Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 206 (2007) 927–949
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Numerical study of modified Adomian's method applied to Burgers equation

Mário Basto[a,*], Viriato Semiao[b], Francisco L. Calheiros[c]

[a]*Mathematic Group, Escola Superior de Gestão, IPCA, Barcelos, Portugal*
[b]*Mechanical Engineering Department, Instituto Superior Técnico, Technical University of Lisbon, Portugal*
[c]*Civil Engineering Department, Faculdade de Engenharia, University of Porto, Portugal*

Received 5 April 2006; received in revised form 4 September 2006

Abstract

The application of Adomian's decomposition method to partial differential equations, when the exact solution is not reached, demands the use of truncated series. But the solution's series may have small convergence radius and the truncated series may be inaccurate in many regions. In order to enlarge the convergence domain of the truncated series, Padé approximants (PAs) to the Adomian's series solution have been tested and applied to partial and ordinary differential equations, with good results. In this paper, PAs, both in x and t directions, applied to the truncated series solution given by Adomian's decomposition technique for Burgers equation, are tested. Numerical and graphical illustrations show that this technique can improve the accuracy and enlarge the domain of convergence of the solution. It is also shown in this paper, that the application of Adomian's method to the ordinary differential equations set arising from the discretization of the spatial derivatives by finite differences, the so-called method of lines, may reduce the convergence domain of the solution's series.

MSC: 35C10; 65M20; 41A21

Keywords: Burgers equation; Adomian's decomposition method; Padé approximants; Method of lines

1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics, in mechanics and biology. Hence, explicit solutions to the nonlinear equations are of fundamental importance to preserve the actual physical character of the problem and to understand thoroughly the described process. In the open literature the advantages of the Adomian's decomposition method applied to many linear and nonlinear problems have been emphasized by many authors. Some analytic procedures to solve nonlinear differential equations, linearize the system or assume that the nonlinearities are relatively insignificant. However, this procedure may change the real solution of the mathematical model that represents the physical reality, sometimes in a serious way. Generally, the numerical methods are based on discretization techniques, and allow only for the calculation of the approximate solution for some values of time and space variables, exhibiting therefore the disadvantage of leading us to overlook some important phenomena occurring in very small time and

* Corresponding author.
E-mail address: mbasto@ipca.pt (M. Basto).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.09.001

space intervals, such as chaos and bifurcations. Various methods for obtaining explicit solutions to nonlinear evolution equations are described in the literature, such as the similarity methods, the generalized separation of variables, the tanh method, the sine–cosine method, the Painlevé method, the Darboux transformation, the inverse scattering transform, the Bäcklund transformation (like the Hopf–Cole transformation in the Burgers equation [41] and [29]) among others [44,30,57].

Adomian's method does not require discretization of the variables, and, therefore, it is not affected by errors associated to discretization. Also, the decomposition method does not require linearization or perturbation and, consequently, does not change the true solution of the problem, being very efficient on determining an approximate or even exact solution in a closed form, for both linear and nonlinear problems [59–63,31].

Another advantage of the Adomian's decomposition technique is on the provision of a rapidly convergent series, attribute manifested by several authors [26,3–6,27,15], as it is only required a small number of terms to obtain an approximation of the solution with high accuracy [45,32,25].

One can find in the literature, among others, applications to stochastic problems [14,9,34,35], ordinary differential equations [39,58] partial differential equations, [66,48,42] and nonlinear equations [4,18,17,8,20] being the benefits of Adomian's method almost always emphasized. However, some embarrassment may appear that are not quite stressed. The good results presented include exact Taylor series solutions identified from the truncated series. This is achieved either because it is easy to identify the exact solution from the truncated series or because the real solution derived by other methods is known in advance [67,46,47] or because the presence of the so-called noisy terms [62,46,47]. Otherwise, it would not be so straightforward the finding of such closed form solutions. Moreover, the solution's series may have small convergence radius and the truncated series solution may be inaccurate in some regions, as described in [43,23,33,1,2]. Also, difficulties can still be encountered on the application of Adomian's method to initial-boundary value problems. These type of problems have been usually solved based only on the imposition of the initial conditions. However, when initial and boundary conditions have to be imposed, the Adomian's series may converge (or not) to a solution with the wrong boundary conditions. To overcome this problem, Adomian [11] and Lesnic [51–53] proposed a method to solve linear initial-boundary problems, with possible extensions to higher-dimensional, inhomogeneous and nonlinear problems. Ngarhasta et al. [56], to solve some PDEs, suggest taking in account both initial and boundary conditions in the canonical form, by choosing the first term of Adomian's series verifying all conditions.

One way of improving the mathematical structure, gaining more information and enlarging the convergence domain of the truncated series solution, is to define Padé approximants (PAs), converting the polynomial approximation into a ratio of two polynomials, a rational function. Some examples applied to ordinary differential equations can be found in [43,16,64,65] and more recently examples applied to partial differential equations [1,2]. Generally, PAs can enlarge the convergence domain of the truncated Taylor series and improve the convergence rate of the truncated series [43]. PAs will also converge on the entire real axis if the function is free of real singularities.

Cheng et al. [23] in order to overcome the divergent difficulties arising from small convergence radius, suggested a piecewise solution technique, by dividing the solution space into regions.

In El-Tawil et al. [33] the Adomian's method approximated to the first two terms is applied to the matrix Riccati differential equation in a recursive form, in n subintervals on the time horizon, which can be used to obtain the solution for the whole time considered. The authors named the method, as the multistage Adomian's decomposition method (MADM), and justified it on the fact that the first two terms give a good approximation for the Riccati matrix equation, only in the neighborhood of the initial time. The accuracy of the method can be increased either by choosing small time intervals or by adding more terms [33].

In the present work, PAs are implemented to the series solution given by Adomian's decomposition technique applied to nonlinear partial differential equations, in particular to Burgers equation. Numerical and graphical illustrations show that this technique can improve the convergence rate and enlarge the domain of convergence of the solution, even when the actual solution cannot be expressed as the ratio of two polynomials. However a disadvantage can come through: the rational approximation may create inaccurate solutions near its poles when the real solution is not a rational polynomial function with respect to the variable considered in the Padé approximation.

Another way of trying to overcome the limitations of the small convergence radius of Adomian's method, which has not yet been tested, would be to develop a time analytical and spacial numerical method, approximating the spatial derivatives by finite differences and then solve analytically the resulting set of ordinary differential equations by applying Adomian's decomposition method and PAs or by applying the MADM. However, it is shown in this paper, that the application of Adomian's method to the ordinary differential equations set arising from the discretization of the spatial derivatives by finite differences, the so-called method of lines, may reduce the convergence domain of the solution's series.

To illustrate these results the Burgers equation [21] is used, as this equation is one of the simplest unidimensional nonlinear partial differential equations and one of the few nonlinear equations that admits analytic solution for arbitrary initial conditions [41,29] and [68,36,37] having been used along the years to test numerical methods, the exact solution serving as validation criterium. Such equation has also been used to explore 'turbulence' [21], in spite of the intrinsic three-dimensionality of the phenomenon. The turbulence in Navier–Stokes equations is incorporated in the quadratic term, nonlinear term of convection, and therefore Burgers equation can run as a starting point for the study of turbulence, as it possesses a nonlinear quadratic term. Other phenomena as wave processes, traffic flow, acoustic transmission, shocks and gas dynamics can also be studied starting from this equation [30,36,40].

2. Description of the method

For the sake of generality, the Adomian's method is described as applied to a nonlinear differential equation $Fu = g$, where F represents a nonlinear differential operator. The technique consists on decomposing the linear part of F into $L + R$, where L is an operator easily invertible and R is the remaining part. Representing the nonlinear term by N, the equation in *canonical form* is

$$Lu + Ru + Nu = g. \tag{1}$$

Representing the inverse of the operator L as L^{-1}, one gets the following equivalent equation:

$$L^{-1}Lu = g - L^{-1}Ru - L^{-1}Nu. \tag{2}$$

Being L the operator derivative of order n, one represents L^{-1} as the n-fold integration operator. Thus, $L^{-1}Lu = u + a$, where a is the term emerging from the integration and one gets

$$u = g - a - L^{-1}Ru - L^{-1}Nu. \tag{3}$$

A series solution $u = \sum_{n=0}^{\infty} u_n$ is looked for. Identifying u_0 as $g - a$, the rest of the terms u_n, $n > 0$, will further be settled by a recursive relation. The key of the method is to decompose the nonlinear term Nu in the equation (3), into a particular series of polynomials $Nu = \sum_{n=0}^{\infty} A_n$, being A_n the so-called *Adomian's polynomials*. Each polynomial A_n depends only on $u_0, u_1, \ldots, u_n$. Adomian introduced formulae to generate these polynomials for all kinds of nonlinearities [15,12,10,13]. It has also been shown that the sum of the Adomian's polynomials is a generalization of the Taylor series in a neighborhood of a function u_0 rather than a point

$$Nu = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{1}{n!}(u - u_0)^n N^{(n)}(u_0) \tag{4}$$

tending the general term of the series to zero very fast, as $1/(mq)!$, according to the optimal choice of the initial term, for m terms and q the order of the linear operator L [15] and [27].

Substituting $u = \sum_{n=0}^{\infty} u_n$ and $Nu = \sum_{n=0}^{\infty} A_n$ into Eq. (3) one gets

$$\sum_{n=0}^{\infty} u_n = g - a - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{5}$$

To determine the components $u_n(x, t)$, $n = 0, 1, 2, \ldots,$ one can employ the recursive relation

$$\begin{aligned}
&u_0 = g - a,\\
&u_1 = -L^{-1}Ru_0 - L^{-1}A_0,\\
&u_2 = -L^{-1}Ru_1 - L^{-1}A_1,\\
&\vdots\\
&u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n.\\
&\vdots
\end{aligned} \tag{6}$$

Adomian's polynomials were formally introduced in [26,3,12,10,13], and expressed as

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{7}$$

This formula is obtained by introducing, for the sake of convenience, the parameter λ, and writing

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n, \tag{8}$$

$$N(u(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n. \tag{9}$$

Expanding in a Taylor series $N \circ u$ in a neighborhood of $\lambda = 0$ one gets

$$\begin{aligned} N(u(\lambda)) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N(u(\lambda))\right]_{\lambda=0} \lambda^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} \lambda^n \end{aligned} \tag{10}$$

from which Eq. (7) follows immediately.

Other methods have been developed for the calculation of Adomian's polynomials A_n [3,5–7]. The next theorem, Theorem 2.1 [3,22], allows the infinite series representing the Adomian's polynomials, A_n, to be substituted by a finite sum, fact that allows its computation.

