 2\sqrt{-1}\,\partial\bar{\partial}\rho = 2\sigma$. ∎

Let me add to this that the vector fields ϑ and $J\vartheta$ are usually not holomorphic.

The symplectic form on Y being exact, every symplectic action of K on Y has a momentum map Ψ. It turns out that the flow of the vector field $-\operatorname{grad}\rho$ has the peculiar property that it pushes forward under Ψ to a flow on $\mathfrak{k}^*$, which retracts the image of Ψ exponentially to a single point in $\mathfrak{z}(\mathfrak{k})^*$, where $\mathfrak{z}(\mathfrak{k})$ is the centre of $\mathfrak{k}$.

PROPOSITION 5.2. *Let Y and ρ be as in Lemma* 5.1. *Suppose that K acts holomorphically on Y, leaving ρ invariant. Put $\alpha = -\sqrt{-1}\,\bar{\partial}\rho$ and $\Psi^\xi = \iota_{\xi_Y}\alpha$ for all $\xi \in \mathfrak{k}$. Let $\mathfrak{G}(t,\cdot) = \mathfrak{G}_t(\cdot)$ denote the flow of $-J\vartheta = -\operatorname{grad}\rho$. Then*

1. *the functions Ψ^ξ are the components of an equivariant momentum map Ψ for the K-action on Y with respect to the symplectic form σ;*

2. *there exists a character c of $\mathfrak{k}$ such that $\mathscr{L}_{J\vartheta}\Psi = 2(\Psi + c)$ (cf.* [10], §3). *Therefore*

$$\mathfrak{G}_t^*\Psi = \mathrm{e}^{-2t}\Psi + (\mathrm{e}^{-2t} - 1)\,c \tag{5.2}$$

for all t such that $\mathfrak{G}_t$ is defined. It follows that $\mathfrak{G}_t$ maps fibres of Ψ to fibres of Ψ. Moreover, if $\mathfrak{G}_t$ is defined for all $t \geqslant 0$, then $\lim_{t\to\infty}\mathfrak{G}_t^*\Psi$ is the constant map $-c$;

3. *if $c = 0$, then the critical set of ρ is contained in the fibre $\Psi^{-1}(0)$. The converse holds if ρ has at least one critical point.*

Proof. Note first that $\sigma = -d\alpha$. Further, since ρ is K-invariant and K acts holomorphically, α is K-invariant. Consequently,

$$d\Psi^\xi = d\iota_{\xi_Y}\alpha = \mathscr{L}_{\xi_Y}\alpha - \iota_{\xi_Y}d\alpha = -\iota_{\xi_Y}d\alpha = \iota_{\xi_Y}\sigma,$$

so ξ_Y is the Hamiltonian vector field of the function Ψ^ξ. An easy calculation shows that $\{\Psi^\xi, \Psi^\eta\} = \Psi^{[\xi,\eta]}$, so Ψ is K-equivariant. This proves 1.

Note that since ϑ is the Hamiltonian vector field of the K-invariant function ρ, the induced vector field ξ_Y commutes with ϑ for all $\xi \in \mathfrak{k}$. Being holomorphic, ξ_Y therefore commutes with $J\vartheta$ as well, so by 1

$$\begin{aligned} d\mathscr{L}_{J\vartheta}\Psi^\xi &= \mathscr{L}_{J\vartheta}\,d\Psi^\xi = \mathscr{L}_{J\vartheta}\iota_{\xi_Y}\sigma = (\iota_{[J\vartheta,\xi_Y]} + \iota_{\xi_Y}\mathscr{L}_{J\vartheta})\,\sigma \\ &= \iota_{\xi_Y}\mathscr{L}_{J\vartheta}\sigma = \iota_{\xi_Y}2\sigma = 2\,d\Psi^\xi. \end{aligned}$$

This implies the function $\mathscr{L}_{J\vartheta}\Psi^\xi - 2\Psi^\xi$ is a constant, say $2c(\xi)$, for all $\xi \in \mathfrak{k}$. It is evidently linear in ξ. From the equivariance of Ψ and the fact that $[J\vartheta, \xi_Y]=0$ it is now easy to deduce that $c([\xi,\eta])=0$ for all ξ and η in $\mathfrak{k}$. This proves the first assertion in 2. Integrating the equation $\mathscr{L}_{J\vartheta}\Psi=2(\Psi+c)$ yields (5.2). The last two assertions are obvious.

Assume $c=0$. Let y be a critical point of ρ. Then 2 implies that $2\Psi(y)=(\mathscr{L}_{\operatorname{grad}\rho(y)}\Psi)(y)=(\mathscr{L}_0\Psi)(y)=0$, so $y\in\Psi^{-1}(0)$. Conversely, assume the critical set of ρ is nonempty and is contained in the fibre $\Psi^{-1}(0)$. Let y be a critical point of ρ. Then from (5.2) we obtain

$$0=\Psi(y)=\Psi(\mathfrak{G}(t, y))=\mathrm{e}^{-2t}\Psi(y)+(\mathrm{e}^{-2t}-1)\,c=(\mathrm{e}^{-2t}-1)\,c$$

for $t\geqslant 0$, and so $c=0$. ▮

The set-up of this proposition is functorial in the following sense. Let Z be a K-invariant closed complex submanifold of Y and let $\rho_Z=\rho|_Z$ be the restriction of ρ to Z. Put $\alpha_Z=-\sqrt{-1}\,\bar\partial\rho$, $\sigma_Z=-d\alpha$ and $\Psi_Z^\xi=\iota_{\xi_Z}\alpha_Z$ for $\xi\in\mathfrak{k}$. Then $\alpha_Z=\alpha|_Z$, $\sigma_Z=\sigma|_Z$ and $\Psi_Z=\Psi|_Z$. Of course, the Hamiltonian vector field ϑ_Z of ρ_Z is not the restriction of ϑ to Z, unless ϑ happens to be tangent to Z.

These observations apply to the pair of manifolds $Y=V$ and $Z=X$, where V is a G-representation space with a K-invariant inner product as in Section 4, and X a G-stable closed nonsingular algebraic subvariety of V. We take ρ to be the function $\rho(v)=\|v\|^2/2$. Clearly, $\sigma=\sqrt{-1}\,\partial\bar\partial\rho$ is the standard symplectic form ω_V, Ψ is the quadratic momentum map Φ_V given by (2.3), and $J\vartheta=\operatorname{grad}\rho$ is the radial vector field $v\,\partial/\partial v$ on V. The idea to use the length function as a tool in invariant theory is due to Kempf and Ness [12]. I shall frequently refer to their main result (see also [27]):

THEOREM 5.3. *For all v in V the following conditions are equivalent*:

1. *the orbit Gv is closed*;
2. *the restriction of ρ to Gv has a stationary point*;
3. *Gv intersects the zero level set of the momentum map Φ_V.*

If v is a stationary point of $\rho|_{Gv}$, then: *$\rho|_{Gv}$ takes on its minimum at v*; *for all $w\in Gv$, $\rho(w)=\rho(v)$ implies $w\in Kv$*; *and $G_v=(K_v)^{\mathbb{C}}$.*

Note that $\operatorname{grad}\rho$ is tangent to the subvariety X only if X is invariant under the standard $\mathbb{C}^\times$-action on V. Because X is closed, the restriction of ρ to X, ρ_X, is a proper function, so the forward trajectories of $-J\vartheta_X=-\operatorname{grad}\rho_X$ are bounded and the flow $\mathfrak{G}_X(t,\cdot)$ is defined for all $t\geqslant 0$. Since ρ_X is real-analytic, $\lim_{t\to\infty}\mathfrak{G}_X(t,x)$ exists for all x in X. The properness of ρ_X implies that the the flow retracts the stable set of every critical level continuously onto the critical set. (See Section 2.5.) Furthermore, ρ_X always has critical points, for example minima. If x is a critical point of ρ_X, it is a critical point of the restriction of ρ_X to the orbit Gx, and therefore

$\Phi_X(x)=0$ by Theorem 5.3. Hence, the character c in Proposition 5.2 is 0, so that

$$\mathscr{L}_{J\vartheta_X}\Phi_X=2\Phi_X \qquad \text{and} \qquad (\mathfrak{G}_X)_t^*\Phi_X=\mathrm{e}^{-2t}\Phi_X. \tag{5.3}$$

THEOREM 5.4. *Let X be a G-stable closed nonsingular algebraic subvariety of V. Suppose that ρ_X has a unique critical level. Let $\mathscr{U}$ be a basis of neighbourhoods (in the classical topology on X) of the critical set of ρ_X. Then the sets $\Delta(U)$, where $U\in\mathscr{U}$, form a basis of neighbourhoods of the vertex 0 of the momentum cone $\Delta(X)$. In particular, the cone spanned by $\Delta(U)$ is equal to $\Delta(X)$ for every $U\in\mathscr{U}$.*

Proof. Let B_ε denote the closed ball of radius ε about the origin in V and let $\delta=\min\{\|x\|:x\in X\}$ be the distance from X to the origin. The assumption on ρ_X implies that its only critical level is the global minimum, $\delta^2/2$. Therefore, the sets $X\cap B_\varepsilon$, where $\varepsilon>\delta$, are a basis of neighbourhoods of the critical set $X\cap B_\delta$ of ρ_X. So it suffices to prove that the sets $\Delta(X\cap B_\varepsilon)$, where $\varepsilon>\delta$, form a basis of neighbourhoods of the vertex 0 of the momentum cone $\Delta(X)$. The proof has three parts: first I show that for some $\varepsilon>\delta$ the set $\Delta(X\cap B_\varepsilon)$ is a neighbourhood of the vertex in $\Delta(X)$. Then I show that the same is true for every $\varepsilon>\delta$. Lastly, I prove that for every ball D about the origin in $\mathfrak{k}^*$ there exists an $\varepsilon>\delta$ such that $\Delta(X\cap B_\varepsilon)\subset D$.

