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On the complex zeros of $H_\mu(z)$, $J'_\mu(z)$, $J''_\mu(z)$ for real or complex order

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Abstract

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Propositions about the nonexistence of complex zeros of the functions $H_\mu(z)=\alpha J_\mu(z)+zJ'_\mu(z)$, $J'_\mu(z)$, $J''_\mu(z)$, where $J'_\mu(z)$ and $J''_\mu(z)$ are the first two derivatives of the Bessel functions $J_\mu(z)$, for μ in general complex, are proved. Bounds for the purely imaginary zeros of the above functions assuming their existence are given. Thus for the range of values for which these bounds are violated there are no purely imaginary zeros of the above functions. Finally, some known results from previous work are generalized in the present paper.

Keywords: Mixed Bessel functions, zeros of derivatives of Bessel functions.

1. Introduction

Let $J_\mu(z)$ be the ordinary Bessel function of order μ, $J'_\mu(z)$ and $J''_\mu(z)$ are the first and second derivatives of $J_\mu(z)$ and $H_\mu(z)=\alpha J_\mu(z)+zJ'_\mu(z)$, $\alpha\in\mathbb{C}$.

The study of the above functions is motivated by several problems arising from the solution of the wave or the heat equation with appropriate boundary conditions and the fact that they are related to the zeros of many interesting mixed Bessel functions.

Several methods have been used in order to study the properties of zeros of these functions. In this paper we use a method which has been presented in [2] and is based on the study of operators in an abstract Hilbert space and also the well-known Mittag–Leffler expansion [4, p.497].

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In the present paper we give some propositions about the nonexistence of complex zeros of the functions $H_\mu(z)$, $J'_\mu(z)$, $J''_\mu(z)$, for μ in general complex. We give also some bounds for the purely imaginary zeros of the above functions, assuming their existence. So, if the given inequalities do not hold, then there are no purely imaginary zeros of the above functions. These bounds for the purely imaginary zeros of $H_\mu(z)$ are

$$\rho_2^2 > -\frac{2(\mu_1+1)|\alpha+\mu|^2}{\mu_1+\alpha_1}, \quad -1<\mu_1<-\alpha_1, \tag{1.1}$$

$$|\rho_2| > \frac{|\mu+\alpha|\sqrt{|\mu+\alpha|^2+2\mu_2(\mu_2+\alpha_2)}-|\mu+\alpha|^2}{|\mu_2+\alpha_2|},$$

$$\mu_2>\max\{0,-\alpha_2\} \text{ or } \mu_2<\min\{0,-\alpha_2\}, \tag{1.2}$$

where ρ_1, ρ_2, μ_1, μ_2, α_1, α_2 are the real and imaginary parts of the zeros, order and α of $H_\mu(z)$, respectively. From (1.1) and (1.2) for $\alpha_1=\alpha_2=0$, we obtain the bounds (2.15) and (2.20), for the purely imaginary zeros of $J'_\mu(z)$.

For the measure of the purely imaginary zeros of $J''_\mu(z)$ the following bounds are found:

$$|\rho_2| > \left[\frac{2\mu_1(1-\mu_1^2)}{1+2\mu_1}\right]^{1/2}, \qquad -1<\mu_1<-\tfrac{1}{2},\ \mu_2=0, \tag{1.3}$$

$$|\rho_2| > \left[\frac{\mu_2^2+|\mu_2|\sqrt{8+\mu_2^2}}{2}\right]^{1/2}, \quad \mu_1=0, \tag{1.4}$$

$$|\rho_2| > |\mu_2|(\sqrt{3}-1), \qquad \mu_1=0, \tag{1.5}$$

as well as the bounds (2.26) and (2.29), holding for complex μ with $\mu_1>-1$, $\mu_2\neq 0$ and $-1<\mu_1<-\frac{1}{2}$, $\mu_2\neq 0$, respectively.

Some results of [3] are generalized in the present paper.