Theorem 2.1. *Adomian's polynomials A_n may be computed by the formula*

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{11}$$

3. Convergence of Adomian's decomposition method

The convergence of Adomian's method has been subject of investigation by several authors [26,3–6,27] and [24,28,54,38,19]. One important observation is that the series solution of Adomian's decomposition technique usually converges very fast to the exact solution, if there is only one, and to one of the solutions, if several exist, tending the general term of the series solution to zero very fast, as $1/(mq)!$, for m terms and the qth order of the linear operator L [27,15].

Consider the Hilbert space $H = L^2((\alpha, \beta) \times [0, T])$ defined by the set of applications

$$u : (\alpha, \beta) \times [0, T] \to \mathbb{R} \quad \text{with} \int_{(\alpha,\beta)\times[0,T]} u^2(s, \tau)\,\mathrm{d}s\,\mathrm{d}\tau < +\infty. \tag{12}$$

Consider the differential equation

$$Lu + Ru + Nu = g, \tag{13}$$

with

$$Tu = Ru + Nu. \tag{14}$$

For T hemicontinuous operator, consider the following hypothesis:

(H_1) $(T(u) - T(v), u - v) \geqslant k\|u - v\|^2$, $k > 0$, $\forall u, v \in H$
(H_2) $\forall M > 0, \exists C(M) > 0 : \|u\| \leqslant M, \|v\| \leqslant M \Longrightarrow (T(u) - T(v), w) \leqslant C(M)\|u - v\|\|w\|, \forall w \in H$

If the above hypothesis are satisfied, the Adomian's method is convergent [24,54,55]. These hypothesis applied to Burgers equation can be verified using the same scheme of proof of [47,56,54,55,49].

Theorem 3.1 (*sufficient condition of convergence*). *The Adomian's decomposition method applied to the Burgers equation*

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \delta\frac{\partial^2 u}{\partial x^2}. \tag{15}$$

without initial and boundary conditions, converges towards a particular solution.

Proof. For Eq. (15) let $Lu = \partial u/\partial t$, $Ru = u(\partial u/\partial x)$ and $Nu = -\delta(\partial^2 u/\partial x^2)$. One has then:

$$Tu = u\frac{\partial u}{\partial x} - \delta\frac{\partial^2 u}{\partial x^2}. \tag{16}$$

This operator T is hemicontinuous.

Verification of hypothesis H_1:

$$\begin{aligned} T(u) - T(v) &= \left(u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x}\right) - \delta\frac{\partial^2}{\partial x^2}(u - v) \\ &= \frac{1}{2}\frac{\partial}{\partial x}(u^2 - v^2) - \delta\frac{\partial^2}{\partial x^2}(u - v), \end{aligned} \tag{17}$$

$$\begin{aligned} &(T(u) - T(v), u - v) \\ &\quad = \left(\frac{1}{2}\frac{\partial}{\partial x}(u^2 - v^2) - \delta\frac{\partial^2}{\partial x^2}(u - v), u - v\right) \\ &\quad = \left(-\delta\frac{\partial^2}{\partial x^2}(u - v), u - v\right) + \frac{1}{2}\left(\frac{\partial}{\partial x}(u^2 - v^2), u - v\right). \end{aligned} \tag{18}$$

In addition, there exists $\alpha, \sigma > 0$ such that

$$\left(-\frac{\partial^2}{\partial x^2}(u - v), u - v\right) \geqslant \alpha\|u - v\|^2, \tag{19}$$

$$\left(\frac{\partial}{\partial x}(u^2 - v^2), u - v\right) \leqslant \sigma\|u + v\|\|u - v\|^2 \leqslant 2M\sigma\|u - v\|^2, \tag{20}$$

where $\|u\| \leqslant M$, $\|v\| \leqslant M$.

Therefore,

$$(T(u) - T(v), u - v) \geqslant (\alpha\delta - 2M\sigma)\|u - v\|^2 \tag{21}$$

Setting $k = \alpha\delta - 2M\sigma$ (with $\alpha\delta > 2M\sigma$), hypothesis H_1 is verified.

For hypothesis H_2:

$$\begin{aligned}(T(u)-T(v),w) &= \left(\frac{1}{2}\frac{\partial}{\partial x}(u^2-v^2)-\delta\frac{\partial^2}{\partial x^2}(u-v),w\right)\\ &= \frac{1}{2}\left(\frac{\partial}{\partial x}(u^2-v^2),w\right)+\delta\left(-\frac{\partial^2}{\partial x^2}(u-v),w\right)\\ &\leqslant M\|u-v\|\|w\|+\delta\|u-v\|\|w\|\\ &\leqslant (M+\delta)\,\|u-v\|\|w\| \end{aligned} \tag{22}$$

being $C(M)=M+\delta$ and therefore hypothesis H_2 is verified. □

4. Application of PAs to Burgers equation

Now PAs are tested and applied to the series solution given by Adomian's decomposition technique for Burgers equation. PAs are usually superior to Taylor expansions when functions contain poles, because the use of rational functions allows them to be well represented. One will compute essentially the symmetric PA, because the diagonal approximants are the most accurate one [64]. Numerical and graphical illustrations illustrate the study.

Consider Burgers equation (15) with the following initial and Dirichlet boundary conditions:

$$u(x,0)=u_0(x), \tag{23}$$

$$u(0,t)=f_0(t),\quad u(1,t)=f_1(t). \tag{24}$$

Following Adomian, the linear operators expressed by Eqs. (25) are defined.

$$L_t(\cdot)=\frac{\partial}{\partial t}(\cdot),\quad L_{xx}(\cdot)=\frac{\partial^2}{\partial x^2}(\cdot). \tag{25}$$

Applying the inverse operator of $L_t(\cdot)=\frac{\partial}{\partial t}(\cdot)$, $L_t^{-1}(\cdot)=\int_0^t(\cdot)\,\mathrm{d}t$, to both sides of Eq. (15) one obtains

$$u(x,t)=u_0(x)+L_t^{-1}\left(-u\frac{\partial u}{\partial x}+\delta\frac{\partial^2 u}{\partial x^2}\right). \tag{26}$$

According to Adomian's method, one assumes that the unknown function $u(x,t)$ can be expressed by an infinite sum of components of the form

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{27}$$

and the nonlinear term $u(\partial u/\partial x)$ into an infinite series of Adomian's polynomials

$$u\frac{\partial u}{\partial x}=\sum_{n=0}^{\infty}A_n. \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (26) one obtains

$$\sum_{n=0}^{\infty}u_n=u_0(x)+L_t^{-1}\left(\delta\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}A_n\right). \tag{29}$$

To determine the components of $u_n(x,t)$, $n=0,1,2,\ldots$, Adomian's technique can employ the recursive relation defined by

$$
\begin{aligned}
&u_0 = u_0(x),\\
&u_1 = L_t^{-1}\left(\delta\frac{\partial^2}{\partial x^2}u_0 - A_0\right),\\
&u_2 = L_t^{-1}\left(\delta\frac{\partial^2}{\partial x^2}u_1 - A_1\right),\\
&\vdots\\
&u_n = L_t^{-1}\left(\delta\frac{\partial^2}{\partial x^2}u_{n-1} - A_{n-1}\right).\\
&\vdots
\end{aligned}
\tag{30}
$$

The Adomian's polynomials depend on the particular nonlinearity. In this case, the A_n polynomials are given by

$$
\begin{aligned}
&A_0 = u_0\frac{\partial}{\partial x}u_0,\\
&A_1 = u_0\frac{\partial}{\partial x}u_1 + u_1\frac{\partial}{\partial x}u_0,\\
&A_2 = u_0\frac{\partial}{\partial x}u_2 + u_1\frac{\partial}{\partial x}u_1 + u_2\frac{\partial}{\partial x}u_0,\\
&A_3 = u_0\frac{\partial}{\partial x}u_3 + u_1\frac{\partial}{\partial x}u_2 + u_2\frac{\partial}{\partial x}u_1 + u_3\frac{\partial}{\partial x}u_0.\\
&\vdots
\end{aligned}
\tag{31}
$$

Applying Adomian's decomposition method in the x direction

$$
u(x,t) = u(0,t) + x\frac{\partial}{\partial x}u(0,t) + \frac{1}{\delta}L_{xx}^{-1}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right],
\tag{32}
$$

with $L_{xx}^{-1}(\cdot) = \int_0^x \mathrm{d}x' \int_0^{x'} (\cdot)\,\mathrm{d}x''$. The series solution is given by

$$
\sum_{n=0}^{\infty} u_n = u(0,t) + x\frac{\partial}{\partial x}u(0,t) + \frac{1}{\delta}L_{xx}^{-1}\left(\frac{\partial}{\partial t}\sum_{n=0}^{\infty}u_n + \sum_{n=0}^{\infty}A_n\right).
\tag{33}
$$

To determine the components of $u_n(x,t)$, $n=0,1,2,\ldots$, Adomian's technique can define the following recursive relation:

$$
\begin{aligned}
&u_0 = u(0,t) + x\frac{\partial u}{\partial x}(0,t),\\
&u_1 = \frac{1}{\delta}L_{xx}^{-1}\left[\frac{\partial}{\partial t}u_0 + A_0\right],\\
&u_2 = \frac{1}{\delta}L_{xx}^{-1}\left[\frac{\partial}{\partial t}u_1 + A_1\right],\\
&\vdots\\
&u_n = \frac{1}{\delta}L_{xx}^{-1}\left[\frac{\partial}{\partial t}u_{n-1} + A_{n-1}\right].\\
&\vdots
\end{aligned}
\tag{34}
$$

In the following examples the truncated series solution $\phi_n(x,t) = \sum_{k=0}^{n} u_k(x,t)$ is considered.

4.1. Example 1

Consider Burgers equation (15) with the particular initial and Dirichlet homogeneous boundary conditions given by

$$u_0(x) = \sin(\pi x), \tag{35}$$

$$u(0,t) = u(1,t) = 0. \tag{36}$$

The exact solution is given by [29]

$$u(x,t) = \frac{4\pi\delta\sum_{k=1}^{\infty} kI_k(1/2\pi\delta)\sin(k\pi x)\ \exp(-k^2\pi^2\delta t)}{I_0(1/2\pi\delta) + 2\sum_{k=1}^{\infty} I_k(1/2\pi\delta)\ \cos(k\pi x)\ \exp(-k^2\pi^2\delta t)} \tag{37}$$

being I_k the modified Bessel functions of first kind.

Consider $\delta = 1$. For such δ one can find, as an illustrative example, the first four terms of the Adomian's series solution:

$$\begin{aligned}
u_0 &= \sin(\pi x),\\
u_1 &= -\sin(\pi x)(\pi + \cos(\pi x))\pi t,\\
u_2 &= \tfrac{1}{2}\sin(\pi x)(\pi^2 + 6\cos(\pi x)\pi + 3(\cos(\pi x))^2 - 1)\pi^2 t^2\\
u_3 &= -\tfrac{1}{6}\sin(\pi x)\pi^3 t^3 \Big[28\pi^2\ \cos(\pi x) + \pi^3 - 15\pi\\
&\qquad +51(\cos(\pi x))^2\pi + 16(\cos(\pi x))^3 - 10\cos(\pi x)\Big].
\end{aligned}$$

Figs. 1 and 2 show the results for the exact solution and the one obtained by application of Adomian's method with $n = 10$ and $n = 14$. Fig. 3 shows the three-dimensional plot for the solution obtained by Adomian's method for $n = 20$. One can see that the solution diverges for values of t greater than a value elsewhere between $t = 0.03$ and $t = 0.04$.