Part 1. The flow $\mathfrak{G}_X$ extends to a deformation retraction

$$\bar{\mathfrak{G}}_X\colon X\times[0,\infty]\to X \tag{5.4}$$

of X onto the set $X\cap B_\delta\subset\Phi_X^{-1}(0)$. Now take any $\eta>\delta$. Then for every x in X the trajectory $\mathfrak{G}_X(t,x)$ is contained in B_η for sufficiently large t. Moreover, by (5.3), $\Phi_X(\mathfrak{G}_X(t,x))=\mathrm{e}^{-2t}\Phi_X(x)$. This implies that

$$\text{the cone spanned by } \Delta(X\cap B_\eta) \text{ is the whole of } \Delta(X). \tag{5.5}$$

Now consider the subset $S=G\cdot(X\cap B_\eta)$ of X. I assert that

$$\text{for every } x\in S \text{ the affine variety } \overline{Gx} \text{ is contained in } S. \tag{5.6}$$

Indeed, take any x in S and any y in $\overline{Gx}$. We have to show that y is in S. Let $\mathfrak{F}_t(\cdot)$ be the gradient flow of the function $-|\Phi_V|^2$. The limit map $\mathfrak{F}_\infty=\lim_{t\to\infty}\mathfrak{F}_t$ is continuous, it retracts S onto $\Phi_X^{-1}(0)\cap B_\eta$, and it retracts $\overline{Gx}$ onto the K-orbit $K(\mathfrak{F}_\infty x)\subset B_\eta$. Moreover, ρ_X is decreasing along the flow lines, so the restriction of ρ_X to $\overline{Gx}$ takes on its minimum at $\mathfrak{F}_\infty(x)$. (See [25] and [27].) This implies $\mathfrak{F}_\infty(y)$ is in the K-orbit through $\mathfrak{F}_\infty(x)$. There are two possibilities: either $\mathfrak{F}_\infty(x)$ is in the interior of the ball B_η, or it is on the boundary. In the first case, the G-orbit Gy intersects the interior of B_η, so $y \in G \cdot (X \cap B_\eta) = S$. In the second case, since $\mathfrak{F}_\infty(x)$ is the point closest to the origin on $\overline{Gx}$ and by assumption the orbit Gx intersects B_η, we see that $\mathfrak{F}_\infty(x)$ lies on Gx. By Theorem 5.3 every G-orbit intersecting $\Phi_V^{-1}(0)$ is closed, and therefore $y \in \overline{Gx} = Gx \subset S$. This proves (5.6).

Next, I assert that

$$\Delta(S) = \Delta(X). \tag{5.7}$$

To see this, let λ be any point in $\Delta(X)$. Then $b\lambda \in \Delta(X \cap B_\eta)$ for some $b > 0$ by (5.5). Take $x \in X \cap B_\eta$ such that $\Phi_X(x) = b\lambda$. Then $\Delta(\overline{Gx})$ contains the ray through $b\lambda$ by Theorem 4.9. But $\overline{Gx} \subset S$ by 5.6, so $\lambda \in \Delta(\overline{Gx}) \subset \Delta(S)$. This proves (5.7).

Now let D be any ball about the origin in $\mathfrak{k}^*$. Then $\Phi_V^{-1}(D) \cap S$ is a bounded subset of V by Lemma 4.10. This means we can find $\varepsilon > \delta$ such that the ball B_ε contains $\Phi_V^{-1}(D) \cap S = \Phi_X^{-1}(D) \cap S$. Then $\Phi_X(X \cap B_\varepsilon) \supset \Phi_X(\Phi_X^{-1}(D) \cap S)$. By (5.7) above, Φ_X maps S surjectively onto $\Phi_X(X)$, and therefore $\Phi_X(\Phi_X^{-1}(D) \cap S) = D \cap \Phi_X(X)$. Consequently, $\Delta(X \cap B_\varepsilon)$ contains $\Delta(X) \cap D$ and is therefore a neighbourhood of the vertex in $\Delta(X)$.

Part 2. Suppose $\Delta(X \cap B_\varepsilon)$ is a neighbourhood of the vertex in $\Delta(X)$ for a certain $\varepsilon > \delta$. Then $e^{-2t}\, \Delta(X \cap B_\varepsilon)$ is a neighbourhood of the vertex for all t. Choose an arbitrary ε' with $\delta < \varepsilon' < \varepsilon$. By the continuity of the retraction (5.4) and the compactness of $X \cap B_\varepsilon$ there exists a t such that $\mathfrak{G}_t(X \cap B_\varepsilon)$ is a subset of $X \cap B_{\varepsilon'}$. So by (5.3), $e^{-2t}\, \Delta(X \cap B_\varepsilon) = \Delta(\mathfrak{G}_t(X \cap B_\varepsilon))$ is a subset of $\Delta(X \cap B_{\varepsilon'})$, so $\Delta(X \cap B_{\varepsilon'})$ is a neighbourhood of the vertex in $\Delta(X)$.

Part 3. Let D be any ball about the origin in $\mathfrak{k}^*$. Take any $\varepsilon > \delta$; then there exists a t such that $e^{-2}\, \Delta(X \cap B_\varepsilon) = \Delta(\mathfrak{G}_t(X \cap B_\varepsilon))$ is contained in D. Again by continuity and compactness, there exists an ε' with $\delta < \varepsilon' < \varepsilon$ such that $X \cap B_{\varepsilon'} \subset \mathfrak{G}_t(X \cap B_\varepsilon)$. But then $\Delta(X \cap B_{\varepsilon'})$ is contained in D. ∎

This proof gives no information on the shape of the set $\Delta(X \cap B_\varepsilon)$ away from the vertex. It seems not unlikely that $\Delta(X \cap B_\varepsilon)$ is convex.

Example 5.5 (homogeneous vector bundles). Let L be a closed subgroup of K and let F be the reductive subgroup $L^{\mathbb{C}}$ of G. Let W be a unitary L-module and let $X = G \times^F W$. There exists an orthogonal (real) representation V_1 of K containing a vector v_0 with stabilizer $K_{v_0} = L$. The map $k \mapsto kv_0$ therefore induces an embedding $K/L \to V_1$. The complexification of this map is an embedding of $K^{\mathbb{C}}/L^{\mathbb{C}} = G/F$ into the unitary K-module $V_1^{\mathbb{C}}$. There also exists an L-equivariant isometric embedding of W into a unitary K-module V_2. Then the map $X \to V_1^{\mathbb{C}} \oplus V_2$ defined by $[g, w] \mapsto gv_0 + gw$ is a G-equivariant closed embedding of X into $V = V_1^{\mathbb{C}} \oplus V_2$. (See Lemmas 1.16 and 1.18 of [29] for a proof of these facts.) Let $\rho(v) = \|v\|^2/2$, where $\|\cdot\|$ denotes the length function with respect to the direct sum metric on V. Let us identify X with its image in V. Using Theorem 5.3 one can easily show that ρ_X has a unique critical level, which is a minimum, and that the critical set is the compact orbit Kv_0. Hence, by Theorem 5.4, the sets $\Delta(U)$, where U ranges over the neighbourhoods of Kv_0 in X, form a basis of neighbourhoods of the vertex of $\Delta(X)$.

Here is an example of a singular variety for which the conclusion of Theorem 5.4 holds.