2. Main results

First of all we note that $H_\mu(z)$, $J'_\mu(z)$ and $J''_\mu(z)$ are particular cases of the function

$$M_\mu(z)=(\beta z^2+\alpha)J_\mu(z)+zJ'_\mu(z).$$

In [1,3] was proved that $\rho\neq 0$ is a zero of the function

$$(\beta z^2+\alpha)J_\mu(z)+zJ'_\mu(z), \tag{2.1}$$

if and only if there exists an element $u\neq 0$ in an abstract Hilbert space H, such that

$$(C_0+\bar{\mu})u-\tfrac{1}{2}\bar{\rho}T_0u=-\tfrac{1}{2}\bar{\rho}^2e_1 \tag{2.2}$$

and

$$(u,e_1)=-\left(\overline{(\beta\rho^2+\alpha)}+\bar{\mu}\right), \tag{2.3}$$

where C_0 is the diagonal operator: $C_0 e_n = n e_n$, $n \geqslant 1$; and $T_0 = V + V^*$, V and V^* being the shift operator with respect to the basis e_n and the adjoint of V, respectively. From (2.2) and (2.3) we find

$$2(C_0 u, u) + 2\bar{\mu} \| u \|^2 - \bar{\rho}(T_0 u, u) = \beta | \rho |^4 + (\alpha + \mu)\rho^2. \tag{2.4}$$

For $\rho = \rho_1 + \mathrm{i}\rho_2$, $\mu = \mu_1 + \mathrm{i}\mu_2$, $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ and $\beta = \beta_1 + \mathrm{i}\beta_2$, equating the real and imaginary parts of (2.4), we obtain

$$2(C_0 u, u) + 2\mu_1 \| u \|^2 - \rho_1(T_0 u, u) = \beta_1 | \rho |^4 + (\alpha_1 + \mu_1)(\rho_1^2 - \rho_2^2) + 2\rho_1\rho_2(\alpha_2 + \mu_2) \tag{2.5}$$

and

$$-2\mu_2 \| u \|^2 + \rho_2(T_0 u, u) = \beta_2 | \rho |^4 + (\alpha_2 + \mu_2)(\rho_1^2 - \rho_2^2) - 2\rho_1\rho_2(\alpha_1 + \mu_1). \tag{2.6}$$

Since $u = \sum_{n=1}^{\infty} (u, e_n)e_n$, we find

$$(C_0 u, u) > \| u \|^2, \tag{2.7}$$

and from (2.3) we get:

$$\| u \|^2 > |(u, e_1)|^2 = | \beta\rho^2 + \alpha + \mu |^2. \tag{2.8}$$

We also know from [2]:

$$|(T_0 u, u)| \leqslant \| T_0 \| \, \| u \|^2 = 2 \| u \|^2. \tag{2.9}$$

Multiplying both sides of (2.5) and (2.6) by ρ_2 and ρ_1, respectively, and adding, we get

$$2\rho_2(C_0 u, u) + 2(\mu_1\rho_2 - \mu_2\rho_1) \| u \|^2 = (\beta_1\rho_2 + \beta_2\rho_1) | \rho |^4 + \left[(\mu_2 + \alpha_2)\rho_1 - (\mu_1 + \alpha_1)\rho_2\right] | \rho |^2. \tag{2.10}$$

Proposition 2.1. *For* $\mu_1 > \max\{-1, -\alpha_1\}$ *and* $\mu_2 > \max\{0, -\alpha_2\}$ (*or* $\mu_2 < \min\{-\alpha_2, 0\}$) *the function* $H_\mu(z)$ *has no complex zeros in the second and fourth* (*or first and third*) *quadrants.*

Proof. For $\rho_2 \neq 0$ and $\beta_1 = \beta_2 = 0$ from (2.10) and due to (2.7) we have

$$\tfrac{1}{2} | \rho |^2 \left[(\mu_2 + \alpha_2)\frac{\rho_1}{\rho_2} - (\mu_1 + \alpha_1)\right] > \left[\mu_1 - \mu_2\frac{\rho_1}{\rho_2} + 1\right] \| u \|^2. \tag{2.11}$$

If $\mu_1 \geqslant \max\{-\alpha_1, -1\}$ and $\mu_2 > \max\{-\alpha_2, 0\}$, $\rho_1\rho_2 < 0$ (or $\mu_2 < \min\{-\alpha_2, 0\}$, $\rho_1\rho_2 > 0$), (2.11) does not hold, which proves the proposition. □