Figs. 4–6 show the results for the exact solution and the one obtained by application of PAs [5/5], [5/4] and [7/7] to Adomian's solution. Fig. 7 show the three-dimensional plot for the solution obtained by a [7/7] PA.

Tables 1 and 2 show the errors (difference between the exact solution and the approximate solution) for the solution given by Adomian's method with $n = 14$, and for the solution modified by PA [7/7].

The more accurate solution and the improved region of convergence is patent by using PAs. However, a drawback of using PAs that can be clearly seen in Fig. 4 and also in Fig. 6, occur due to the existence of poles in the rational approximation.

4.2. Example 2

Consider Burgers equation (15) with initial and boundary conditions given by

$$u(x,1) = x - \pi\tanh\left(\frac{\pi x}{2\delta}\right), \tag{38}$$

$$u(0,t) = 0, \qquad \frac{\partial}{\partial x}u(0,t) = \frac{1}{t} - \frac{\pi^2}{2\delta t^2}. \tag{39}$$

The exact solution is

$$u(x,t) = \frac{x}{t} - \frac{\pi}{t}\tanh\left(\frac{\pi x}{2\delta t}\right). \tag{40}$$

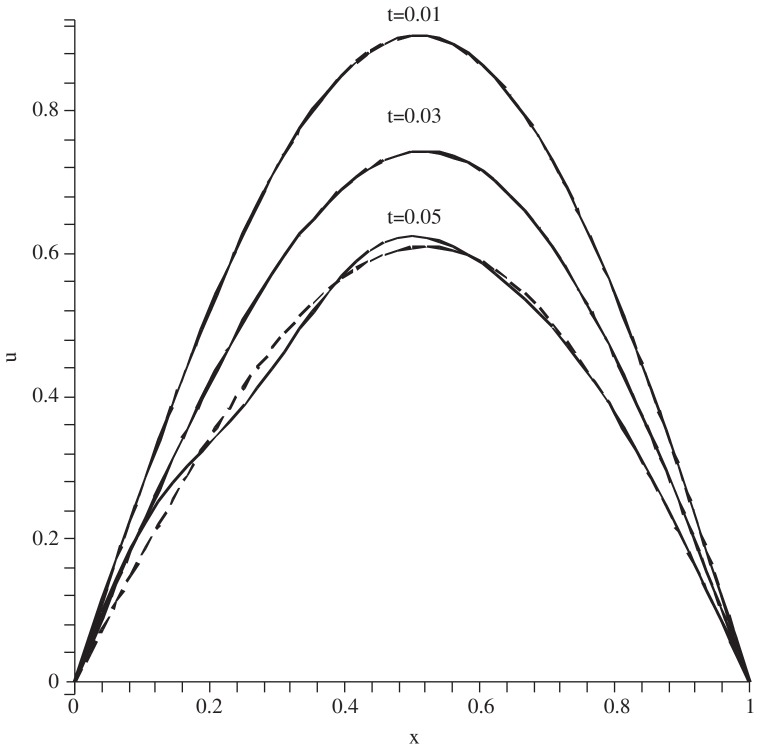

Fig. 1. Example 1: results obtained for the exact solution (dashed line) and the one obtained by application of Adomian's method (solid line) with $n = 10$.

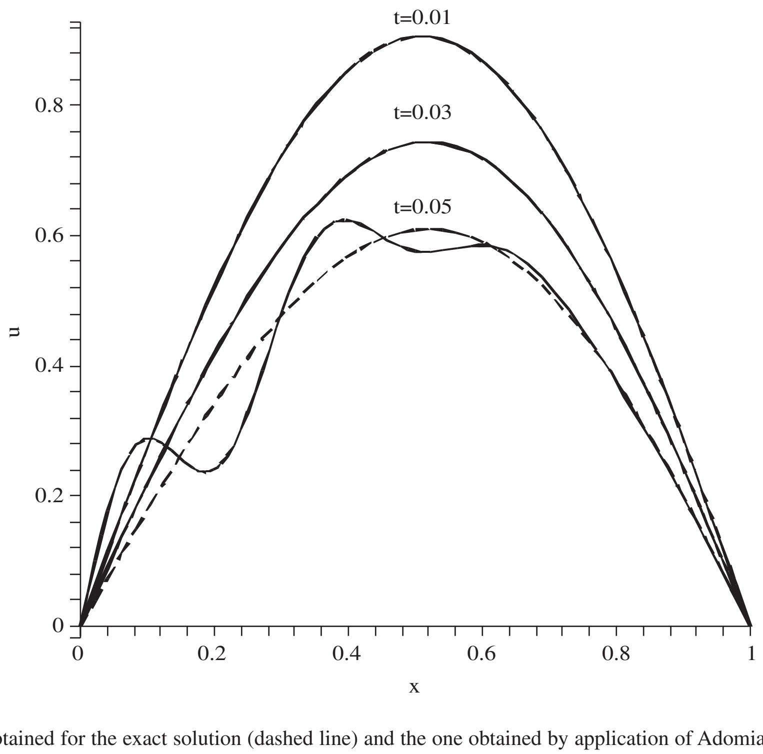

Fig. 2. Example 1: results obtained for the exact solution (dashed line) and the one obtained by application of Adomian's method (solid line) with $n = 14$.

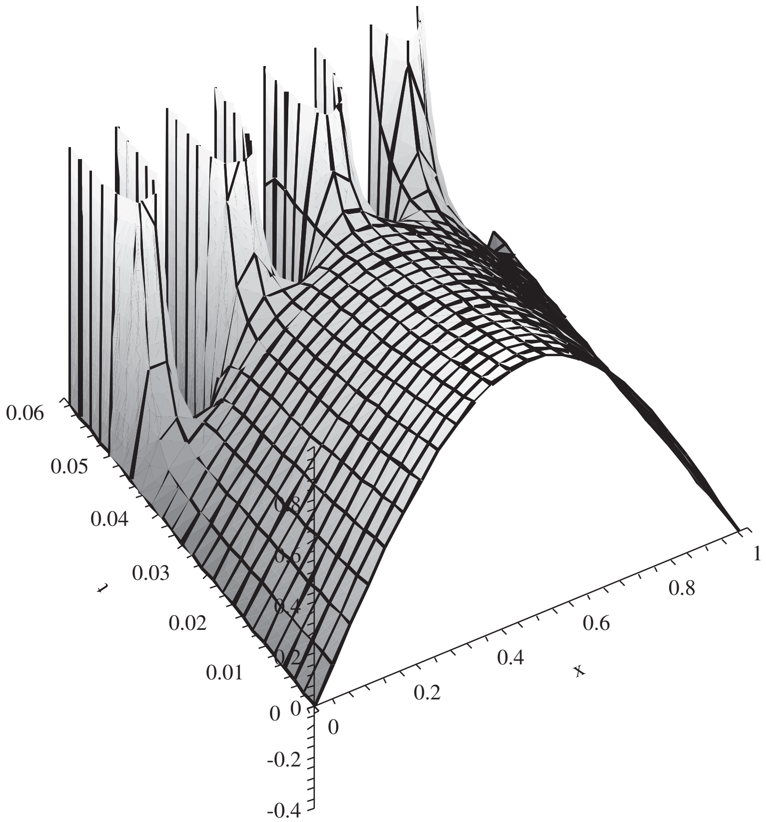

Fig. 3. Example 1: three-dimensional plot for the solution obtained by Adomian's method for $n=20$.

This example was worked by [50]. By Adomian's decomposition method applied in the x direction, the solution is given by

$$u(x,t)=u(0,t)+x\frac{\partial}{\partial x}u(0,t)+\frac{1}{\delta}L_{xx}^{-1}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right] \tag{41}$$

being $L_{xx}^{-1}(\cdot)=\int_0^x \mathrm{d}x'\int_0^{x'}(\cdot)\,\mathrm{d}x''$.

So, as performed in [50] one obtains

$$u_0=\frac{x}{t}-\frac{x\pi^2}{2\delta t^2},$$

$$u_1=\frac{\pi^4x^3}{24\delta^3t^4},$$

$$u_2=-\frac{\pi^6x^5}{240\delta^5t^6},$$

$$u_3=\frac{17\pi^8x^7}{40320\delta^7t^8},$$

$$u_4=-\frac{31\pi^{10}x^9}{725760\delta^9t^{10}}.$$

The series obtained is the Taylor expansion of the real solution (40) around $x=0$.

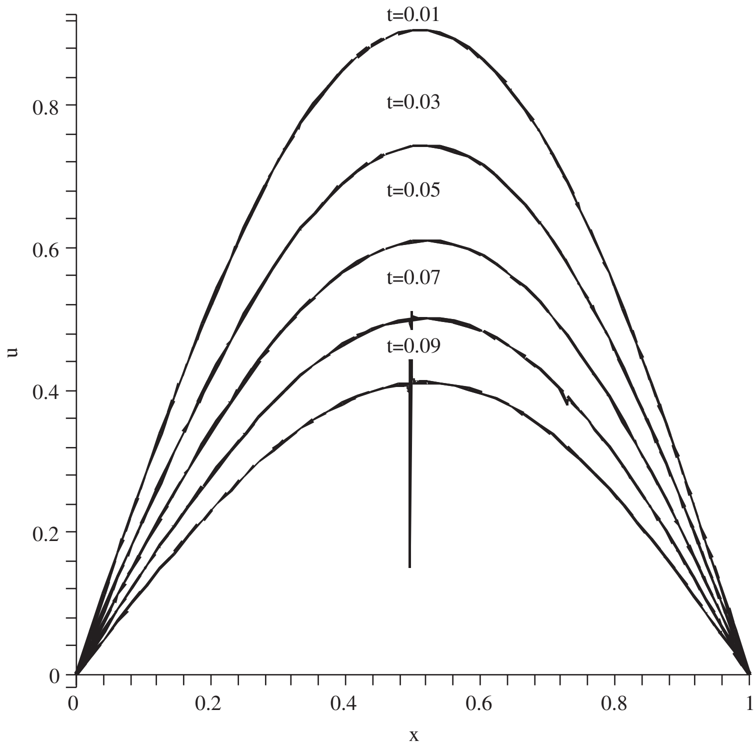

Fig. 4. Example 1: results obtained for the exact solution (dashed line) and the one obtained by application of a Padé approximant [5/5] to Adomian's series solution (solid line).