Example 5.6. Let F, W and X be as in the previous example, and let Y be an F-invariant affine cone in W. Let X' be the affine subvariety $G \times^F Y$ of the vector bundle $X = G \times^F W$ and embed X into a G-module V as in the previous example. Because $Y \subset W$ is invariant under dilations, the subvariety X' is invariant under the gradient flow of ρ_X. It follows that the restriction of the flow $\mathfrak{G}_X$ to X' retracts X' onto the compact orbit Kv_0. Exactly the same proof as that of Theorem 5.4 now shows that the sets $\Delta(U')$, where U' ranges over the neighbourhoods of Kv_0 in X', form a basis of neighbourhoods of the vertex of the cone $\Delta(X')$.

6. HAMILTONIAN ACTIONS AND CONVEXITY

In this section I explain how the previous, mainly algebro-geometric, results can be generalized to arbitrary Hamiltonian actions. The basic idea is that every symplectic manifold with a Hamiltonian K-action can locally near every orbit in the zero fibre of the momentum map be identified with a germ of a complex affine G-variety. I then state the main result of the paper, Theorem 6.7. First recall the following standard definition.

Definition 6.1. Let M be a Hamiltonian K-manifold with momentum map Φ. For every $\mu \in \mathfrak{k}^*$ the (*Meyer–Marsden–Weinstein*) *reduced space* or *symplectic quotient* at level μ is the space $\Phi^{-1}(K\mu)/K$. It is denoted by $M_{\mu, K}$, or by M_μ, if the group K is clear from the context.

By the results of [30], the symplectic quotient is a stratified space carrying natural symplectic forms on the strata, which satisfy certain compatibility conditions. For most μ in $\mathfrak{k}^*$, M_μ is actually a symplectic V-manifold.

One application of symplectic reduction is the construction of "local models," which I now briefly explain. See [24] or [7] for details. Let μ be any vector in $\mathfrak{k}^*$ and let L be any closed subgroup of K_μ. Let W be a symplectic representation of L and let Φ_W be the standard quadratic L-momentum map on W. Let $\mathfrak{z}_\mu$ be the centre of $\mathfrak{k}_\mu$. Then the zero-weight space in $\mathfrak{k}$ under the adjoint action of $\mathfrak{z}_\mu$ is exactly $\mathfrak{k}_\mu$, so $\mathfrak{k}_\mu$ has a natural $\mathfrak{k}_\mu$-invariant complement in $\mathfrak{k}$. This means that the principal fibre bundle $K_\mu \to K \to K\mu$ comes equipped with a natural connection. Furthermore, the Levi decomposition $\mathfrak{k}_\mu = \mathfrak{z}_\mu \oplus [\mathfrak{k}_\mu, \mathfrak{k}_\mu]$ shows that $\mathfrak{z}_\mu$ is a direct summand of $\mathfrak{k}_\mu$. We can therefore view $\mathfrak{z}_\mu^*$ as a subspace of $\mathfrak{k}_\mu^*$ and $\mathfrak{k}_\mu^*$ as a subspace of $\mathfrak{k}^*$. Under these natural identifications, μ is an element of $\mathfrak{z}_\mu^* \subset \mathfrak{k}_\mu^*$.

The manifold $K \times \mathfrak{k}_\mu^*$ carries a natural closed two-form (the minimal-coupling form defined by the symplectic form on $K\mu$ and the connection on $K \to K\mu$), which is nondegenerate in a K-invariant neighbourhood of $K \times \{0\}$. Now consider the manifold $X = K \times \mathfrak{k}_\mu^* \times T^*K_\mu \times W$ and identify T^*K_μ with $K_\mu \times \mathfrak{k}_\mu^*$ by means of left-translations. The action of $K_\mu \times L$ on X defined by

$$(k, l) \cdot (g, \kappa, h, v, w) = (gk^{-1}, k\kappa, khl^{-1}, lv, lw)$$

is Hamiltonian with momentum map $\Psi\colon X \to \mathfrak{k}_\mu^* \times \mathfrak{l}^*$ given by

$$\Psi(g, \kappa, h, v, w) = (-\kappa + hv,\ -v|_{\mathfrak{l}} + \Phi_W(w)).$$

Definition 6.2. $X(\mu, L, W) = \Psi^{-1}(-\mu, 0)/(K_\mu \times L)$ is the symplectic quotient of X by the $K_\mu \times L$-action at the value $(-\mu, 0) \in \mathfrak{k}_\mu^* \times \mathfrak{l}^*$.

It turns out that $X(\mu, L, W)$ is smooth. Consider the K-action on X defined by left-multiplication on the first factor. It is Hamiltonian as well and, moreover, it commutes with the $K_\mu \times L$-action. It descends therefore to a Hamiltonian K-action on $X(\mu, L, W)$. The easiest way to write the K-momentum map on $X(\mu, L, W)$ is as follows. Put $\mathfrak{m} = \mathfrak{k}_\mu/\mathfrak{l}$. Choose an L-invariant complement of $\mathfrak{l}$ in $\mathfrak{k}_\mu$ and identify it with $\mathfrak{m}$. Then $\mathfrak{m}^*$ is a subspace of $\mathfrak{k}_\mu^*$. The map φ from the product $K \times \mathfrak{m}^* \times W$ to $\Psi^{-1}(-\mu, 0)$ defined by

$$\varphi(g, v, w) = (g, v + \Phi_W(w) + \mu, 1, v + \Phi_W(w), w)$$

descends to a K-equivariant diffeomorphism

$$\bar{\varphi}\colon K \times^L (\mathfrak{m}^* \times W) \to X(\mu, L, W).$$

Via this diffeomorphism, the associated bundle $K \times^L (\mathfrak{m}^* \times W)$ acquires a closed two-form that is symplectic in a neighbourhood of the zero section. Note that the definition of $\bar{\varphi}$ depends only on the choice of a complement of $\mathfrak{l}$ in $\mathfrak{k}_\mu$ (which is equivalent to the choice of a connection on the principal fibre bundle $L \to K \to K/L$). We shall henceforth identify $X(\mu, L, W)$ with $K \times^L (\mathfrak{m}^* \times W)$ through $\bar{\varphi}$. The K-momentum map on $K \times^L (\mathfrak{m}^* \times W)$ is given by

$$\Phi([g, v, w]) = g(v + \Phi_W(w) + \mu). \tag{6.1}$$

It is useful to consider the restriction of Φ to a K-invariant neighbourhood of the point $[1, 0, 0]$. If we perform the reduction of X by the $K_\mu \times L$-action in stages, first with respect to L and then with respect to K_μ, we see that $X(\mu, L, W)$ is an "iterated" associated bundle: it is a bundle

$$X(\mu, L, W) \cong K \times^{K_\mu} (K_\mu \times^L (\mathfrak{m}^* \times W)) = K \times^{K_\mu} Y, \tag{6.2}$$

over the coadjoint orbit $K\mu \cong K/K_\mu$ with fibre the Hamiltonian K_μ-space

$$Y = K_\mu \times^L (\mathfrak{m}^* \times W). \tag{6.3}$$

The K_μ-momentum map $Y \to \mathfrak{k}_\mu^*$ is the restriction of Φ to Y. We can now write Φ as the composition of two maps,

$$K \times^{K_\mu} Y \to K \times^{K_\mu} \mathfrak{k}_\mu^* \xrightarrow{\iota} \mathfrak{k}^*, \tag{6.4}$$

the first of which is the unique bundle map extending the K_μ-momentum map on the fibre Y, and the second of which is defined by $\iota([g, v]) = gv$.

Recall that $\mathfrak{k}_\mu^*$ is a *slice* at μ for the coadjoint action on $\mathfrak{k}^*$: the restriction of ι to a sufficiently small K-invariant open neighbourhood of $[1, \mu]$ is a K-equivariant embedding onto an open neighbourhood of μ. This implies that the restriction of Φ to a sufficiently small K-invariant neighbourhood U of $[1, 0, 0]$ is a bundle map of associated bundles over $K\mu$. Consequently, $U \cap \Phi^{-1}(\mathfrak{k}_\mu^*) = U \cap Y$ and the image $\Phi(U)$ is a bundle over $K\mu$ with fibre $\Phi(U \cap Y)$.

If μ happens to be in $\mathfrak{t}_+^*$, then $\mathfrak{t}^*$ is a subset of $\mathfrak{k}_\mu^*$, and $\mathfrak{t}_+^*$ is contained in $\mathfrak{t}_{+,\mu}^*$, the positive Weyl chamber of $\mathfrak{k}_\mu^*$. In fact, $\mathfrak{t}_{+,\mu}^* \cap D = \mathfrak{t}_+^* \cap D$ for a sufficiently small neighbourhood D of μ in $\mathfrak{k}^*$. It follows from this that if U is small enough

$$\Delta(U) = \Phi(U) \cap \mathfrak{t}_+^* = \Phi(U \cap Y) \cap \mathfrak{t}_+^* = \Phi(U \cap Y) \cap \mathfrak{t}_{+,\mu}^* = \Delta(U \cap Y), \tag{6.5}$$

where $\Delta(U \cap Y)$ stands for the momentum set of $U \cap Y$ considered as a Hamiltonian K_μ-space.

