An Effective Criterion for Congruence of Real Symmetric Matrix Pairs

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ABSTRACT

This paper gives a rational method of determining the congruence of $m\times m$ real symmetric pairs over the reals R. If (S_1,T_1) and (S_2,T_2) are nonsingular pairs, then (S_1,T_1) is congruent to (S_2,T_2) over R if and only if $S_1^{-1}T_1$ is similar to $S_2^{-1}T_2$ and the signatures of $S_1f(S_1^{-1}T_1)^k\ g(S_1^{-1}T_1)$ and $S_2f(S_2^{-1}T_2)^k g(S_2^{-1}T_2)$ are equal for $k=0,1,2,\dots,m-1$ and for all $g(x)$ in P, where P is a relatively small set of real polynomials and $f(x)$ is a fixed polynomial. This result is then extended to singular pairs using theorems on minimal indices.

We say that a pair of $m\times m$ matrices (A,B) is a nonsingular pair if A is nonsingular; otherwise the pair is called singular. Two pairs of $m\times m$ real symmetric matrices, (S_1,T_1) and (S_2,T_2), are congruent if and only if there exists a real nonsingular matrix C such that $C'S_1C=S_2$ and $C'T_1C=T_2$, where C' is the transpose of C.

Define $\mathcal{C}=\{(S,T)/S,T$ are $m\times m$ real symmetric matrices and (S,T) is a nonsingular pair$\}$. Now consider two pairs (S_1,T_1) and (S_2,T_2) in $\mathcal{C}$ such that $S_1^{-1}T_1$ is similar to $S_2^{-1}T_2$. Let $p(x)$ be the characteristic polynomial of $S_1^{-1}T_1$ and $S_2^{-1}T_2$, and gcd $(p(x),p'(x))$ be the greatest common divisor of $p(x)$ and $p'(x)$, where $p'(x)$ is the derivative of $p(x)$. Define $f(x)=p(x)/\gcd(p(x),p'(x))$. Then we have $f(x)=(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$, where the λ_i's are the distinct real characteristic roots of $S_1^{-1}T_1$ and $S_2^{-1}T_2$ for $i=1,2,\dots,r$, and are the distinct nonreal roots for $i=r+1,\ r+2,\dots,n$. Let us assume that $\lambda_1>\lambda_2>\cdots\lambda_r$, and consider

$$f(x+t)=\sum_{i=0}^{n}f_i(x)t^{n-i}.$$

Here the f_j's are the $(n-j)$th derivatives of f divided by a positive constant. Let

$$P=\left\{ g(x) \middle| g(x)=\prod_{i=1}^{n-1} f_i(x)^{j_i}, \qquad j_i\in\{0,1,2\}, \quad i=1,2,\dots,n-1 \right\}. \quad (1)$$

Now we will state an effective criterion for congruence of pairs in $\mathcal{C}$ in the following theorem:

THEOREM. *Let (S_1,T_1), (S_2,T_2) be in $\mathcal{C}$; then (S_1,T_1) is congruent to (S_2,T_2) if and only if $S_1^{-1}T_1$ is similar to $S_2^{-1}T_2$ and*

$$\operatorname{sig} S_1 f(S_1^{-1}T_1)^k g(S_1^{-1}T_1)=\operatorname{sig} S_2 f(S_2^{-1}T_2)^k g(S_2^{-1}T_2). \qquad (*)$$

for all $g(x)\in P$ and $k=0,1,2,\dots,m-1$, where $\operatorname{sig}A$ denotes the signature of symmetric matrix A.

The corresponding theorem for the Hermitian case is also true, and the proof is similar to the real case. The following lemmas will be used in the proof of this theorem.

LEMMA 1. *Let (S,T) be in $\mathcal{C}$ and $h(x)\in R[x]$, where R is the field of real numbers. Then $S\,h(S^{-1}T)$ is real symmetric.*

The lemma is easily proved by noting that $[S(S^{-1}T)^k]'=S(S^{-1}T)^k$ for any natural number k and that $h(A)'=h(A')$ for each square matrix A.