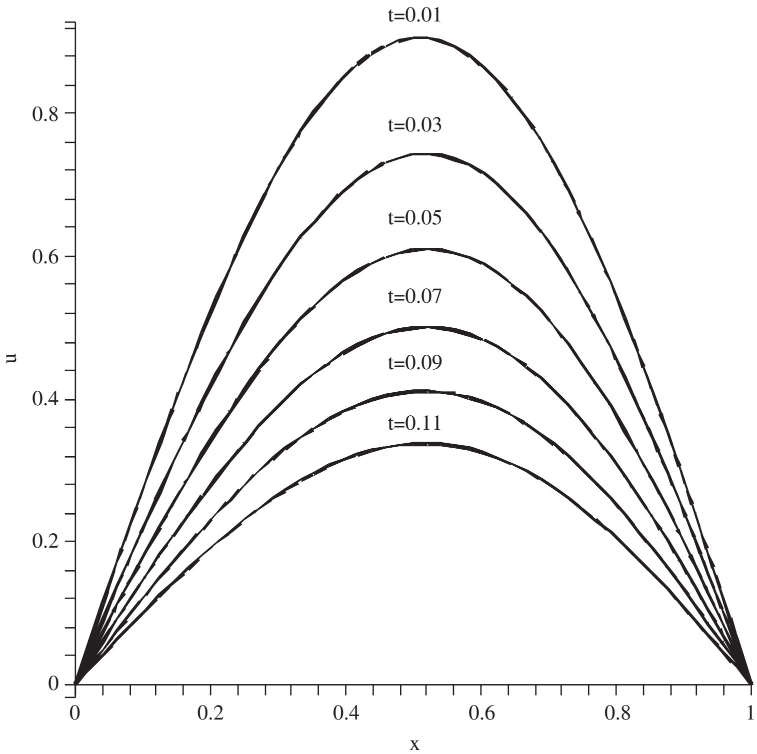

Fig. 5. Example 1: results obtained for the exact solution (dashed line) and the one obtained by application of a Padé approximant [5/4] to Adomian's series solution (solid line).

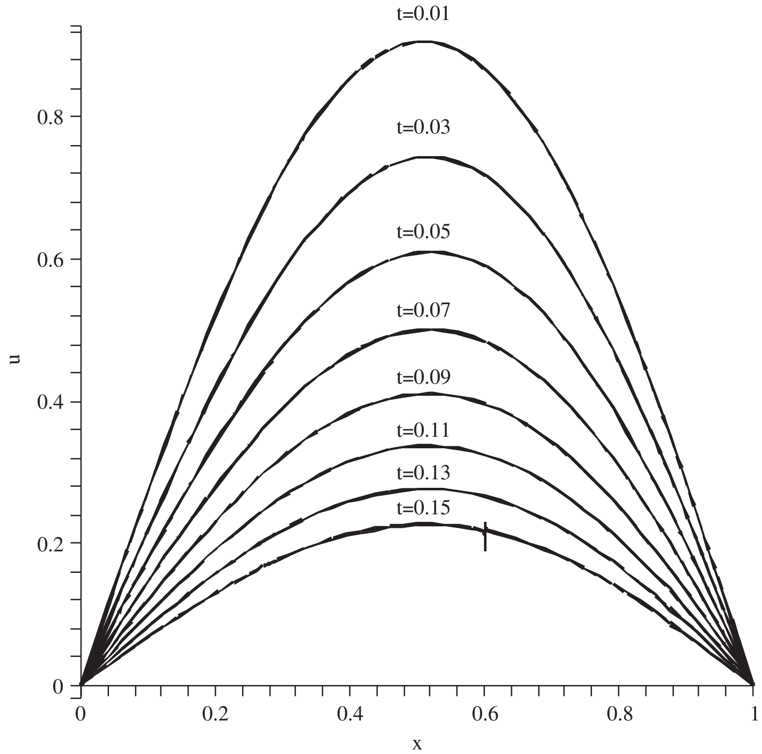

Fig. 6. Example 1: results obtained for the exact solution (dashed line) and the one obtained by application of a Padé approximant [7/7] to Adomian's series solution (solid line).

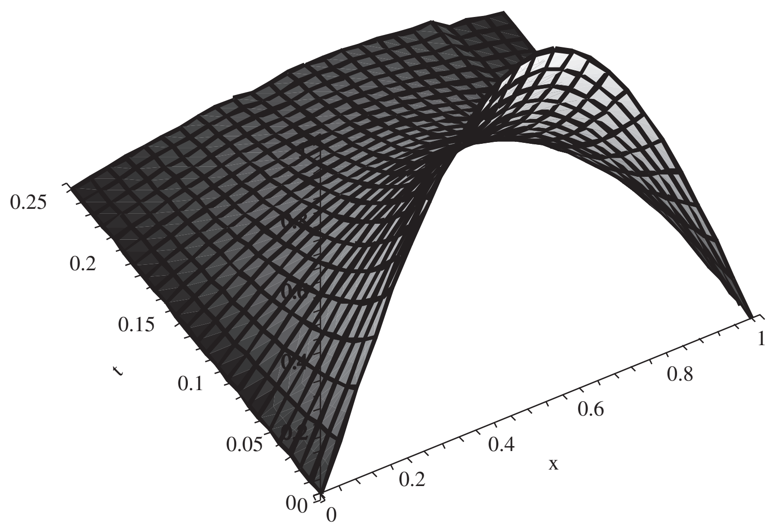

Fig. 7. Example 1: three-dimensional plot for the solution obtained by application of a Padé approximant [7/7] to Adomian's series solution.

The expansion of the function $\tanh(x)$ being

$$\tanh(x) = \sum_{n=1}^{\infty} B_{2n} \frac{4^n(4^n-1)}{(2n)!} x^{2n-1}, \quad |x| < \frac{\pi}{2} \tag{42}$$

Table 1
Example 1: errors, differences between the exact solution and the approximate solution given by Adomian's method with $n = 14$

$t_i \backslash x_j$	0.05	0.15	0.25	0.35	0.45
0.1	−7.51745E − 12	8.46701E − 13	4.91018E − 12	−4.38956E − 12	4.50247E − 13
0.2	−1.98818E − 07	1.75001E − 08	1.35282E − 07	−1.15055E − 07	6.11515E − 09
0.3	−7.30473E − 05	5.18810E − 06	5.09930E − 05	−4.18427E − 05	7.06584E − 07
0.4	−4.71039E − 03	2.75777E − 04	3.34633E − 03	−2.67247E − 03	−2.88965E − 05
0.5	−1.17644E − 01	5.76254E − 03	8.46449E − 02	−6.61789E − 02	−2.16060E − 03
0.6	−1.61678E + 00	6.69537E − 02	1.17453E + 00	−9.02702E − 01	−4.54042E − 02
0.7	−1.47351E + 01	5.19643E − 01	1.07855E + 01	−8.17325E + 00	−5.30109E − 01
$t_i \backslash x_j$	0.55	0.65	0.75	0.85	0.95
0.1	1.54630E − 12	−8.34916E − 13	−1.34394E − 13	2.16168E − 13	1.83671E − 14
0.2	4.43314E − 08	−2.12349E − 08	−5.21332E − 09	6.05034E − 09	9.07751E − 10
0.3	1.70554E − 05	−7.46111E − 06	−2.35344E − 06	2.27028E − 06	4.52373E − 07
0.4	1.13185E − 03	−4.60881E − 04	−1.72366E − 04	1.47078E − 04	3.49075E − 05
0.5	2.88133E − 02	−1.10676E − 02	−4.68945E − 03	3.66155E − 03	9.79461E − 04
0.6	4.01285E − 01	−1.46841E − 01	−6.84936E − 02	4.99722E − 02	1.45870E − 02
0.7	3.69265E + 00	−1.29695E + 00	−6.53125E − 01	4.51534E − 01	1.40891E − 01

Table 2
Example 1: errors, differences between the exact solution and the approximate solution given by application of a Padé approximant [7/7] to Adomian's series solution

$t_i \backslash x_j$	0.05	0.15	0.25	0.35	0.45
0.1	7.72079E − 17	4.15110E − 16	−4.60375E − 15	1.85715E − 15	−3.08566E − 15
0.2	6.64533E − 13	2.97967E − 12	4.79233E − 11	1.15523E − 11	−2.68941E − 11
0.3	8.83448E − 11	3.53281E − 10	2.15023E − 09	1.18129E − 09	−3.97138E − 09
0.4	2.24271E − 09	8.27523E − 09	3.44407E − 08	2.41130E − 08	−1.24531E − 07
0.5	2.35585E − 08	8.18779E − 08	2.72689E − 07	2.11025E − 07	−1.92525E − 06
0.6	1.43608E − 07	4.76656E − 07	1.36455E − 06	1.10350E − 06	−2.77392E − 05
0.7	6.06647E − 07	1.94200E − 06	4.97574E − 06	4.09749E − 06	2.09511E − 04
0.8	1.97133E − 06	6.13120E − 06	1.44267E − 05	1.19450E − 05	1.68476E − 04
1.0	1.20971E − 05	3.60752E − 05	7.51463E − 05	6.19395E − 05	3.97112E − 04
1.5	1.90141E − 04	5.37610E − 04	9.53013E − 04	7.69280E − 04	2.52636E − 03
2.0	8.60069E − 04	2.38495E − 03	3.93444E − 03	3.15794E − 03	7.90138E − 03
3.0	3.99455E − 03	1.09768E − 02	1.71344E − 02	1.38859E − 02	2.73590E − 02
4.0	8.34046E − 03	2.29107E − 02	3.50999E − 02	2.87099E − 02	5.12215E − 02
$t_i \backslash x_j$	0.55	0.65	0.75	0.85	0.95
0.1	3.12547E − 15	4.52255E − 16	−4.48250E − 16	−9.95009E − 16	−8.52856E − 15
0.2	1.51485E − 11	2.45257E − 12	−2.64229E − 12	−8.25245E − 12	6.56919E − 11
0.3	1.28440E − 09	2.10130E − 10	−2.48538E − 10	−1.00155E − 09	3.02544E − 09
0.4	2.25029E − 08	3.53289E − 09	−4.64875E − 09	−2.28691E − 08	4.31081E − 08
0.5	1.73013E − 07	2.51916E − 08	−3.75399E − 08	−2.17176E − 07	2.93990E − 07
0.6	8.08725E − 07	1.06053E − 07	−1.83231E − 07	−1.21422E − 06	1.25877E − 06
0.7	2.72097E − 06	3.11320E − 07	−6.43426E − 07	−4.79189E − 06	3.94343E − 06
0.8	7.26666E − 06	6.96330E − 07	−1.79757E − 06	−1.48379E − 05	9.91050E − 06
1.0	3.24437E − 05	1.73659E − 06	−8.88349E − 06	−8.77082E − 05	4.00941E − 05
1.5	3.05102E − 04	−2.56678E − 05	−1.14953E − 04	−1.73076E − 03	3.24537E − 04
2.0	1.01625E − 03	−2.84541E − 04	−5.55498E − 04	−1.69520E − 02	1.02811E − 03
3.0	3.04366E − 03	−2.95330E − 03	−3.65196E − 03	6.38294E − 02	3.43774E − 03
4.0	3.77849E − 03	−1.00792E − 02	−1.08112E − 02	6.11860E − 02	6.22718E − 03