Proposition 2.2. *For* $\mu = \mu_1 + \mathrm{i}\mu_2$, $\mu_2 > \max\{0, -\alpha_2\}$ (*or* $\mu_2 < \min\{0, -\alpha_2\}$) $H_\mu(z)$ *has no real zeros.*

Proof. From (2.6) for $\beta_1 = \beta_2 = \rho_2 = 0$, we get

$$-\mu_2 \| u \|^2 = \tfrac{1}{2}\rho_1^2(\alpha_2 + \mu_2).$$

If $\mu_2 > \max\{0, -\alpha_2\}$ or $\mu_2 < \min\{0, -\alpha_2\}$, the above equation does not hold. □

Proposition 2.3. *For $\mu_1 > -1$, $\mu_2 = 0$, $\alpha_2 > 0$ (or $\alpha_2 < 0$) $H_\mu(z)$ has no complex zeros in the second and fourth (or first and third) quadrants.*

Proof. The zeros of $M_\mu(z)$ satisfy the relation

$$\beta z^2 + \alpha + \mu = \frac{zJ_{\mu+1}(z)}{J_\mu(z)}, \tag{2.12}$$

as follows from

$$(\beta z^2 + \alpha) J_\mu(z) + zJ_\mu'(z) = 0$$

and the well-known relation [4, p.45]

$$zJ_\mu'(z) = \mu J_\mu(z) - zJ_{\mu+1}(z).$$

Because of the well-known Mittag–Leffler expansion [4, p.497]

$$\frac{zJ_{\mu+1}(z)}{J_\mu(z)} = 2z^2 \sum_{n=1}^{\infty} \frac{1}{j_{\mu,n}^2 - z^2}, \quad z \neq \pm j_{\mu,n}, \ n = 1, 2, \ldots,$$

(2.12) becomes

$$\beta z^2 + \alpha + \mu = 2z^2 \sum_{n=1}^{\infty} \frac{1}{j_{\mu,n}^2 - z^2}. \tag{2.13}$$

From (2.13), for $\beta = 0$, $\alpha = \alpha_1 + i\alpha_2$, $\mu = \mu_1 + i\mu_2$, $z = \rho_1 + i\rho_2$ and equating the imaginary parts of the two members of (2.13), we find

$$\alpha_2 = 4\rho_1\rho_2 \sum_{n=1}^{\infty} \frac{j_{\mu,n}^2}{\left| j_{\mu,n}^2 - z^2 \right|^2},$$

which does not hold for $\alpha_2 > 0$, $\rho_1\rho_2 < 0$ or $\alpha_2 < 0$, $\rho_1\rho_2 > 0$, so the desired result is established. Also, from the last relation, we obtain that the function $H_\mu(z)$ cannot have purely imaginary zeros for $\mu_1 > -1$, $\mu_2 = 0$, $\alpha_2 \neq 0$, a result which was proved in [3] by a different method. □

Proposition 2.4. *For $\mu = \mu_1 + i\mu_2$, $\alpha = \alpha_1 + i\alpha_2$, $-1 < \mu_1 < -\alpha_1$, the purely imaginary zeros of $H_\mu(z)$, if they exist, satisfy the relation*

$$\rho_2^2 > -\frac{2(\mu_1 + 1)|\alpha + \mu|^2}{\mu_1 + \alpha_1}. \tag{2.14}$$

Proof. This follows from (2.11) and (2.8) for $\rho_1 = 0$ and $\beta = 0$. □

Remark 2.5. (i) For $\alpha_1 = \alpha_2 = 0$, $\mu = \mu_1 + i\mu_2$, $-1 < \mu_1 < 0$, from (2.14) we obtain

$$\rho_2^2 > -\frac{2(\mu_1 + 1)|\mu|^2}{\mu_1}, \tag{2.15}$$

for the purely imaginary zeros of $J_\mu'(z)$, if they exist.

(ii) For $\alpha_2 = \mu_2 = 0$, (2.14) gives the bound: $\rho_2^2 > -2(\mu_1 + 1)(\mu_1 + \alpha_1)$, found in [3].