Now let M be an arbitrary Hamiltonian K-manifold with momentum map Φ. Marle [24] and Guillemin and Sternberg [7] have shown that locally at any orbit M is isomorphic to some $X(\mu, L, W)$.

THEOREM 6.3 (symplectic slices). *Let m be any point in M. Let $L = K_m$ be the stabilizer of m, let $\mu = \Phi(m)$, and let*

$$W = T_m(Km)^\omega / (T_m(Km) \cap T_m(Km)^\omega).$$

Then there exist a K-invariant neighbourhood U_1 of m in M, a K-invariant neighbourhood U_2 of $[1, 0, 0]$ in $X(\mu, L, W)$, and a map $f: U_1 \to U_2$ with the following properties: f is a K-equivariant symplectomorphism; f intertwines the momentum maps on U_1 and U_2; and $f(m) = [1, 0, 0]$.

The symplectic vector space W is called the *symplectic slice* at m. Note that in the situation of the theorem $\mathfrak{m}$ is simply the tangent space to $K_\mu m$ at m. An immediate consequence of the theorem is that the germ at m of $\Phi^{-1}(\mathfrak{k}_\mu^*)$, called a *local cross-section* of M, is a smooth K_μ-invariant symplectic submanifold, because in the local model $X(\mu, L, W)$ it is equal to the germ of $Y = K_\mu \times^L (\mathfrak{m}^* \times W)$ at $[1, 0, 0]$. Furthermore, if m has the property that $\mu = \Phi(m) \in \mathfrak{t}_+^*$, then for any small K-invariant neighbourhood U of m we have an equality of momentum sets

$$\Delta(U) = \Delta(U \cap \Phi^{-1}(\mathfrak{k}_\mu^*)), \tag{6.6}$$

because by (6.5) the same is true in the local model. An analogue of (6.6) for projective varieties was proved by Brion (Proposition 4.1 in [3]).

DEFINITION 6.4. Let m be any point in M. The *local momentum cone* at m is the set $\Delta_m = \mu + \Delta(Y_m)$. Here Y_m is the complex affine $(K_\mu)^{\mathbb{C}}$-variety $(K_\mu)^{\mathbb{C}} \times^{(K_m)^{\mathbb{C}}} W$, with $\mu = \Phi(m)$ and with W being the symplectic slice at m furnished with a compatible K_m-invariant complex structure.

So Δ_m is a convex polyhedral cone with vertex μ and is contained in $\mathfrak{t}_{+,\mu}^*$, the positive Weyl chamber of K_μ. Up to a translation by μ it is a rational cone.

From the symplectic slice theorem we deduce the following "local" convexity theorem.

THEOREM 6.5. 1. *For every m in M such that $\mu = \Phi(m)$ is contained in $\mathfrak{t}_+^*$ and for every sufficiently small K-invariant neighbourhood U of m the set $\Delta(U)$ is a neighbourhood of the vertex of the local momentum cone $\Delta_m \subset \mathfrak{t}_{+,\mu}^*$. In particular, the cone with vertex μ spanned by $\Delta(U)$ is equal to Δ_m;*

2. *for every $\mu \in \mathfrak{k}^*$ and for every connected component C of the fibre $\Phi^{-1}(\mu)$, the local momentum cone Δ_m is independent of the point $m \in C$.*

Proof. 1. First we treat the case $\mu=0$. At points in the zero fibre of Φ there is an alternative local model for M as follows. Put $F=L^{\mathbb{C}}=(K_m)^{\mathbb{C}}$ and choose an F-invariant compatible complex structure on the symplectic slice W at m. Consider the affine G-variety $Z=G\times^F W$. Embed it into a unitary K-module V as in Example 5.5, and regard it as a Hamiltonian K-space with the symplectic structure and momentum map inherited from V. The stabilizer of the point $[1,0]\in Z$ under the K-action is equal to L. The G-orbit through $[1,0]$ is closed, so by Theorem 5.3, $[1,0]$ is contained in $\Phi_Z^{-1}(0)$. The symplectic slice at $[1,0]$ is simply the (Hermitian) orthogonal complement to the G-orbit through $[1,0]$ and is therefore equal to W. Theorem 6.3 now allows us to conclude that the K-invariant germ of M at m is isomorphic as a Hamiltonian K-manifold to the K-invariant germ of Z at $[1,0]$.

Putting this information together with Theorems 4.9 and 5.4 and Example 5.5, we see that for every small K-invariant neighbourhood U of m the set $\Delta(U)=\Phi(U)\cap\mathfrak{t}^*_+$ is a neighbourhood of the vertex of the cone $\Delta_m=\Delta(G\times^F W)$.

If $\mu\neq 0$, we may without loss of generality assume that $\mu\in\mathfrak{t}^*_+$, because every K-orbit in $\mathfrak{k}^*$ intersects $\mathfrak{t}^*_+$. This case can then be reduced to the case $\mu=0$ by using (6.6) and shifting the K_μ-momentum map on the space Y in (6.3) by the vector $-\mu$. This shifted map is still an equivariant K_μ-momentum map, because μ is in $\mathfrak{z}^*_\mu$.

2. Again, it suffices to prove this for $\mu=0$. If M is an affine G-variety, then by Theorem 5.3 the points in $\Phi^{-1}(0)$ are exactly those whose G-orbits are closed. Corollary 4.24 then says that the local momentum cones Δ_m are constant along the fibre $\Phi^{-1}(0)$. For general M, this argument combined with the symplectic slice theorem shows that Δ_m is locally constant along the fibre $\Phi^{-1}(0)$. ∎

Remark 6.6. It follows from the Cartan decomposition of G that there exists a global K-equivariant diffeomorphism between the two local models $K\times^L(\mathfrak{m}^*\times W)$ and $G\times^F W$, but I don't know if there is a global symplectomorphism.

The following generalization of Kirwan's convexity theorem [14] is the main result of this paper. The first part describes $\Delta(M)$ as a locally finite intersection of polyhedral cones, each of which is determined by local data on M. (The fact that $\Delta(M)$ is a convex set when Φ is proper was also proved in [11].) The second part is a necessary condition for a point to be a vertex of $\Delta(M)$. It generalizes the well-known fact that for torus actions vertices arise as images of fixed points. The third part states that the points where this necessary condition is fulfilled form a discrete subset of $\mathfrak{t}^*_+$.

THEOREM 6.7. *Assume that the momentum map $\Phi\colon M \to \mathfrak{k}^*$ is proper.*

1. *$\Delta(M)$ is the intersection of local momentum cones:*

$$\Delta(M) = \bigcap_{m \in \Phi^{-1}(\mathfrak{t}_+^*)} \Delta_m. \tag{6.7}$$

This intersection is locally finite and therefore $\Delta(M)$ is a closed convex polyhedral subset of $\mathfrak{t}_+^*$;

2. *if μ is a vertex of $\Delta(M)$ and m is any point in the fibre $\Phi^{-1}(\mu)$, then $\mathfrak{k}_\mu = [\mathfrak{k}_\mu, \mathfrak{k}_\mu] + \mathfrak{k}_m$, or, equivalently, $K_\mu = [K_\mu, K_\mu]\, K_m$. In particular, if μ is a vertex of $\Delta(M)$ lying in the interior of $\mathfrak{t}_+^*$, then T fixes m;*

3. *let E be the subset of M consisting of all points m such that $\mu \in \mathfrak{t}_+^*$ and $\mathfrak{k}_\mu = [\mathfrak{k}_\mu, \mathfrak{k}_\mu] + \mathfrak{k}_m$, where $\mu = \Phi(m)$. The image $\Phi(E)$ is a discrete subset of $\mathfrak{t}_+^*$. If M is compact, then $\Delta(M)$ is the convex hull of $\Phi(E)$.*

Proof. 1. The assumption that Φ is proper implies that its image $\Phi(M)$ is closed and, by the argument outlined in Section 2.5, that its fibres are connected. Consequently, by Theorem 6.5.2, for every $\mu \in \Delta(M)$ the cone Δ_m is the same for all points $m \in \Phi^{-1}(\mu)$. It is now easy to deduce from Theorem 6.5.1 plus the fact that Φ is proper that for every $\mu \in \Delta(M)$ there exists an open subset D of $\mathfrak{t}^*$ containing μ such that

$$\Delta(M) \cap D = \Delta_m \cap D, \tag{6.8}$$

where m is any point in the fibre $\Phi^{-1}(\mu)$. This means that $\Delta(M)$ is locally convex. But every closed locally convex set is convex, so $\Delta(M)$ is convex. Since every closed convex set is the intersection of all closed cones containing it, the equality (6.8) also implies (6.7). Furthermore, applying Theorem 6.5 to the Hamiltonian K-manifold $\Phi^{-1}(D)$, we find that for every $\mu' \in D$ and every $m' \in \Phi^{-1}(\mu')$ the local momentum cone $\Delta_{m'}$ is equal to the cone with vertex μ' spanned by Δ_m. Since Δ_m is a polyhedral cone, it follows from this that as μ' ranges over D, only finitely many different cones $\Delta_{m'}$ can occur. In other words, the collection of cones appearing in the intersection (6.7) is locally finite on $\mathfrak{t}_+^*$. This means that $\Delta(M)$ is a polyhedron.