Notation. E_q is the $q\times q$ matrix with 1's on the antidiagonal [i.e., the (i,j)th positions for which $i+j=q+1$] and 0's elsewhere; J_q is the $q\times q$ matrix with 1's on the first superdiagonal and 0's elsewhere; I_q is the $q\times q$ identity matrix.

Recall that for $a\in R$, $\operatorname{sgn} a$ is $a/|a|$ if $a\neq 0$ and 0 if $a=0$.

LEMMA 2. *Let $h(x),m(x)\in R[x]$ and $\lambda\in R$ with $h(\lambda)=0\neq h'(\lambda)$. Let $E=E_q$, $J=J_q$, $I=I_q$. Then $h(\lambda I+J)^k=0$ for $k\geqslant q$, and*

$$\operatorname{sig} E h(\lambda I+J)^{q-1} m(\lambda I+J)=\operatorname{sgn} h'(\lambda)^{q-1} m(\lambda).$$

Proof. By Taylor's Theorem, we have [since $J^q=0$ and $h(\lambda I)=h(\lambda)I$]

$$h(\lambda I+J)=\sum_{l=1}^{q-1}\frac{h^{(l)}(\lambda I)}{l!}J^l$$

$$=Jh'(\lambda)\sum_{l=0}^{q-2} c_l J^l,$$

where $c_l = h^{(l+1)}(\lambda)/[(l+1)!h'(\lambda)]$, $l=1,\dots,q-2$ (in particular, $c_0=1$), and $h^{(l)}(x)$ denotes the lth derivative of $h(x)$. Hence $h(\lambda I+J)^k=0$ for all $k \geqslant q$. Let $m(x)\in R[x]$. Then routine calculation gives

$$Eh(\lambda I+J)^{q-1}m(\lambda I+J)=h'(\lambda)^{q-1}m(\lambda)EJ^{q-1}. \tag{2}$$

The matrix (2) is a $q\times q$ matrix whose (q,q)th entry is $h'(\lambda)^{q-1}m(\lambda)$ and whose other entries are all zero. Hence the signature of the matrix (2) equals $\operatorname{sgn} h'(\lambda)^{q-1}m(\lambda)$. ■

LEMMA 3. *Let (S_1,T_1), (S_2,T_2) be in $\mathcal{C}$. If (S_1,T_1) is congruent to (S_2,T_2), then $S_1\, h(S_1^{-1}T_1)$ is congruent to $S_2h(S_2^{-1}T_2)$ for all $h(x)\in R[x]$.*

The corresponding result is also true for the Hermitian case.

LEMMA 4. *If (S,T) is an $m\times m$ nonsingular Hermitian pair such that $S^{-1}T$ has no real characteristic roots, and $h(x)\in R[x]$, then $\operatorname{sig} Sh(S^{-1}T)=0$.*

Proof. Without loss of generality [2], we may assume that $S^{-1}T$ has only one pair of complex conjugate nonreal elementary divisors, $(x-\lambda)^q$ and $(x-\bar{\lambda})^q$. Then (S,T) is conjunctive with

$$\left(\begin{bmatrix} 0 & E_q \\ E_q & 0 \end{bmatrix}, \begin{bmatrix} 0 & E_q(\bar{\lambda}I_q+J_q) \\ E_q(\lambda I_q+J_q) & 0 \end{bmatrix}\right).$$

Let $h(x)\in R[x]$. Then $S\; h(S^{-1}T)$ is conjunctive with

$$M=\begin{bmatrix} 0 & K^* \\ K & 0 \end{bmatrix},$$

where $K=E_qh(\lambda I_q+J_q)$ and K^* is the complex conjugate transpose of K. Since M is conjunctive with $-M$, we have $\operatorname{sig} Sh(S^{-1}T)=\operatorname{sig} M=0$. ■

COROLLARY 5. *Let (S,T) be an $m\times m$ nonsingular Hermitian pair. If (S,T) is conjunctive with $(A_1\oplus A_2,\ B_1\oplus B_2)$, where A_1,A_2,B_1,B_2 are matrices such that $A_1^{-1}B_1$ has only real roots and $A_2^{-1}B_2$ has only nonreal roots, then $\operatorname{sig} Sh(S^{-1}T)=\operatorname{sig} A_1h(A_1^{-1}B_1)$ for all $h(x)\in R[x]$.*

Notation. F denotes any field; $A=(a_{ij})$ denotes an $n\times m$ matrix whose (i,j)th element is a_{ij}; when A is over R, $\operatorname{sgn} A$ denotes the matrix $B=(b_{ij})$ such that $b_{ij}=\operatorname{sgn} a_{ij}$ for all i,j.