Table 3
Example 2: the true results obtained for the error between the exact solution and the one obtained by application of Adomian's method with $n = 20$

$t_i \backslash x_j$	0.01	0.02	0.03	0.04	0.05
0.1	2.00000E + 01	7.03687E + 13	1.31303E + 21	1.82050E + 26	1.74903E + 30
0.2	1.81899E − 12	1.00000E + 01	2.29652E + 08	3.51844E + 13	3.56543E + 17
0.3	3.65567E − 20	2.47351E − 07	6.66667E + 00	1.13138E + 06	1.22236E + 10
0.4	1.21644E − 25	9.09495E − 13	2.71528E − 05	5.00000E + 00	5.73412E + 04
0.5	6.76623E − 30	5.33595E − 17	1.69846E − 09	3.31983E − 04	4.00000E + 00

one gets for the real solution (40) expanded in Taylor series around $x = 0$:

$$\frac{x}{t} - \frac{\pi}{t}\tanh\left(\frac{\pi x}{2\delta t}\right) = \frac{x}{t} - \sum_{n=1}^{\infty} B_{2n}\frac{4^n(4^n-1)}{(2n)!}\left(\frac{\pi}{t}\right)^{2n}\left(\frac{x}{2\delta}\right)^{2n-1}, \quad |x| < \delta t \tag{43}$$

where B_n are the Bernoulli numbers. This expansion coincides with the one obtained by Adomian's method

$$\frac{x}{t} - \frac{\pi}{t}\tanh\left(\frac{\pi x}{2\delta t}\right) = \frac{x}{t} - \frac{\pi^2 x}{2\delta t^2} + \frac{\pi^4}{24\delta^3 t^4}x^3 - \frac{\pi^6}{240\delta^5 t^6}x^5 + \frac{17\pi^8}{40320\delta^7 t^8}x^7 - \frac{31\pi^{10}}{725760\delta^9 t^{10}}x^9 + \cdots. \tag{44}$$

Keeping this in mind, one can conclude that the Adomian's truncated series solution is only a good approximation for the real solution when $|x| < \delta t$, diverging for all other values of x and t. Representing the truncated series solution by $\phi_n = \sum_{k=0}^{n} u_k$, the numerical comparison for $\delta = 0.1$, performed in [50] for the truncated series ϕ_{20} given by Adomian's method with the analytical solution (40) is not accurate. For this value of δ, one should have $10x < t$ and the values taken by those authors were $0.1 \leqslant t_i \leqslant 0.5$ and $0.01 \leqslant x_j \leqslant 0.05$. Table 3 shows the precise results obtained for the error between the exact solution and the one obtained by application of Adomian's method with $n = 20$, for $\delta = 0.1$. For the majority of those values one can see that the solution diverges. In order to be accurate, the values presented in [50] should obey to the relation $\max x_j < \delta \min t_i$, that is, those authors may have performed their calculations for $\delta > 0.5$.

Consider again $\delta = 0.1$. Fig. 8 shows the results obtained for the exact solution and the one obtained by application of Adomian's method with $n = 20$. Fig. 9 shows the results obtained for the exact solution and the one obtained by application of PA [10/10] to Adomian's series solution. The results obtained for the error between the exact solution and the one obtained by application of PA [10/10] to Adomian's series solution are shown in Table 4. The more accurate solution and the improved region of convergence is patent by using PAs.

4.3. Example 3

Consider once more Burgers equation (15) with the following particular initial and boundary conditions:

$$u(x, 1) = \frac{x}{1 + e^{x^2/4\delta}}, \tag{45}$$

$$u(0, t) = 0, \quad u(1, t) = \frac{1}{t(1 + \sqrt{t}e^{1/4\delta t})}. \tag{46}$$

The exact solution, which can be verified by direct substitution in Burgers equation (15), is

$$u(x, t) = \frac{x}{t(1 + \sqrt{t}e^{x^2/4\delta t})}. \tag{47}$$

Consider $\delta = 0.1$. Figs. 10 and 11 show the three-dimensional results for the solution obtained by application of Adomian's method with $n = 10$ and the one obtained by application of a PA [3/3] to Adomian's series solution.

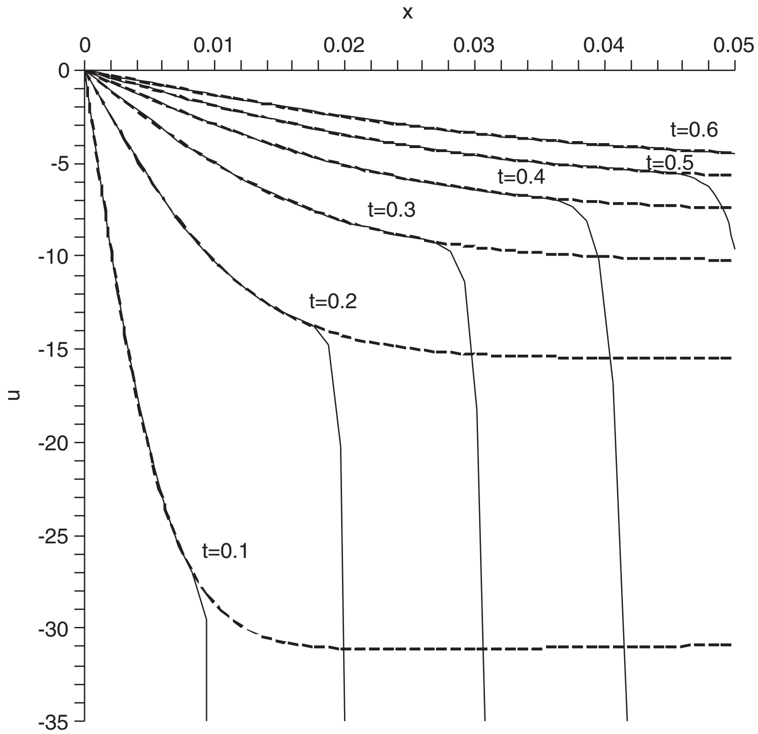

Fig. 8. Example 2: results for $t = 0.1$, $t = 0.2$, $t = 0.3$, $t = 0.4$, $t = 0.5$ and $t = 0.6$, obtained for the exact solution (dashed line) and the one obtained by application of Adomian's method with $n = 20$ (solid line).

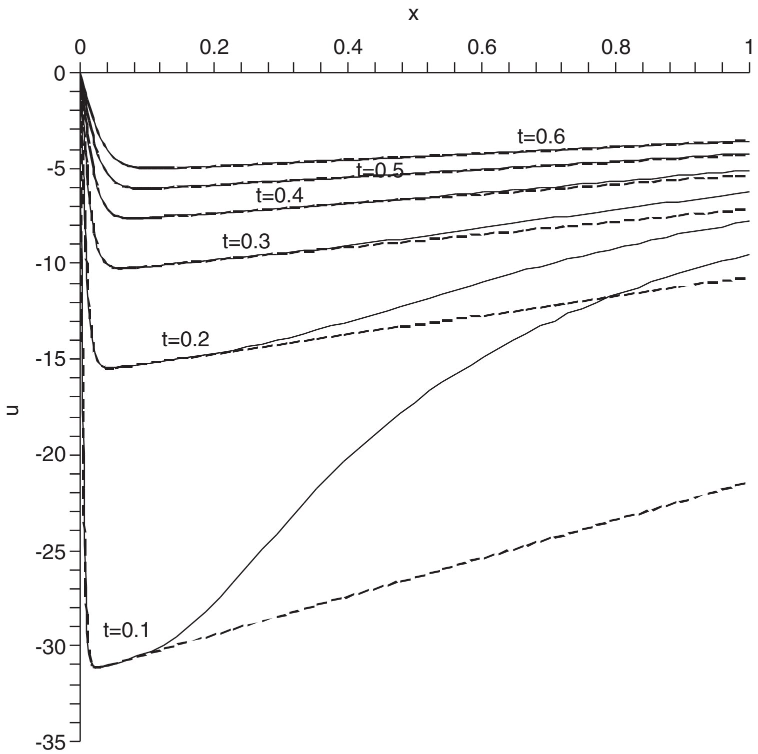

Fig. 9. Example 2: results for $t = 0.1$, $t = 0.2$, $t = 0.3$, $t = 0.4$, $t = 0.5$ and $t = 0.6$, obtained for the exact solution (dashed line) and the one obtained by application of a Padé approximant [10/10] to Adomian's series solution (solid line).

Table 4
Example 2: results obtained for the error between the exact solution and the one obtained by application of a Padé approximant [10/10] to Adomian's series solution

$t_i \backslash x_j$	0.01	0.02	0.03	0.04	0.05
0.1	−7.80799E − 15	−1.08753E − 09	−4.15470E − 07	−1.63449E − 05	−1.99498E − 04
0.2	−5.74753E − 21	−3.66689E − 15	−4.69991E − 12	−5.09706E − 10	−1.48886E − 08
0.3	−9.38304E − 25	−1.03846E − 18	−2.24700E − 15	−3.82994E − 13	−1.66668E − 11
0.4	−1.65326E − 27	−2.34680E − 21	−6.73391E − 18	−1.49469E − 15	−8.24634E − 14
0.5	−1.06739E − 29	−1.72494E − 23	−5.83936E − 20	−1.53467E − 17	−9.92477E − 16
$t_i \backslash x_j$	0.06	0.07	0.08	0.09	0.10
0.1	−1.22003E − 03	−4.78901E − 03	−1.39084E − 02	−3.26205E − 02	−6.54008E − 02
0.2	−1.94092E − 07	−1.47243E − 06	−7.60301E − 06	−2.95326E − 05	−9.23204E − 05
0.3	−3.11483E − 10	−3.28583E − 09	−2.29811E − 08	−1.18092E − 07	−4.77972E − 07
0.4	−1.91039E − 12	−2.45529E − 11	−2.06381E − 10	−1.26007E − 09	−5.99905E − 09
0.5	−2.66083E − 14	−3.91240E − 13	−3.72707E − 12	−2.55932E − 11	−1.36165E − 10
$t_i \backslash x_j$	0.11	0.12	0.13	0.14	0.15
0.1	−1.16485E − 01	−1.89345E − 01	−2.86389E − 01	−4.08878E − 01	−5.56990E − 01
0.2	−2.43553E − 04	−5.61274E − 04	−1.15936E − 03	−2.18899E − 03	−3.83582E − 03
0.3	−1.60074E − 06	−4.59913E − 06	−1.16494E − 05	−2.65664E − 05	−5.54553E − 05
0.4	−2.34166E − 08	−7.77501E − 08	−2.25780E − 07	−5.85883E − 07	−1.38180E − 06
0.5	−5.90685E − 10	−2.16882E − 09	−6.93288E − 09	−1.97193E − 08	−5.07715E − 08