2. This follows from (6.8) and Theorem 4.25 (applied to the group $G = (K_\mu)^{\mathbb{C}}$ and the variety $X = (K_\mu)^{\mathbb{C}} \times^{(K_m)^{\mathbb{C}}} W$).

3. First I prove that $\Phi(E)$ is discrete. Since Φ is proper, it suffices to show that every point m in M possesses a neighbourhood U such that $\Phi(E \cap U)$ is discrete. By the symplectic slice theorem, we may therefore assume M is an affine variety. But it follows from Lemma 6.8 below that for an affine G-variety the set $\Phi(E)$ consists of the origin in $\mathfrak{t}_+^*$ only.

Finally, the second statement in 3 follows immediately from 2 and the fact that a compact convex set is the convex hull of its extreme points. ∎

LEMMA 6.8. *Let V be a unitary K-module and suppose that $v \in V$ satisfies the condition $K_\mu = [K_\mu, K_\mu]\, K_v$, where $\mu = \Phi_V(v)$. Then $\mu = 0$.*

Proof. We want to prove that $\mu(\xi) = \Phi_V^\xi(v) = (\sqrt{-1}/2)\langle \xi v, v\rangle = 0$ for all $\xi \in \mathfrak{k}$. It suffices to show this for all $\xi \in \mathfrak{k}_\mu$. The condition on v says that $\xi = [\chi, \zeta] + \eta$ for some $\chi, \zeta \in [\mathfrak{k}_\mu, \mathfrak{k}_\mu]$ and $\eta \in \mathfrak{k}_v$. Now $\eta v = 0$, so $\mu(\eta) = 0$; and $\mu(\chi, \zeta) = \mu(\operatorname{ad} \chi(\zeta)) = (\operatorname{ad}^* \chi)\, \mu(\zeta) = 0$. We conclude that $\mu(\xi) = 0$. ∎

Theorem 6.7 is clearly not optimal. As Theorem 4.9 shows, it is not always necessary to assume that Φ is proper, nor even that M is nonsingular. See Section 7 for further examples of convexity for noncompact or singular spaces.

On the other hand, the necessary condition 2 for a point to be mapped to a vertex of $\Delta(M)$ is optimal in the following sense.

PROPOSITION 6.9. *For every $\mu \in \mathfrak{t}_+^*$ and for every closed subgroup L of K_μ such that $K_\mu = [K_\mu, K_\mu]\, L$, there exists a Hamiltonian K-manifold (M, ω, K, Φ) with a point $m \in M$ satisfying the following properties: $K_m = L$, $\Phi(m) = \mu$, and λ is a vertex of $\Delta(M)$.*

The proof will be given in Section 7.3.

Theorem 6.7.1 implies that for every μ in $\Delta(M)$ the cone with vertex μ spanned by $\Delta(M)$ is equal to the local momentum cone Δ_m, where m is any point in $\Phi^{-1}(\mu)$. In other words, the local shape of $\Delta(M)$ near μ is determined by the representation of the isotropy subgroup K_m on the symplectic slice W at m. Calculating Δ_m boils down to finding generators for the monoid of highest weights of the homogeneous vector bundle $(K_\mu)^{\mathbb{C}} \times^{(K_m)^{\mathbb{C}}} W$. If μ is contained in the boundary of the positive Weyl chamber, this is usually an arduous task. However, the situation is more manageable if the fibre $\Phi^{-1}(\mu)$ contains a point m that is fixed under K_μ. This means that $K_m = K_\mu$, or, equivalently, that the restriction of Φ to the orbit Km is a symplectic isomorphism onto the coadjoint orbit $K\mu$. If this is the case, the vector space $\mathfrak{m}$ in the local model (6.2) is 0, so the tangent space at m to the symplectic cross-section $\Phi^{-1}(\mathfrak{t}_\mu^*)$ is equal to the symplectic slice W at m, and W is simply the symplectic orthogonal complement of $T_m(Km)$ inside $T_m M$. In other words,

$$T_m(\Phi^{-1}(\mathfrak{t}_\mu^*)) = W = T_m M / T_m(Km) \cong (T_m(Km))^\omega. \tag{6.9}$$

It follows that $\Delta_m = \Delta(W)$, where W is regarded as a K_μ-module. In some examples this enables one to determine the entire momentum set $\Delta(M)$; see for instance Section 7.1.

7. EXAMPLES

7.1. *Actions on Projective Space*

Let V be a finite-dimensional unitary K-module. If V is irreducible and has highest weight λ, it follows from (2.6) that the momentum polytope of the projective space $\mathbb{P}V$ for the T-action is the convex hull of the Weyl group orbit through λ^*. Similarly, if $\lambda_1, \lambda_2, ..., \lambda_k$ are the highest weights of the irreducible submodules of V, then the T-momentum map image of $\mathbb{P}V$ is the convex hull of the union of the $\mathfrak{W}$-orbits through $\lambda_1^*, \lambda_2^*, ..., \lambda_k^*$. This implies $\Delta(\mathbb{P}V)$ is a subset of

$$\mathfrak{t}_+^* \cap \operatorname{hull}(\mathfrak{W}\lambda_1^* \cup \mathfrak{W}\lambda_2^* \cup \cdots \cup \mathfrak{W}\lambda_k^*).$$

Arnal and Ludwig [1] and Wildberger [31] determined this subset for "most" V that are irreducible. I shall calculate $\Delta(\mathbb{P}V)$ in some (but not all) of the remaining cases.

For simplicity I assume K to be semisimple, although the results can easily be generalized to arbitrary compact groups. Let Ψ be the root system of $(\mathfrak{k}, \mathfrak{t})$ and let $\mathfrak{g} = \bigoplus_{\alpha \in \Psi} \mathbb{C}E_\alpha$ be the root space decomposition of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$. The following result can be used to find a "lower bound" for the polytope $\Delta(\mathbb{P}V)$.

LEMMA 7.1. 1. *Let v be any vector in V and let $[v]$ be the ray through v. Then $\Phi_{\mathbb{P}V}([v]) \in \mathfrak{t}^*$ if and only if $\langle E_\alpha v, v\rangle = 0$ for all roots α.*

2. *Let $v_1, ..., v_l$ be weight vectors in V with weights $\nu_1, ..., \nu_l$. Assume that $\langle E_\alpha v_i, v_j\rangle = 0$ for all roots α and for all i and j with $1 \leqslant i < j \leqslant l$. (This is for instance the case if for all i and j the difference $\nu_i - \nu_j$ is not a root.) Then the subspace spanned by $v_1, ..., v_l$ is contained in $\Phi_V^{-1}(\mathfrak{t}^*)$, and the intersection $\mathfrak{t}_+^* \cap \operatorname{hull}\{\nu_1^*, ..., \nu_l^*\}$ is contained in $\Delta(\mathbb{P}V)$.*

Proof. 1. Let $\xi_\alpha = E_\alpha - E_{-\alpha}$ and $\eta_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. By (2.5), $\Phi_{\mathbb{P}V}([v]) \in \mathfrak{t}^*$ if and only if $\langle \xi_\alpha v, v\rangle = \langle \eta_\alpha v, v\rangle = 0$ for all roots α. Since $E_{-\alpha}$ (viewed as an operator on V) is the adjoint of E_α, this is equivalent to $\langle E_\alpha v, v\rangle = 0$ for all α.

2. Let W denote the linear span of the v_i. Let α be any root. The assumption implies $\langle E_\alpha v, v\rangle = 0$ whenever v is in W. It now follows from 1 that $\Phi_V(W)$ is contained in $\mathfrak{t}^*$.