Proposition 2.6. *For* $\mu_2 > \max\{0, -\alpha_2\}$ *or* $\mu_2 < \min\{0, -\alpha_2\}$, *the purely imaginary zeros of* $H_\mu(z)$, *if they exist, satisfy the relation*

$$|\rho_2| > \frac{-|\alpha+\mu|^2 + |\alpha+\mu|\sqrt{|\alpha+\mu|^2 + 2\mu_2(\alpha_2+\mu_2)}}{|\alpha_2+\mu_2|}. \tag{2.16}$$

Proof. From (2.6), for $\beta_2 = \rho_1 = 0$, $\rho_2 \neq 0$ and due to (2.9) we obtain

$$-1 + \frac{\mu_2}{\rho_2} \leqslant \frac{\rho_2(\mu_2+\alpha_2)}{2\|u\|^2} \leqslant 1 + \frac{\mu_2}{\rho_2}. \tag{2.17}$$

From (2.8) for $\beta = 0$, and from (2.17), we have: if $\rho_2 > 0$, $\mu_2+\alpha_2 > 0$ or $\rho_2 < 0$, $\mu_2+\alpha_2 < 0$,

$$-1 + \frac{\mu_2}{\rho_2} < \frac{\rho_2(\mu_2+\alpha_2)}{2|\mu+\alpha|^2}; \tag{2.18}$$

if $\rho_2 > 0$, $\mu_2+\alpha_2 < 0$ or $\rho_2 < 0$, $\mu_2+\alpha_2 > 0$,

$$1 + \frac{\mu_2}{\rho_2} > \frac{\rho_2(\mu_2+\alpha_2)}{2|\mu+\alpha|^2}. \tag{2.19}$$

After some algebra, we get from (2.18):

$$\rho_2 > \frac{-|\alpha+\mu|^2 + |\alpha+\mu|\sqrt{|\alpha+\mu|^2 + 2\mu_2(\alpha_2+\mu_2)}}{\alpha_2+\mu_2}, \quad \text{for } \mu_2+\alpha_2 > 0,$$

$$\rho_2 < \frac{-|\alpha+\mu|^2 + |\alpha+\mu|\sqrt{|\alpha+\mu|^2 + 2\mu_2(\alpha_2+\mu_2)}}{\alpha_2+\mu_2}, \quad \text{for } \mu_2+\alpha_2 < 0.$$

And from (2.19):

$$\rho_2 > \frac{|\alpha+\mu|^2 - |\alpha+\mu|\sqrt{|\alpha+\mu|^2 + 2\mu_2(\alpha_2+\mu_2)}}{\alpha_2+\mu_2}, \quad \text{for } \mu_2+\alpha_2 < 0,$$

$$\rho_2 < \frac{|\alpha+\mu|^2 - |\alpha+\mu|\sqrt{|\alpha+\mu|^2 + 2\mu_2(\alpha_2+\mu_2)}}{\alpha_2+\mu_2}, \quad \text{for } \mu_2+\alpha_2 > 0.$$

Hence we have

$$|\rho_2| > \frac{-|\alpha+\mu|^2 + |\alpha+\mu|\sqrt{|\alpha+\mu|^2 + 2\mu_2(\alpha_2+\mu_2)}}{\alpha_2+\mu_2}, \quad \text{for } \mu_2+\alpha_2 > 0,$$

and

$$|\rho_2| > \frac{|\alpha+\mu|^2 - |\alpha+\mu|\sqrt{|\alpha+\mu|^2 + 2\mu_2(\alpha_2+\mu_2)}}{\alpha_2+\mu_2}, \quad \text{for } \mu_2+\alpha_2 < 0.$$

From these inequalities we have the desired result. The discriminant is positive due to the conditions for μ_2. □

Corollary 2.7. *For $\mu=\mu_1+\mathrm{i}\mu_2$, $\mu_1<0$, the purely imaginary zeros of $J_\mu'(z)$, if they exist, satisfy the relation*

$$|\rho_2|>\frac{|\mu|\sqrt{|\mu|^2+2\mu_2^2}-|\mu|^2}{|\mu_2|}. \tag{2.20}$$

Proof. We make the assumption that $\mu_1<0$, because from (2.10) for $\mu_2\neq 0$, $\rho_1=\alpha_1=\alpha_2=\beta_1=\beta_2=0$ and $\mu_1\geqslant 0$, $J_\mu'(z)$ cannot have purely imaginary zeros. We obtain (2.20) from (2.16) for $\alpha_1=\alpha_2=0$. □