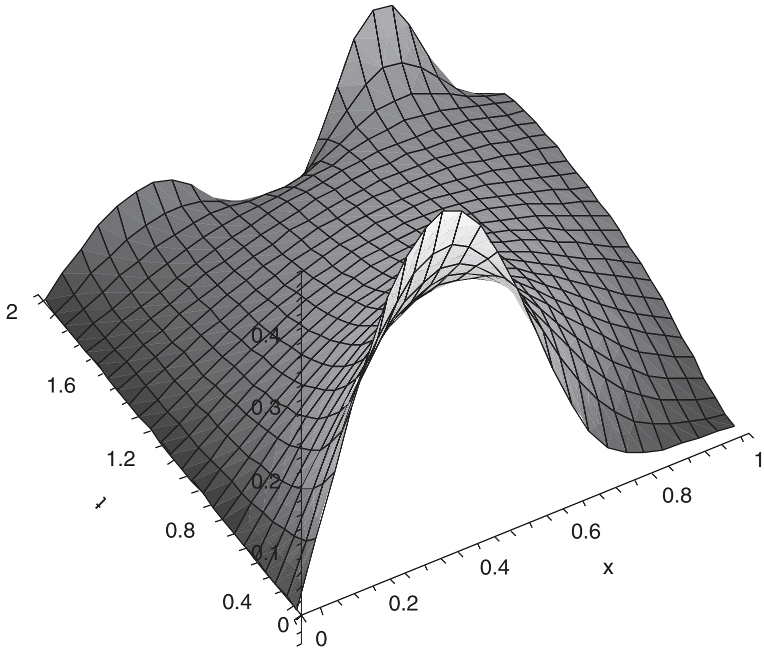

Fig. 10. Example 3: three-dimensional plot, for $t = 0.3$ to $t = 2$, for the solution obtained by Adomian's method for $n = 10$.

The symmetric PA correspondent to $n = 10$ is a [5/5] PA, but the much too demanding computation associated to it led one to use instead the less demanding [3/3] PA.

Table 5 shows the results obtained for the error between exact solution and the one obtained by application of Adomian's method with $n = 10$. The results obtained for the error between the exact solution and the one obtained by application of PA [3/3] to Adomian's series solution is shown in Table 6.

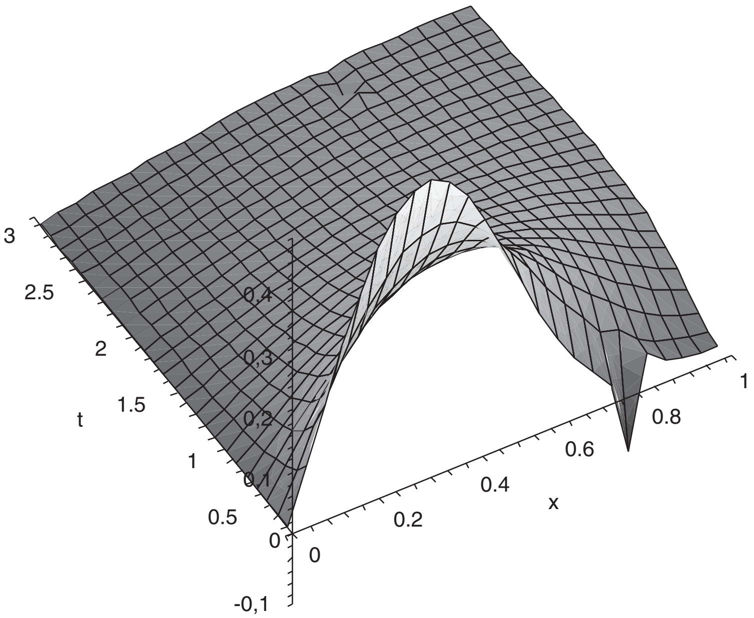

Fig. 11. Example 3: three-dimensional plot, for $t = 0.3$ to $t = 3$, for the solution obtained by application of a Padé approximant [3/3] to Adomian's series solution.

Table 5
Example 3: results obtained for the error between the exact solution and the one obtained by application of Adomian's method with $n = 10$

$t_i \backslash x_j$	0.1	0.2	0.3	0.4	0.5
0.1	2.43833E − 01	2.61487E − 01	−1.53871E − 01	−4.21187E − 01	−9.57961E − 02
0.3	5.14780E − 03	7.11723E − 03	1.86708E − 03	−9.74435E − 03	−1.25352E − 02
0.5	7.61650E − 05	1.09543E − 04	4.59579E − 05	−1.20577E − 04	−2.16753E − 04
0.7	1.97159E − 07	2.88136E − 07	1.39259E − 07	−2.78553E − 07	−5.76015E − 07
0.9	8.64972E − 13	1.27512E − 12	6.60120E − 13	−1.13337E − 12	−2.53511E − 12
1.1	−7.07303E − 13	−1.04838E − 12	−5.65256E − 13	8.78828E − 13	2.06821E − 12
1.3	−1.05975E − 07	−1.57665E − 07	−8.73234E − 08	1.26589E − 07	3.08767E − 07
1.5	−2.53072E − 05	−3.77535E − 05	−2.13128E − 05	2.93269E − 05	7.34644E − 05
1.7	−9.04018E − 04	−1.35142E − 03	−7.73855E − 04	1.02270E − 03	2.61537E − 03
1.9	−1.28341E − 02	−1.92170E − 02	−1.11263E − 02	1.42377E − 02	3.70166E − 02

$t_i \backslash x_j$	0.6	0.7	0.8	0.9	1.0
0.1	2.59384E − 01	2.08289E − 01	−5.01249E − 02	−1.33354E − 01	−4.61523E − 02
0.3	3.31525E − 03	1.34845E − 02	3.18653E − 03	−6.41433E − 03	−4.46725E − 03
0.5	−2.30201E − 05	2.22107E − 04	1.22349E − 04	−9.01966E − 05	−1.03979E − 04
0.7	−1.59242E − 07	5.48580E − 07	4.25550E − 07	−1.80266E − 07	−3.14395E − 07
0.9	−9.40572E − 13	2.25722E − 12	2.11536E − 12	−5.71407E − 13	−1.45578E − 12
1.1	8.89948E − 13	−1.74225E − 12	−1.84047E − 12	3.25633E − 13	1.20793E − 12
1.3	1.45327E − 07	−2.48709E − 07	−2.85370E − 07	3.25616E − 08	1.80701E − 07
1.5	3.67200E − 05	−5.70584E − 05	−6.95540E − 05	4.79428E − 06	4.28002E − 05
1.7	1.36483E − 03	−1.97130E − 03	−2.51681E − 03	8.79493E − 05	1.51269E − 03
1.9	1.99534E − 02	−2.72113E − 02	−3.60346E − 02	3.06051E − 04	2.12348E − 02

The most accurate solution and the improved region of convergence is patent by using PAs in spite of the comparison being performed between a [3/3] PA and the series solution with $n = 10$. Only for small values of time t and in the neighborhoods of $x = 0.8$ this is not true, because of the appearance of poles in the solution near $x = 0.8$, as it is shown in Figs. 11 and 12.

Table 6
Example 3: results obtained for the error between the exact solution and the one obtained by application of a Padé approximant [3/3] to Adomian's series solution

$t_i \backslash x_j$	0.1	0.2	0.3	0.4	0.5
0.1	−8.60461E − 03	−4.13796E − 02	−1.79268E − 01	−2.61018E − 02	2.20208E − 01
0.3	−4.73900E − 05	−1.66618E − 04	−8.16442E − 04	−8.71155E − 04	1.31269E − 03
0.5	−6.84560E − 07	−2.13900E − 06	−8.94407E − 06	−1.08418E − 05	8.03014E − 06
0.7	−4.94078E − 09	−1.44641E − 08	−5.37449E − 08	−6.72593E − 08	2.37770E − 08
0.9	−7.70704E − 13	−2.16225E − 12	−7.37029E − 12	−9.34334E − 12	1.14841E − 12
1.1	3.13951E − 13	8.54371E − 13	2.72755E − 12	3.48157E − 12	1.06377E − 13
1.3	3.16534E − 10	8.41686E − 10	2.55197E − 09	3.27169E − 09	4.60525E − 10
1.5	5.71408E − 09	1.49182E − 08	4.33790E − 08	5.57863E − 08	1.25124E − 08
1.7	3.26833E − 08	8.40620E − 08	2.36098E − 07	3.04354E − 07	8.83149E − 08
1.9	1.09029E − 07	2.76943E − 07	7.55334E − 07	9.75586E − 07	3.35335E − 07
3.0	2.69059E − 06	6.53941E − 06	1.60435E − 05	2.08624E − 05	1.09247E − 05
4.0	9.42320E − 06	2.23926E − 05	5.19886E − 05	6.78483E − 05	4.14633E − 05
6.0	3.26520E − 05	7.56064E − 05	1.64632E − 04	2.15799E − 04	1.53060E − 04

$t_i \backslash x_j$	0.6	0.7	0.8	0.9	1.0
0.1	2.26184E − 01	1.75848E − 01	2.57468E − 01	−2.01302E − 01	−6.01716E − 02
0.3	6.15424E − 03	1.41896E − 02	1.73301E + 00	−1.00293E − 02	−6.03234E − 03
0.5	8.59361E − 05	4.13946E − 04	−7.93718E − 04	−2.41646E − 04	−2.11008E − 04
0.7	5.15066E − 07	4.31281E − 06	−3.70338E − 06	−1.84955E − 06	−1.98855E − 06
0.9	6.75362E − 11	9.98495E − 10	−4.20469E − 10	−2.64645E − 10	−3.20265E − 10
1.1	−2.38213E − 11	−8.06939E − 10	1.31928E − 10	9.47361E − 11	1.22714E − 10
1.3	−2.13499E − 08	1.13939E − 05	1.06848E − 07	8.33902E − 08	1.12384E − 07
1.5	−3.49795E − 07	1.19630E − 05	1.60163E − 06	1.32262E − 06	1.82453E − 06
1.7	−1.84535E − 06	3.42814E − 05	7.81115E − 06	6.71618E − 06	9.39248E − 06
1.9	−5.74947E − 06	7.53380E − 05	2.26973E − 05	2.01117E − 05	2.83473E − 05
3.0	−1.11354E − 04	6.34877E − 04	3.33750E − 04	3.21491E − 04	4.55076E − 04
4.0	−3.45297E − 04	1.42551E − 03	8.85082E − 04	8.79852E − 04	1.23382E − 03
6.0	−1.04284E − 03	3.08856E − 03	2.21164E − 03	2.26387E − 03	3.11921E − 03

5. The method of lines

Another possible way of trying to bypass the small convergence radius of Adomian's method would be to develop a time analytical and space numerical method, approximating the spatial derivatives by finite differences and then solve analytically the resulting set of ordinary differential equations by applying Adomian's decomposition method and PAs or by applying the MADM. It is shown herein, illustrated by an example, that the application of Adomian's method to the ordinary differential equations set arising from the discretization of the spatial derivatives by finite differences, may reduce the convergence domain of the solution's series, and therefore the above bypassing approach becomes useless. Actually, the convergence radius of the series solution may decrease with the number of discretization points, and, although the numerical results obtained by application of PAs to the Adominan's series solution enlarge the convergence domain, this improvement also decreases with the number of spatial discretization points.