From this fact and from (2.5) we infer that

$$\Phi_{\mathbb{P}V}(g[c_1 v_1 + \cdots + c_l v_l]) = \frac{|c_1|^2 \nu_1^* + \cdots + |c_l|^2 \nu_l^*}{|c_1|^2 + \cdots + |c_l|^2},$$

where g is any element of the normalizer of T representing $w_0 \in \mathfrak{W}$. Hence $\Delta(\mathbb{P}V) = \Phi_{\mathbb{P}V}(\mathbb{P}V) \cap \mathfrak{t}_+^*$ contains the set $\mathfrak{t}_+^* \cap \operatorname{hull}\{v_1^*, ..., v_l^*\}$. ∎

Put $\check{\mu} = 2\mu/(\mu, \mu)$ for any μ in $(\mathfrak{t}^{\mathbb{C}})^*$. Let $\alpha_1, ..., \alpha_r$ be the simple roots of $\mathfrak{k}$ and $\pi_1, ..., \pi_r$ the corresponding fundamental weights, that is, the basis of $\mathfrak{t}^*$ dual to $\check{\alpha}_1, ..., \check{\alpha}_r$. Then $\lambda = \sum_{i=1}^r (\lambda, \check{\alpha}_i)\, \pi_i$, where the coefficients $(\lambda, \check{\alpha}_i)$ are nonnegative integers. The next result gives an upper bound for the polytope $\Delta(\mathbb{P}V)$ for irreducible V.

PROPOSITION 7.2. *Assume V is irreducible and has highest weight λ. Let Π_λ be the set of weights of V that are not of the form $\lambda - \alpha$ for any positive root α such that $(\lambda, \check{\alpha}) = 1$. Then $\Delta(\mathbb{P}V)$ is contained in $\mathfrak{t}_+^* \cap \operatorname{hull} \Pi_\lambda^*$.*

Proof. Let v_λ be a highest-weight vector in V. It suffices to show that the local momentum cone $\Delta_{[v_\lambda]}$ of $\mathbb{P}V$ at the point $[v_\lambda]$ (see Definition 6.4) is contained in the cone with vertex $-\lambda$ spanned by the set $-\Pi_\lambda$. To this end let us compute the tangent space to the symplectic cross-section at $[v_\lambda]$ and the weights of the T-action on it. The vector v_λ is an eigenvector for K_λ, so $K_{[v_\lambda]} = K_\lambda$. In view of (6.9) this implies that we have an isomorphism of K_λ-modules

$$T_{[v_\lambda]}(\Phi^{-1}(\mathfrak{k}_\lambda^*)) = W \cong T_{[v_\lambda]}(\mathbb{P}V)/T_{[v_\lambda]}(K[v_\lambda]), \tag{7.1}$$

W being the symplectic slice at $[v_\lambda]$. Define Π to be the subset of the weight lattice Λ^* consisting of 0 and of all weights occurring in W under the action of the maximal torus T of K_λ. From (7.1) we get:

$$\Delta_{[v_\lambda]} \subset -(\lambda + \operatorname{cone} \Pi). \tag{7.2}$$

(If λ is strictly dominant, so that $K_\lambda = T$, then this inclusion is an equality.) I assert that

$$\operatorname{cone} \Pi = \operatorname{cone}(-\lambda + \Pi_\lambda). \tag{7.3}$$

Because of (7.2), this will finish the proof. Let $V = \bigoplus_{\nu \in \Lambda^*} V_\nu$ be the weight space decomposition of V and let $\mathbb{C}_{-\lambda}$ be the one-dimensional representation of K_λ defined by the character $-\lambda \in \mathfrak{z}_\lambda^*$. Then the quotient map $V - \{0\} \to \mathbb{P}V$ induces an isomorphism of K_λ-modules

$$T_{[v_\lambda]}(\mathbb{P}V) \cong \bigoplus_{\nu \in \Lambda^* - \{\lambda\}} V_\nu \otimes \mathbb{C}_{-\lambda}. \tag{7.4}$$

Since the maximal unipotent subgroup N fixes v_λ, the complex stabilizer $G_{[v_\lambda]}$ is the parabolic subgroup $P_\lambda = (K_\lambda)^{\mathbb{C}} N$. This implies that the real orbit $K[v_\lambda]$ is equal to the complex orbit $G[v_\lambda]$ and therefore we have natural isomorphisms of complex K_λ-representations

$$T_{[v_\lambda]}(K[v_\lambda]) \cong \mathfrak{p}_\lambda^\circ \cong \bigoplus_{\substack{\alpha \in \Psi^- \\ (\lambda, \alpha) \neq 0}} \mathbb{C} E_\alpha, \tag{7.5}$$

where $\mathfrak{p}_\lambda^\circ$ denotes the annihilator of $\mathfrak{p}_\lambda$ in $\mathfrak{g}^*$. (The second isomorphism is induced by the Killing form on $\mathfrak{g}$.) Let $\Lambda_r^* \subset \Lambda^*$ denote the root lattice of the pair $(\mathfrak{k}, \mathfrak{t})$. There are two inclusions,

$$(\Lambda_r^* \backslash \Psi^-) \cap (-\lambda + \operatorname{hull} \mathfrak{W}\lambda) \subset \Pi \subset \Lambda_r^* \cap (-\lambda + \operatorname{hull} \mathfrak{W}\lambda),$$

the second of which follows from (7.4) and the first of which follows from (7.1) and (7.5). Moreover, by the definition of Π_λ, the same inclusions hold with Π replaced by $-\lambda + \Pi_\lambda$. Therefore, to establish (7.3), it suffices to prove the following statement for every positive root α: if $-\alpha \in \Pi$, then $-\alpha \in -\lambda + \Pi_\lambda$; and if $-\alpha \in -\lambda + \Pi_\lambda$, then some multiple of $-\alpha$ is in Π. There are three possibilities:

1. $(\lambda, \breve{\alpha}) = 0$. Then $\lambda - \alpha$ is not a weight of V, so $-\alpha$ is contained in neither Π nor $-\lambda + \Pi_\lambda$.

2. $(\lambda, \breve{\alpha}) = 1$. Then $\lambda - \alpha \notin \Pi_\lambda$ by definition. Also, $\lambda - \alpha$ is a weight of V with multiplicity one, so by (7.5) $-\alpha$ is not a weight of W, so $-\alpha \notin \Pi$.

3. $(\lambda, \breve{\alpha}) = l > 1$. Then the weights $\lambda, \lambda - \alpha, \ldots, \lambda - l\alpha$ occur in V. Since $(\lambda, \breve{\alpha}) \neq 1$, all these weights are also in Π_λ. The weight $-\alpha$ may or may not occur in W (depending on whether the multiplicity of $\lambda - \alpha$ in V is greater than one), but at any rate $-l\alpha$ is a weight of W. We conclude that $-\alpha \in -\lambda + \Pi_\lambda$ and $-l\alpha \in \Pi$.

In sum, we have shown that $\Pi \subset -\lambda + \Pi_\lambda$ and $-\lambda + \Pi_\lambda \subset \operatorname{cone} \Pi$. This proves (7.3). ∎

The following result follows easily from Lemma 7.1 and Proposition 7.2.

PROPOSITION 7.3 ([1], [31]). *Assume V is irreducible and has highest weight λ.*

1. *Suppose that $(\lambda, \breve{\alpha}_i) \neq 1$ for $i = 1, 2, \ldots, r$. Then $\Delta(\mathbb{P}V) = \mathfrak{t}_+^* \cap \operatorname{hull} \mathfrak{W}\lambda^*$;*

2. *if $(\lambda, \breve{\alpha}) = 1$ for some positive root α, then $\Delta(\mathbb{P}V)$ is not equal to $\mathfrak{t}_+^* \cap \operatorname{hull} \mathfrak{W}\lambda^*$.*

3. *$\Phi_{\mathbb{P}V}(\mathbb{P}V) \cap \mathfrak{t}^*$ is convex if and only if $(\lambda, \breve{\alpha}_i) \neq 1$ for $i = 1, 2, \ldots, r$.*

If $(\lambda, \breve{\alpha}) = 1$ for some positive root α, Proposition 7.3 does not give an explicit description of the polytope $\Delta(\mathbb{P}V)$. Using Lemma 7.1 one can easily check that the upper bound given by Proposition 7.2 is sharp e.g., in the following cases.

PROPOSITION 7.4. *The equality* $\Delta(\mathbb{P}V) = \mathfrak{t}_+^* \cap \operatorname{hull} \Pi_\lambda^*$ *holds if*

1. $K = \mathrm{SU}(4)$ *and* $\lambda = \pi_1, \pi_2, \pi_3$, *or* $\pi_1 + \pi_2 + \pi_3$;
2. K *has rank two and* $\lambda \in \mathfrak{t}_+^*$ *is arbitrary.*

Figures 1–3 illustrate Proposition 7.4. They show the convex hull of $\mathfrak{W}\lambda^*$ and the polytope $\Delta(\mathbb{P}V)$ (shaded). The intersection $\mathfrak{t}_+^* \cap \operatorname{hull} \mathfrak{W}\lambda^*$ is indicated in light shading. The dominant weights occurring in V are denoted by black circles. These are exactly the images of the T-fixed points in $\mathbb{P}V$ intersected with $\mathfrak{t}_+^*$. Notice that few of the vertices on the walls of $\mathfrak{t}_+^*$ arise as images of T-fixed points. In those cases where they are not weights of V, the fundamental weights of $\mathfrak{k}$ are indicated by black squares.