Remark 2.8. The bound (2.15) is sharper than (2.20) for $(-2-\sqrt{3})/(3+\sqrt{3})<\mu_1<0$, i.e., for $-0.788675<\mu_1<0$. In fact, if we take the ratio of the quantities on the right-hand side of (2.15) and (2.20), we have

$$A=\frac{\left[-2-\dfrac{2}{\mu_1}\right]^{1/2}|\mu_2|}{\sqrt{|\mu|^2+2\mu_2^2}\,|\mu|}=\frac{\left[-2-\dfrac{2}{\mu_1}\right]^{1/2}|\mu_2|\left(\sqrt{|\mu|^2+2\mu_2^2}+|\mu|\right)}{2\mu_2^2}$$

or

$$A^2=\frac{\left[-1-\dfrac{1}{\mu_1}\right]\left(\sqrt{|\mu|^2+2\mu_2^2}+|\mu|^2\right)^2}{2\mu_2^2}.$$

We have

$$\frac{\left(\sqrt{|\mu|^2+2\mu_2^2}+|\mu|\right)^2}{2\mu_2^2}>\frac{(\sqrt{3}+1)^2\mu_2^2}{2\mu_2^2}=2+\sqrt{3}, \tag{2.21}$$

$$\text{if } \frac{-2-\sqrt{3}}{3+\sqrt{3}}<\mu_1<0, \quad \text{then } -1-\frac{1}{\mu_1}>\frac{1}{2+\sqrt{3}}. \tag{2.22}$$

Multiplying (2.21) and (2.22), we find $A^2>1$ and since $A>0$, it follows that $A>1$, which proves our assertion.

On the contrary, for $\mu_1\to -1$, $\mu_1>-1$ and $|\mu_2|$ not very small, obviously $A^2<1$ and the bound (2.20) is sharper than (2.15). For example, for $-1<\mu_1\leqslant -0.8$, $|\mu_2|\geqslant 2.9$, (2.20) is sharper than (2.15).

Remark 2.9. From (2.15) and (2.20) we find that for $\mu_2\to\pm\infty$, $|\rho_2|\to\infty$.

Proposition 2.10. *For real μ, $0\leqslant\mu\leqslant 1$, the function $J_\mu''(z)$ has no complex zeros.*

Proof. From the Bessel differential equation

$$(z^2-\mu^2)J_\mu(z)+zJ_\mu'(z)=-z^2J_\mu''(z),$$

we see that the zeros of $J''_\mu(z)$ are the same as the zeros of (2.1) for $\beta = 1$ and $\alpha = -\mu^2$. So, from (2.13), putting $\beta = 1$, $\mu_2 = 0$, $z = \rho_1 + i\rho_2$, ρ_1, $\rho_2 \neq 0$, $\alpha_1 = -\mu^2$, $\alpha_2 = 0$ and equating the real and imaginary parts, we have

$$\mu - \mu^2 + (\rho_1^2 - \rho_2^2) = 2(\rho_1^2 - \rho_2^2) \sum_{n=1}^{\infty} \frac{j_{\mu,n}^2}{\left| j_{\mu,n}^2 - z^2 \right|^2} - 2|\rho|^4 \sum_{n=1}^{\infty} \frac{1}{\left| j_{\mu,n}^2 - z^2 \right|^2}, \tag{2.23}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{j_{\mu,n}^2}{\left| j_{\mu,n}^2 - z^2 \right|^2}. \tag{2.24}$$

From (2.23), because of (2.24) we obtain

$$\mu - \mu^2 = -2|\rho|^4 \sum_{n=1}^{\infty} \frac{1}{\left| j_{\mu,n}^2 - z^2 \right|^2},$$

which does not hold for $0 \leqslant \mu \leqslant 1$. □

Proposition 2.11. *For $\mu_2 = 0$, $-1 < \mu_1 < -\frac{1}{2}$, the purely imaginary zeros of $J''_\mu(z)$, if they exist, satisfy the inequality*

$$|\rho_2| > \left[\frac{2\mu_1(1 - \mu_1^2)}{1 + 2\mu_1} \right]^{1/2}. \tag{2.25}$$