LEMMA 6. *Let $X_1, X_2, \ldots, X_r$ be nonzero $n\times 1$ column vectors over F. If the $n\times(r-1)$ matrix $[X_1, X_2, \ldots, X_{r-1}]$ has rank $r-1$ and X_r can be expressed (uniquely) as*

$$X_r = b_1X_1 + b_2X_2 + \cdots + b_{r-1}X_{r-1} \tag{3}$$

for $b_i\in F$ (not all zero), say with $b_j\neq 0$, then for each sequence $\gamma_1, \gamma_2, \ldots, \gamma_r$ in F for which $\gamma_j\neq\gamma_r$, the $2n\times r$ matrix

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_r \\ \gamma_1X_1 & \gamma_2X_2 & \cdots & \gamma_rX_r \end{bmatrix} \tag{4}$$

has rank r.

Proof. We will show that the columns of the matrix (4) are linearly independent. Let $a_1, a_2, \ldots, a_r$ be any sequence in F. Suppose

$$a_1X_1 + a_2X_2 + \cdots + a_rX_r = 0$$

$$a_1\gamma_1X_1 + a_2\gamma_2X_2 + \cdots + a_r\gamma_rX_r = 0. \tag{5}$$

Then $\sum_{i=1}^{r-1}(\gamma_i-\gamma_r)a_iX_i=0$. Therefore $a_i(\gamma_i-\gamma_r)=0$ for all i. But $\gamma_r\neq\gamma_j$; therefore $a_j=0$. From (3) and (5), we have $a_r\sum_{i=1}^{r-1}b_iX_i=-\sum_{i=1}^{r-1}a_iX_i$. Since $X_1, X_2, \ldots, X_{r-1}$ are linearly independent, we have $a_rb_j=-a_j=0$. But $b_j\neq 0$, so $a_r=0$. Consequently $a_i=0$ for all i. ■

LEMMA 7. *Let $X_1, X_2, \ldots, X_r$ be r distinct nonzero $n\times 1$ column vectors over F. Then there exist s row vectors ($1\times r$ matrices) $Y_j=(\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jr})$, $j=1, 2, \ldots, s$ with $s\leqslant 2^{r-1}$, having the following three properties:* (i) $Y_1=(1, 1, \ldots, 1)$, (ii) *Y_j is a (componentwise) product of at most $r-1$ rows of the $n\times r$ matrix $[X_1, X_2, \ldots, X_r]$ for $j\geqslant 2$, and* (iii) *the $sn\times r$ matrix*

$$\begin{bmatrix} \alpha_{11}X_1 & \alpha_{12}X_2 & \cdots & \alpha_{1r}X_r \\ \alpha_{21}X_1 & \alpha_{22}X_2 & \cdots & \alpha_{2r}X_r \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{s1}X_1 & \alpha_{s2}X_2 & \cdots & \alpha_{sr}X_r \end{bmatrix} \tag{6}$$

has rank r.

Proof. The proof is by induction on r. For $r=1$, Lemma 6 is obviously true. We can take $s=1$ and $Y_1=(1)$. For $r=2$, if X_1, X_2 are linearly independent, then $[X_1, X_2]$ has rank 2 and again we can take $s=1$, $Y_1=(1,1)$. If X_1, X_2 are linearly dependent, then consider a row, which we denote by $(\alpha_{21}, \alpha_{22})$, of $[X_1, X_2]$ such that $\alpha_{21} \neq \alpha_{22}$. By Lemma 6 with $r=2$

$$\begin{bmatrix} X_1 & X_2 \\ \alpha_{21}X_1 & \alpha_{22}X_2 \end{bmatrix}$$

has rank 2. Clearly we can take $s=2$ and $Y_2=(\alpha_{21}, \alpha_{22})$, which is (a product of) one row of $[X_1, X_2]$.