Finally, here is a generalization of Proposition 7.3 to reducible representations.

PROPOSITION 7.5. 1. *Let* $\lambda_1, \lambda_2, ..., \lambda_k$ *be the highest weights of the irreducible submodules of* V. *Suppose that* $(\lambda_j, \check{\alpha}_i) \neq 1$ *for all* i *and* j. *Then* $\Delta(\mathbb{P}V) = \mathfrak{t}_+^* \cap \operatorname{hull}(\mathfrak{W}\lambda_1^* \cup \mathfrak{W}\lambda_2^* \cup \cdots \cup \mathfrak{W}\lambda_k^*)$.

2. *Suppose* V *is the direct sum of at least* $r+1$ *copies of a unitary irreducible representation with highest weight* λ, *where* r *is the rank of* K. *Then* $\Delta(\mathbb{P}V) = \mathfrak{t}_+^* \cap \operatorname{hull} \mathfrak{W}\lambda^*$.

Proof 1. This is easy to deduce from Lemma 7.1 and Proposition 7.3.

2. Write $V = \bigoplus_1^k V'$, where $k > r$ and V' is an irreducible module with highest weight λ. Clearly, $\Delta(\mathbb{P}V)$ is a subset of $\mathfrak{t}_+^* \cap \operatorname{hull} \mathfrak{W}\lambda^*$. The

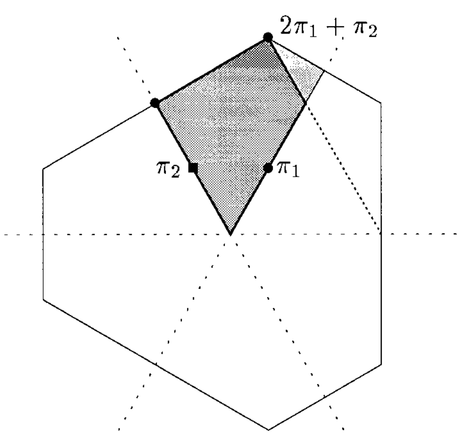

FIG. 1. $K = \mathrm{SU}(3)$ and $\lambda = \pi_1 + 2\pi_2$.

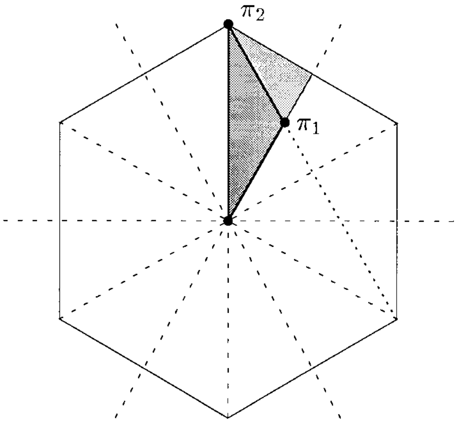

FIG. 2. $K = G_2$ and $\lambda = \pi_2$ (highest weight of complexified adjoint representation, $\mathfrak{g}_2^{\mathbb{C}}$).

weight polytope hull $\mathfrak{W}\lambda$ has exactly r edges containing the vertex λ. Let $v_1, v_2, ..., v_r$ be the opposite endpoints of these edges. Choose weight vectors $v_0, v_1, ..., v_r$, each coming from a different copy of V', and having weights $\lambda, v_1, v_2, ..., v_r$, respectively. Since each copy of V' is K-invariant and they are all mutually orthogonal, $\langle E_\alpha v_i, v_j \rangle = 0$ for all roots α and for all i and j with $0 \leqslant i < j \leqslant r$. Lemma 7.1 therefore tells us that $\mathfrak{t}_+^* \cap \operatorname{hull}\{\lambda^*, v_1^*, ..., v_r^*\}$ is contained in $\Delta(\mathbb{P}V)$. Hence, $\Delta(\mathbb{P}V) = \mathfrak{t}_+^* \cap \operatorname{hull} \mathfrak{W}\lambda^*$. ∎

7.2. *Cotangent Bundles*

Let Q be a connected K-manifold and let M be the cotangent bundle of Q. Points in M will be written as pairs (q, p), where $q \in Q$ and $p \in T_q^* Q$, and tangent vectors to M as pairs $(\delta q, \delta p)$, where $\delta q \in T_q Q$ and $\delta p \in T_p(T_q^* Q)$. The standard one-form α on M is the K-invariant form defined by $\alpha_{(q,p)}(\delta q, \delta p) = p(\delta q)$. The two-form $\omega = -d\alpha$ is symplectic, and the lifted K-action on M is Hamiltonian with momentum map defined by $\Phi^\xi = \iota_{\xi_M} \alpha$, that is, $\Phi^\xi(q, p) = p(\xi_{Q,q})$. Clearly, Φ is homogeneous of degree one in p, so $\Phi^{-1}(0)$ is a conical subset of M. In particular, Φ is not proper (not even if Q is compact) and Theorem 6.7 does not apply. But the homogeneity of Φ also implies that $\Delta(M)$ is equal to the cone on $\Delta(U)$ for any neighbourhood U of the zero section. In view of Theorem 6.5 this means that $\Delta(M) = \Delta_{(q,0)}$, where q is any point in Q. Furthermore, $\Phi(M) = -\Phi(M)$, so $\Delta(M) = \Delta(M)^*$. The symplectic slice W at $(q, 0)$ to the K-action on M is equal to T^*V, where V is the slice $T_q Q / T_q(Kq)$ at q to the K-action on Q. This implies that $W = V + JV = V^{\mathbb{C}}$ for a suitable K_q-invariant complex structure J on W, and so the variety $G \times^{(K_q)^{\mathbb{C}}} W$ is the complexification of the real-algebraic variety $K \times^{K_q} V$. We have proved:

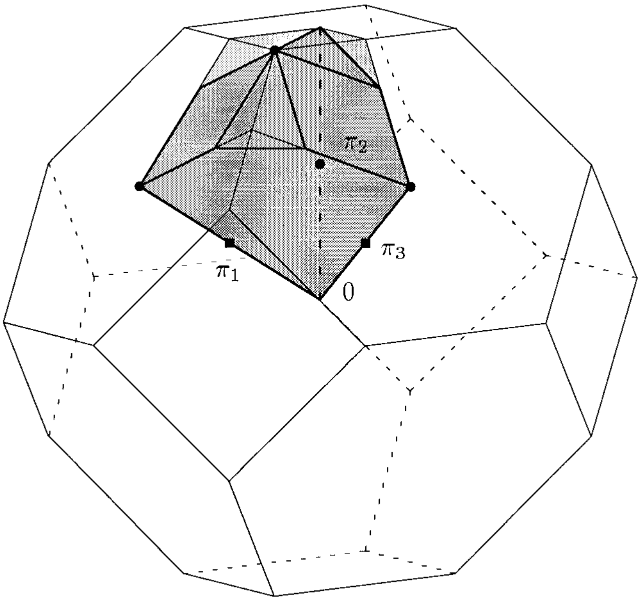

FIG. 3. $K = \mathrm{SU}(4)$ and $\lambda = \pi_1 + \pi_2 + \pi_3 = \frac{1}{2}\sum_{\alpha \in \Psi^+} \alpha$. Vertices of $\Delta(\mathbb{P}V)$ are $\pi_1 + \pi_2 + \pi_3$, 0, $2\pi_1, 2\pi_2, 2\pi_3$, $\frac{4}{3}\pi_1 + \pi_2$, $\frac{4}{3}\pi_3 + \pi_2$, $\pi_1 + \frac{5}{3}\pi_3$ and $\frac{5}{3}\pi_1 + \pi_3$.

THEOREM 7.6. *For every connected K-manifold Q, the set $\Delta(T^*Q)$ is a rational convex polyhedral cone. It is invariant under the involution $*$ and equal to the momentum cone of the complexification of the K-variety $K \times^{K_q} V$. Here $V = T_qQ/T_q(Kq)$ is the slice at an arbitrary point $q \in Q$.*

For instance, let L be a closed subgroup of K and let Q be the homogeneous space K/L. Then the theorem says that $\Delta(T^*(K/L)) = \Delta(G/L^{\mathbb{C}})$.