Proof. From (2.10) for $\beta_1 = 1$, $\alpha_1 = -\mu_1^2$, $\beta_2 = \alpha_2 = \mu_2 = \rho_1 = 0$ and due to (2.7), (2.8) we have

$$(1 + 2\mu_1)\rho_2^4 - (\mu_1 - \mu_1^2)(3 + 4\mu_1)\rho_2^2 + 2(\mu_1 + 1)(\mu_1 - \mu_1^2)^2 < 0.$$

Since $2\mu_1 + 1 < 0$, from the above inequality follows (2.25). □

Remark 2.12. In the case $\mu_2 = 0$, $-\frac{1}{2} < \mu_1 < 0$, in [3, p.165] was proved that the function $J''_\mu(z)$ has no purely imaginary zeros.

Proposition 2.13. *For $\mu = \mu_1 + i\mu_2$, $\mu_1 > -1$, the purely imaginary zeros of $J''_\mu(z)$, if they exist, satisfy the inequality*

$$|\rho_2| > \left[\tfrac{1}{2}\left\{ \mu_2^2 + \mu_1 - \mu_1^2 + \sqrt{(\mu_2^2 + \mu_1 - \mu_1^2) + 8(1 + \mu_1)\mu_2^2(1 - 2\mu_1)^2} \right\} \right]^{1/2}. \tag{2.26}$$

Proof. From (2.10), (2.7) and (2.8) for $\rho_1 = \beta_2 = 0$, $\beta_1 = 1$, $\alpha_1 = \mu_2^2 - \mu_1^2$, $\alpha_2 = -2\mu_1\mu_2$ and if $\mu_1 + 1 > 0$, we obtain

$$\rho_2^4 + (\mu_1^2 - \mu_2^2 - \mu_1)\rho_2^2 - 2(\mu_1 + 1)\mu_2^2(1 - 2\mu_1)^2 > 2(1 + \mu_1)(\rho_2^2 + \mu_1^2 - \mu_2^2 - \mu_1)^2. \tag{2.27}$$

Hence,

$$\rho_2^4 + (\mu_1^2 - \mu_2^2 - \mu_1)\rho_2^2 - 2(\mu_1 + 1)\mu_2^2(1 - 2\mu_1)^2 > 0,$$

from which (2.26) follows. □

Remark 2.14. For $\mu_1 = 0$ the inequality (2.26) becomes

$$|\rho_2| > \left[\tfrac{1}{2}\left\{\mu_2^2 + |\mu_2|\sqrt{\mu_2^2 + 8}\right\}\right]^{1/2}. \tag{2.28}$$

Proposition 2.15. *For $\mu = \mu_1 + i\mu_2$, $-1 < \mu_1 < -\frac{1}{2}$, the purely imaginary zeros of $J_\mu''(z)$, if they exist, satisfy the inequality*

$$|\rho_2| > \left[\frac{-(\mu_1^2 - \mu_2^2 - \mu_1)(3 + 4\mu_1) + \sqrt{\Delta}}{2(1 + 2\mu_1)}\right]^{1/2}, \tag{2.29}$$

where Δ is given by

$$\Delta = (\mu_1 - \mu_1^2 + \mu_2^2)^2 - 8(1 + 2\mu_1)(1 + \mu_1)(\mu_2 - 2\mu_1\mu_2)^2.$$

Proof. For $-1 < \mu_1 < -\frac{1}{2}$, Δ is positive, and since $1 + 2\mu_1 < 0$, the proposition follows from (2.27). □

Remark 2.16. For $\mu_2 = 0$, (2.29) gives (2.25). Note that (2.29) is sharper than (2.26) but has narrower range of validity.

Proposition 2.17. *For $\mu = i\mu_2$, the purely imaginary zeros of $J_\mu''(z)$, if they exist, satisfy the inequality*

$$|\rho_2| > |\mu_2|(\sqrt{3} - 1). \tag{2.30}$$

Proof. From (2.6), due to (2.9) and (2.8), and for $\rho_1 = \mu_1 = \beta_2 = \alpha_2 = 0$, and $\beta_1 = 1$, $\alpha_1 = \mu_2^2$, and proceeding as in the proof of Proposition 2.6 we have the desired inequality. □

Remark 2.18. The bound (2.26), for $\mu_1 = 0$, is sharper than (2.30).

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