Consider Burgers equation (15) with initial and Dirichlet boundary conditions given by

$$u(x,0) = \sin(\pi x), \tag{48}$$

$$u(0,t) = u(1,t) = 0. \tag{49}$$

Consider a uniform grid with $\Delta x = x_{j+1} - x_j$, and approximate the first and second spatial derivatives by the difference schemes of second order

$$\frac{\partial u}{\partial x} = \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \tag{50}$$

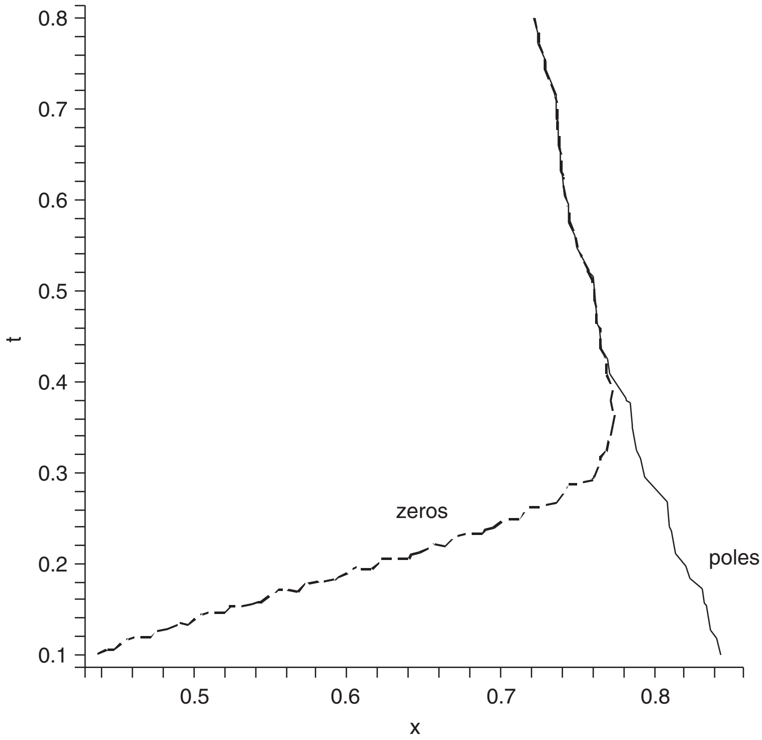

Fig. 12. Example 3: Zeros and poles for the solution obtained by application of a Padé approximant [3/3] to Adomian's series solution.

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1} + u_{j-1} - 2u_j}{\Delta x^2}. \tag{51}$$

By applying the finite differences (50) and (51) to Burgers equation (15) with initial and boundary conditions (48) and (49), one gets for $N-1$ points the following equation:

$$\frac{\partial u_j}{\partial t} = \frac{1}{2\Delta x}(u_j u_{j-1} - u_j u_{j+1}) + \frac{\delta}{\Delta x^2}(u_{j+1} + u_{j-1} - 2u_j).$$
$$1 < j < N-1 \tag{52}$$

or, in conservative form:

$$\frac{\partial u_j}{\partial t} = \frac{1}{4\Delta x}(u_{j-1}^2 - u_{j+1}^2) + \frac{\delta}{\Delta x^2}(u_{j+1} + u_{j-1} - 2u_j), \quad 1 < j < N-1. \tag{53}$$

Adomian's polynomials are given, respectively, by (the superscript of u represents the spatial discretization):

$$A_n^{;j} = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{i=0}^{n} \lambda^i u_i^j\right)\left(\sum_{i=0}^{n} \lambda^i u_i^{j-1}\right)\right]_{\lambda=0}, \tag{54}$$

$$A_n^{j} = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{i=0}^{n} \lambda^i u_i^j\right)\left(\sum_{i=0}^{n} \lambda^i u_i^{j}\right)\right]_{\lambda=0}. \tag{55}$$

Consider $\delta = 1$. For the solution given by Adomian's method with $n = 20$ applied to system (53) with initial and boundary conditions given by Eqs. (48) and (49), for different values of N, considering the convergence radius $r(N)$ the largest value of t for which the difference for all of the considered points, between the exact values of the solution given by (37) and the ones obtained by the truncated series applied to the discretized system (53), is inferior to 10^{-2},

Table 7
Experimental convergence radius in function of the number of subintervals

N	$r(\mathbf{N})$	N	$r(\mathbf{N})$
20	0.013575	100	0.000563
30	0.006024	200	0.000137
40	0.003461	300	0.000062
50	0.002307	400	0.000034
70	0.001161	500	0.000021

one obtains the values showed in Table 7. One can see that the convergence radius $r(N)$ and/or the rate of convergence of the series solution, decreases with the number of subintervals N. The conclusions applied to system (52) are similar.

This behavior is not due to the dimension of the system, but it seems to be related to the way the series solution of Adomian's method is constructed. To determine the components $u_n^j(x,t)$, $n=0,1,2,\ldots,$ $j=1,2,\ldots,N-1$, applied to system (53), Adomian's technique employs the recursive relation defined by

$$
\begin{aligned}
u_0^j &= u(x_j,0),\\
u_1^j &= \int_0^t \left(\frac{1}{4\Delta x}\left[A_0^{j-1}-A_0^{j+1}\right]+\frac{\delta}{\Delta x^2}\left[u_0^{j+1}+u_0^{j-1}-2u_0^j\right]\right),\\
u_2^j &= \int_0^t \left(\frac{1}{4\Delta x}\left[A_1^{j-1}-A_1^{j+1}\right]+\frac{\delta}{\Delta x^2}\left[u_1^{j+1}+u_1^{j-1}-2u_1^j\right]\right),\\
&\vdots\\
u_n^j &= \int_0^t \left(\frac{1}{4\Delta x}[A_{n-1}^{j-1}-A_{n-1}^{j+1}]+\frac{\delta}{\Delta x^2}[u_{n-1}^{j+1}+u_{n-1}^{j-1}-2u_{n-1}^j]\right).\\
&\vdots
\end{aligned}
\tag{56}
$$

For the purpose of clarity, the solution for $n=2$ of the ith equation written in terms of the zero's term of Adomian's series solution (the subscript zero has been omitted), is

$$
\begin{aligned}
\sum_{k=0}^{2} u_k^i = {} & \left[\frac{1}{2}u_{i+2}+3u_i+\frac{1}{2}u_{i-2}-2u_{i-1}-2u_{i+1}\right]t^2\frac{\delta^2}{\Delta x^4}\\
&+\left[-\frac{1}{2}u_{i+2}^2-u_{i+1}u_i+3u_{i+1}^2-u_{i+1}u_{i+2}\right.\\
&\left.+u_{i-1}u_i-3u_{i-1}^2+u_{i-1}u_{i-2}+\frac{1}{2}u_{i-2}^2\right]\frac{t^2}{4}\frac{\delta}{\Delta x^3}\\
&+[16u_{i+1}+16u_{i-1}-32u_i]\frac{t}{16}\frac{\delta}{\Delta x^2}\\
&+[u_{i+1}u_{i+2}^2-u_{i+1}u_i^2-u_{i-1}u_i^2+u_{i-1}u_{i-2}^2]\frac{t^2}{16}\frac{1}{\Delta x^2}\\
&+[u_{i-1}^2-u_{i+1}^2]\frac{t}{4}\frac{1}{\Delta x}+u_i.
\end{aligned}
\tag{57}
$$

Increasing the number of subintervals N, Δx diminishes, the absolute values of the expressions between brackets in Eqs. (56) and (57) should also diminish and the values of the powers of $1/\Delta x$ increase. It seems that the powers of $1/\Delta x$ may have a deeper influence on the convergence radius of the series solution, as shown below. Consider the powers of $1/\Delta x^2$. If one has αN subintervals ($\alpha>1$), $1/\Delta x^2$ equals α^2N^2, which is α^2 times larger than that of $1/\Delta x^2$ if one

Table 8
Theoretical convergence radius in function of the number of subintervals

N	$r(\mathbf{N})$	N	$r(\mathbf{N})$
20	0.013575	100	0.000543
30	0.006033	200	0.000136
40	0.003394	300	0.000060
50	0.002172	400	0.000034
70	0.001108	500	0.000022

had N subintervals. Therefore, the convergence radius may decrease as a function of α^2. Considering that it decreases α^2 times, this means that $r(\alpha N) = (1/\alpha^2)r(N)$ or $r(y) = (N^2/y^2)r(N)$. Fixing the value of $N = 20$, the one found in Table 7, one gets the values showed in Table 8, that are almost the same as those showed in Table 7.

However, in some particular problems, the behavior can be quite different. Consider Burgers equation (15) with $\delta = 1$ and initial and Dirichlet boundary conditions given by

$$u(x, 0) = x, \tag{58}$$

$$u(0, t) = 0, \quad u(1, t) = \frac{1}{1+t}. \tag{59}$$

In this case one has at $x = 1$ a boundary condition not nil. Hence, the computations may become quite hard or even impossible to carry on, due to the successive analytical integrations performed by Adomian's method. Nevertheless, one can substitute $1/(1 + t)$ by its Taylor series $\sum_{k=0}^{\infty} (-1)^k t^k$ and constructing the Adomian's series through the standard procedure, the corresponding terms in both series are identical. This way the computations become easier to perform, and, in this particular case, the solution is independent of the number of lines used, since the series solution obtained is the exact solution given by $x \sum_{k=0}^{\infty} (-1)^k t^k$.

6. Conclusions

When it is not possible to know the explicit solution given by Adomian's method applied to nonlinear partial differential equations, the series obtained may be adequate only in a small region. Techniques to enlarge and/or improve this region may be necessary.

In order to prove numerically whether the application of PAs to Adomian's series solution of Burgers equation leads to better accuracy and larger convergence region, the numerical solution of some examples with and without PAs were evaluated, and one concluded, from the worked examples, that PAs improve the convergence region and the accuracy of the solution, except in the neighborhood of poles that can appear in the rational approximation and are not present in the real solution.