7.3. *Symplectic Quotients*

Let M be a Hamiltonian K-manifold with momentum map $\Phi\colon M \to \mathfrak{k}^*$. Let L be a closed normal subgroup of K. Then M is a Hamiltonian L-space with momentum map $\Phi_{(L)} = \iota^* \circ \Phi\colon M \to \mathfrak{l}^*$, where ι is the inclusion of $\mathfrak{l}$ into $\mathfrak{k}$. Let $\bar{K}$ be the Lie group K/L. The kernel of ι^* can be identified in a natural way with $\bar{\mathfrak{k}}^*$, the dual of the Lie algebra of $\bar{K}$. Let μ be any point in $\mathfrak{z}(\mathfrak{k})^*$, where $\mathfrak{z}(\mathfrak{k})$ denotes the centre of $\mathfrak{k}$. The symplectic quotient of M at the level $\iota^*\mu \in \mathfrak{z}(\mathfrak{l})^*$ with respect to the L-action,

$$M_{\iota^*\mu} = M_{\iota^*\mu,\, L} = \Phi_{(L)}^{-1}(\iota^*\mu)/L = \Phi^{-1}(\mu + \bar{\mathfrak{k}}^*)/L,$$

is a stratified Hamiltonian $\bar{K}$-space. (Cf. [30].) A momentum map $\Phi_{(\bar{K})}: M_{\iota^*\mu} \to \bar{\mathfrak{k}}^* \subset \mathfrak{k}^*$ for the $\bar{K}$-action on $M_{\iota^*\mu}$ is induced by the map $\Phi^{-1}(\mu + \bar{\mathfrak{k}}^*) \to \bar{\mathfrak{k}}^*$ sending m to $\Phi(m) - \mu$. (Up to a shift by an element of $\mathfrak{z}(\bar{\mathfrak{k}})^*$, the map $\Phi_{(\bar{K})}$ only depends on the point $\iota^*\mu$.) It is easy to calculate $\Delta(M_{\iota^*\mu})$ in terms of $\Delta(M)$. Let $\bar{T}$ be the maximal torus $T/(T \cap L)$ of $\bar{K}$. Then $\bar{\mathfrak{t}}^*$ is naturally isomorphic to $\mathfrak{t}^* \cap \bar{\mathfrak{k}}^*$, and the intersection $\bar{\mathfrak{t}}^*_+ = \mathfrak{t}^*_+ \cap \bar{\mathfrak{k}}^*$ is a Weyl chamber of the pair $(\bar{K}, \bar{T})$. The following result is now obvious (regardless of whether $M_{\iota^*\mu}$ is smooth or not).

Proposition 7.7. *$\Delta(M_{\iota^*\mu}) = (-\mu + \Delta(M)) \cap \bar{\mathfrak{k}}^*$. Therefore, if $\Delta(M)$ is a closed convex polyhedral subset of $\mathfrak{t}^*_+$, then $\Delta(M_{\iota^*\mu})$ is a closed convex polyhedral subset of $\bar{\mathfrak{t}}^*_+$.*

For example, consider the Hamiltonian $K \times K$-space $T^*K \cong K \times \mathfrak{k}^*$ with momentum map $(k, v) \mapsto (kv, -v)$. Let L be any closed subgroup of K. The momentum map for the restriction of the action to $K \times L$ is $(k, v) \mapsto (kv, -v|_{\mathfrak{l}})$. The symplectic quotient of T^*K at level 0 with respect to the normal subgroup $\{1\} \times L \subset K \times L$ is isomorphic as a Hamiltonian K-space to the cotangent bundle of the homogeneous space K/L. Proposition 7.7 tells us that the momentum map image of $T^*(K/L)$ is the set $K\{v : v|_{\mathfrak{l}} = 0\}$, and hence

$$\Delta(T^*(K/L)) = K\mathfrak{l}^{\circ} \cap \mathfrak{t}^*_+, \tag{7.6}$$

where $\mathfrak{l}^{\circ}$ is the annihilator of $\mathfrak{l}$ in $\mathfrak{k}^*$. (This can also be seen from the equality $\Delta(T^*(K/L)) = \Delta(G/L^{\mathbb{C}})$ proven in Section 7.2 and the Peter–Weyl Theorem.) Consequently, the set $K\mathfrak{l}^{\circ} \cap \mathfrak{t}^*_+$ is a rational convex polyhedral cone.

Now assume that K is semisimple and take L to be the maximal torus T. Kostant's convexity theorem [16] implies that $K\mathfrak{t}^{\circ} \cap \mathfrak{t}^*_+ = \mathfrak{t}^*_+$. (Consider the natural projection $\iota^*: \mathfrak{k}^* \to \mathfrak{t}^*$. Take any $\mu \in \mathfrak{t}^*_+$. By Kostant's theorem the set $\iota^*(K\mu) \subset \mathfrak{t}^*$ is equal to the convex hull of the Weyl group orbit through μ, which contains the origin in $\mathfrak{t}^*$. Therefore $K\mu \cap \mathfrak{t}^{\circ} = K\mu \cap \ker \iota^*$ is not empty.) We conclude from (7.6) that $\Delta(T^*(K/T)) = \mathfrak{t}^*_+$.

More generally, take L to be the centralizer K_σ of a wall σ of the Weyl chamber $\mathfrak{t}^*_+$. It is not difficult to see from the root space decomposition of the pair $(\mathfrak{k}, \mathfrak{t})$ that $K\mathfrak{k}^{\circ}_\sigma \cap \mathfrak{t}^*_+$ contains the ray through every dominant root that is not perpendicular to the wall σ. This is insufficient information to determine $\Delta(T^*(K/K_\sigma))$ in general, but if K is e.g., of type B_2 or G_2, then this implies that $\Delta(T^*(K/K_\sigma)) = \mathfrak{t}^*_+$ for any wall $\sigma \neq \{0\}$. If $K = \mathrm{SU}(n)$ and σ is the one-dimensional wall spanned by either π_1 or $\pi_{n-1} = \pi_1^*$, one can easily calculate by hand that $\Delta(T^*(K/K_\sigma)) = K\mathfrak{k}^{\circ}_\sigma \cap \mathfrak{t}^*_+$ is equal to the ray spanned by the maximal root $\alpha_1 + \cdots + \alpha_{n-1} = \pi_1 + \pi_{n-1}$.

As another application of Proposition 7.7, I now give a proof of Proposition 6.9. For any $\mu \in \mathfrak{t}^*_+$ and for any closed subgroup L of K_μ, let

M be the space $X(\mu, L, \{0\})$, the local model of Definition 6.2 with trivial symplectic slice W. By (6.2), M is the bundle over the coadjoint orbit $K\mu$ with fibre $T^*(K_\mu/L)$ furnished with the minimal-coupling form. Put $\mathfrak{m} = \mathfrak{k}_\mu/\mathfrak{l}$. Then $\mathfrak{m}^*$ is canonically isomorphic to the annihilator of $\mathfrak{l}$ inside $\mathfrak{k}_\mu^*$. Clearly, the point $m = (1, 0, 0)$ in $K \times^L \mathfrak{m}^* \cong M$ has the property that $K_m = L$ and $\Phi(m) = \mu$. Also, the symplectic cross-section of M at m is just the cotangent bundle $T^*(K_\mu/L)$ with its standard momentum map shifted by $\mu \in \mathfrak{z}_\mu^*$. By (7.6), the local momentum cone of M at m is therefore equal to

$$\Delta_m = \mu + (K_\mu \mathfrak{m}^* \cap \mathfrak{t}^*_{+,\mu}), \tag{7.7}$$

where $\mathfrak{t}^*_{+,\mu}$ denotes the positive Weyl chamber of $\mathfrak{k}_\mu$. Now assume that $K_\mu = [K_\mu, K_\mu]\, L$, or, in other words, $\mathfrak{k}_\mu = [\mathfrak{k}_\mu, \mathfrak{k}_\mu] + \mathfrak{l}$. Because of the decomposition $\mathfrak{k}_\mu^* = [\mathfrak{k}_\mu, \mathfrak{k}_\mu]^* \oplus \mathfrak{z}_\mu^*$, this is equivalent to $\mathfrak{m}^* \cap \mathfrak{z}_\mu^* = \{0\}$, or:

$$K_\mu \mathfrak{m}^* \cap \mathfrak{z}_\mu^* = \{0\}. \tag{7.8}$$

Now the Weyl chamber $\mathfrak{t}^*_{+,\mu}$ is the product of $\mathfrak{z}_\mu^*$ and the Weyl chamber of the semisimple part, $\mathfrak{t}^*_{+,\mu} \cap [\mathfrak{k}_\mu, \mathfrak{k}_\mu]^*$, which is a proper cone. So the cone $K_\mu \mathfrak{m}^* \cap \mathfrak{t}^*_{+,\mu}$ could only fail to be proper if $K_\mu \mathfrak{m}^*$ contained a nontrivial linear subspace of $\mathfrak{z}_\mu^*$. But this is impossible because of (7.8). By (7.7), the point $\Phi(m)$ is therefore a vertex of $\Delta(M)$. This completes the proof of Proposition 6.9.

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