It is also shown that the application of Adomian's method to the ordinary differential equations set arising from the discretization of the spatial derivatives by finite differences, the so-called method of lines, may reduce the convergence domain of the solution's series, behavior that seems to be related to the presence of powers of $1/\Delta x$, that are larger when the spatial interval grid is smaller.

References

[1] T.A. Abassy, M.A. El-Tawil, H.K. Saleh, The solution of KdV and mKdV equations using Adomian Padé approximation, Internat. J. Nonlinear Sci. Numer. Simul. 5 (4) (2004) 327–339.
[2] T. A. Abassy, M. A. El-Tawil, H. K. Saleh, The solution of Burgers' and good boussinesq equations using ADM-Padé technique, Chaos Solitons & Fractals, in press.
[3] K. Abbaoui, Y. Cherruault, Convergence of Adomian's method applied to differential equations, Comput. Math. Appl. 28 (5) (1994) 103–109.
[4] K. Abbaoui, Y. Cherruault, Convergence of Adomian's method applied to nonlinear equations, Math. Comput. Modelling 20 (9) (1994) 69–73.
[5] K. Abbaoui, Y. Cherruault, New ideas for proving convergence of decomposition methods, Comput. Math. Appl. 29 (7) (1995) 103–108.
[6] K. Abbaoui, Y. Cherruault, V. Seng, Practical formulae for the calculus of multivariable Adomian polynomials, Math. Comput. Modelling 22 (1) (1995) 89–93.
[7] K. Abbaoui, Y. Cherruault, The decomposition method applied to the Cauchy problem, Kybernetes 28 (1) (1999) 68–74.

[8] S. Abbasbandy, Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method, Appl. Math. Comput. 145 (2003) 887–893.
[9] G. Adomian, Stochastic Burger's equation, Math. Comput. Modelling 22 (8) (1995) 103–105.
[10] G. Adomian, A new approach to nonlinear partial differential equations, J. Math. Anal. Appl. 102 (1984) 420–434.
[11] G. Adomian, A new approach to the heat equation—an application of the decomposition method, J. Math. Anal. Appl. 113 (1986) 202–209.
[12] G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, New York, 1986.
[13] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Boston, 1994.
[14] G. Adomian, Random volterra integral equations, Math. Comput. Modelling 22 (8) (1995) 101–102.
[15] G. Adomian, Explicit solutions of nonlinear partial differential equations, Appl. Math. Comput. 88 (1997) 117–126.
[16] I.V. Andrianov, V.I. Olevskii, S. Tokarzewski, A modified Adomian's decomposition method, J. Appl. Math. Mech. 62 (2) (1998) 309–314.
[17] E. Babolian, J. Biazar, Solution of nonlinear equations by modified Adomian decomposition method, Appl. Math. Comput. 132 (2002) 167–172.
[18] E. Babolian, A. Davari, Numerical implementation of Adomian decomposition method, Appl. Math. Comput. 153 (2004) 301–305.
[19] T. Badredine, K. Abbaoui, Y. Cherruault, Convergence of Adomian's method applied to integral equations, Kybernetes 28 (5) (1999) 557–564.
[20] M. Basto, V. Semiao, F. Calheiros, A new iterative method to compute nonlinear equations, Appl. Math. Comput. 173 (2006) 468–483.
[21] J.M. Burgers, A mathematical model illustrating the theory of turbulence, Adv. Appl. Mech. 1 (1948) 171–199.
[22] W. Chen, Z. Lu, An algorithm for Adomian decomposition method, Appl. Math. Comput. 159 (2004) 221–235.
[23] M. Cheng, K. Scott, Y. Sun, B. Wu, Explicit solution of nonlinear electrochemical models by the decomposition method, Chem. Eng. Technol. 25 (2002) 1155–1160.
[24] Y. Cherruault, Convergence of Adomian's method, Kybernetes 18 (2) (1989) 31–38.
[25] Y. Cherruault, K. Abbaoui, A computational approach to the wave equations, an application of the decomposition method, Kybernetes 33 (1) (2004) 80–97.
[26] Y. Cherruault, G. Adomian, Decomposition method: a new proof of convergence, Math. Comput. Modelling 18 (12) (1993) 103–106.
[27] Y. Cherruault, G. Adomian, K. Abbaoui, R. Rach, Further remarks on convergence of decomposition method, Internat. J. Bio-Medical Comput. 38 (1995) 89–93.
[28] Y. Cherruault, G. Saccomandi, B. Some, New results for convergence of Adomian's method applied to integral equations, Math. Comput. Modelling 16 (2) (1992) 85–93.
[29] J.D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, Quart. Appl. Math. 9 (3) (1951) 225–236.
[30] L. Debnath, Nonlinear Partial Differential Equations for Scientists and Engineers, second ed., Birkhäuser, Boston, 2004.
[31] S.M. El-Sayed, M.R. Abdel-Aziz, A comparison of Adomian's decomposition method and wavelet-Galerkin method for solving integro-differential equations, Appl. Math. Comput. 136 (2003) 151–159.
[32] S.M. El-Sayed, D. Kaya, On the numerical solution of the system of two-dimensional Burgers' equations by the decomposition method, Appl. Math. Comput. 158 (2004) 101–109.
[33] M.A. El-Tawil, A.A. Bahnasawi, A. A.-Naby, Solving Riccati differential equation using Adomian's decomposition method, Appl. Math. Comput. 157 (2004) 503–514.
[34] M. El-Tawil, F. Saleh, El-Seedy, I. Kalla, Decomposition solution of stochastic nonlinear oscillator, Internat. J. Differential Equations Appl. 6 (4) (2002) 411–422.
[35] M. El-Tawil, F. Saleh, El-Seedy, I. Kalla, Decomposition solution of stochastic nonlinear partial differential equations, Internat. J. Differential Equations Appl. 7 (1) (2003) 29–39.
[36] C.A.J. Fletcher, Numerical Solution of Partial Differential Equations, North-Holland, Amsterdam, 1982.
[37] C.A.J. Fletcher, Computational Galerkin Methods, Springer, New York, 1984.
[38] L. Gabet, The theoretical foundation of the Adomian method, Comput. Math. Appl. 27 (12) (1994) 41–52.
[39] S. Guellal, P. Grimalt, Y. Cherruault, Numerical study of Lorenz's equation by the Adomian method, Comput. Math. Appl. 33 (3) (1997) 25–29.
[40] R. Haberman, Mathematical in Models, SIAM, Philadelphia, PA, 1998.
[41] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 9 (1950) 201–230.
[42] H.N.A. Ismail, K. Raslan, A.A.A. Rabboh, Adomian decomposition method for Burger's-Huxley and Burger's–Fisher equations, Appl. Math. Comput. 159 (2004) 291–301.
[43] Y.C. Jiao, Y. Yamamoto, C. Dang, Y. Hao, An after treatment technique for improving the accuracy of Adomian's decomposition method, Comput. Math. Appl. 43 (2002) 783–798.
[44] E. Kamke, Differentialgleichungen: Losungsmethoden und Losungen, Teubner, Stuttgart, 1983.
[45] D. Kaya, On the solution of a Korteweg-de-Vries like equation by the decomposition method, Internat. J. Comput. Math. 72 (1999) 531–539.
[46] D. Kaya, M. Aassila, An application for a generalized KdV equation by the decomposition method, Phys. Lett. A 299 (2002) 201–206.
[47] D. Kaya, I.E. Inan, A convergence analysis of the ADM and an application, Appl. Math. Comput. 161 (3) (2005) 1015–1025.
[48] D. Kaya, S.M. El-Sayed, On the solution of the coupled Schrödinger-KdV equation by the decomposition method, Phys. Lett. A 313 (2003) 82–88.
[49] D. Kaya, I.E. Inan, A numerical application of the decomposition method for the combined KdV–MKdV equation, Appl. Math. Comput. 168 (2) (2005) 915–926.
[50] D. Kaya, A. Yokus, A numerical comparison of partial solutions in the decomposition method for linear and nonlinear partial differential equations, Math. Comput. Simul. 60 (2002) 507–512.
[51] D. Lesnic, A computational algebraic investigation of the decomposition method for time-dependent problems, Appl. Math. Comput. 119 (2001) 197–206.

[52] D. Lesnic, Convergence of Adomian's decomposition method: Periodic temperatures, Comput. Math. Appl. 44 (2002) 13–24.
[53] D. Lesnic, The decomposition method for forward and backward time-dependent problems, J. Comput. Appl. Math. 147 (2002) 27–39.
[54] T. Mavoungou, Y. Cherruault, Convergence of Adomian's method and applications to non-linear partial differential equations, Kybernetes 21 (6) (1992) 13–25.
[55] T. Mavoungou, Y. Cherruault, Numerical study of Fisher's equation by Adomian's method, Math. Comput. Modelling 19 (1) (1994) 89–95.
[56] N. Ngarhasta, B. Some, K. Abbaoui, Y. Cherruault, New numerical study of Adomian method applied to a diffusion model, Kybernetes 31 (1) (2002) 61–75.
[57] A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, Chapman & Hall/CRC, London, Boca Raton, 2004.
[58] P. Vadasz, S. Olek, Convergence and accuracy of Adomian's decomposition method for the solution of Lorenz equations, Internat. J. Heat and Mass Transfer 43 (2000) 1715–1734.
[59] A.-M. Wazwaz, A comparison between Adomian decomposition method and Taylor series method in the series solutions, Appl. Math. Comput. 97 (1998) 37–44.
[60] A.-M. Wazwaz, The decomposition method for approximate solution of the Goursat problem, Appl. Math. Comput. 69 (1995) 299–311.
[61] A.-M. Wazwaz, A new approach to the nonlinear advection problem: an application of the decomposition method, Appl. Math. Comput. 72 (1995) 175–181.
[62] A.-M. Wazwaz, Necessary conditions for the appearance of noise terms in decomposition solution series, Appl. Math. Comput. 81 (1997) 265–274.
[63] A.-M. Wazwaz, A reliable technique for solving the wave equation in an infinite one-dimensional medium, Appl. Math. Comput. 92 (1998) 1–7.
[64] A.-M. Wazwaz, The modified decomposition method and Padé approximants for solving the Thomas–Fermi equation, Appl. Math. Comput. 105 (1999) 11–19.
[65] A.-M. Wazwaz, Analytica approximations and Padé approximants for Volterra's population model, Appl. Math. Comput. 100 (1999) 13–25.
[66] A.-M. Wazwaz, Construction of solitary wave solutions and rational solutions for the KdV equation by Adomian decomposition method, Chaos Solitons & Fractals 12 (2001) 2283–2293.
[67] A.-M. Wazwaz, A. Gorguis, Exact solutions for heat-like and wave-like equations with variable coefficients, Appl. Math. Comput. 149 (2004) 15–29.
[68] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.