Assume the lemma is true for $r=k$. Suppose we have $k+1$ distinct nonzero vectors, $X_1, X_2, \ldots, X_{k+1}$. By the induction hypothesis, there exist Y_j, $j=1, 2, \ldots, s$, with $Y_1=(1, 1, \ldots, 1)$, such that the $sn \times (k+1)$ matrix $A=[R_1, R_2, \cdots R_{k+1}]$ [A is the matrix in (6) with $r=k+1$] has rank $\geqslant k$, namely with $R_1, R_2, \ldots, R_k$ linearly independent. Also we can take $s \leqslant 2^{k-1}$, and Y_j is a product of at most $k-1$ rows of $[X_1, X_2, \ldots, X_{k+1}]$ for $j \geqslant 2$. If $R_1, \ldots, R_{k+1}$ are linearly independent, then we are done. If not, then $R_{k+1}=\sum_{i=1}^{k} b_i R_i$ for some $b_i \in F$, not all zero; say $b_j \neq 0$. Now consider a row, which we denote by $(\gamma_1, \gamma_2, \ldots, \gamma_{k+1})$, of $[X_1, X_2, \ldots, X_{k+1}]$ such that $\gamma_j \neq \gamma_{k+1}$. Then by Lemma 6 with $r=k+1$, the $2sn \times (k+1)$ matrix

$$\begin{bmatrix} R_1 & R_2 & \cdots & R_{k+1} \\ \gamma_1 R_1 & \gamma_2 R_2 & \cdots & \gamma_{k+1} R_{k+1} \end{bmatrix}$$

has rank $k+1$. The conclusions of the lemma follow immediately. ∎

REMARK. Suppose of the r vectors, $X_1, X_2, \ldots, X_r$ of Lemma 7, the first h vectors are linearly independent; then the s of the conclusion can be chosen $\leqslant 2^{r-h}$. If, in Lemma 7, the characteristic of F is not 2, and if there exist rows $(1, 1, \ldots, 1)$ and $\left(1, -1, 1, \ldots, (-1)^{r-1}\right)$ of $[X_1, X_2, \ldots, X_r]$, then for $r \geqslant 4$, we can take $s \leqslant 2^{(r-1)/2}-1$ if r is odd and $\leqslant 3 \cdot 2^{(r-4)/2}-1$ if r is even. For $r \leqslant 3$, we can take $s=1$.

LEMMA 8. *Let $f(x)=(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \in R[x]$, where λ_i is real for $i=1, 2, \ldots, r$, nonreal for $i=r+1, r+2, \ldots, n$. Assume $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and consider $f(x+t)=\sum_{j=0}^{n} f_j(x) t^{n-j}$. Let*

$$X_i = [\operatorname{sgn}(f_0(\lambda_i), f_1(\lambda_i), \ldots, f_{n-1}(\lambda_i))]' \qquad \text{for } i=1, 2, \ldots, r.$$

Then the X_i's are distinct and in fact are pairwise linearly independent.

Proof. First note that $\operatorname{sgn} f_i(\lambda_i) = \operatorname{sgn} f^{(n-i)}(\lambda_i)$. Since the λ_i's are distinct, $f'(\lambda_i) \neq 0$ for $i = 1, 2, \ldots, r$. Consider λ_p, λ_q where $r \geqslant p > q \geqslant 1$. Then there exists $\beta \in R$, $\lambda_p < \beta < \lambda_q$ such that $f'(\beta) = 0$ (by Rolle's theorem). Now apply Budan's theorem [1] to $f'(x)$ and consider the following sequences:

$$f'(\lambda_p), f^{(2)}(\lambda_p), \ldots, f^{(n)}(\lambda_p); \tag{7}$$

$$f'(\lambda_q), f^{(2)}(\lambda_q), \ldots, f^{(n)}(\lambda_q). \tag{8}$$

Then the numbers of variations of signs of the sequences (7) and (8) are not equal, by Budan's theorem; otherwise it would contradict the existence of β. Therefore $X_p \neq X_q$, and since $f_0(x) = 1$, it follows that X_p and X_q are linearly independent. ■

LEMMA 9. *Let $f_0(x), f_1(x), \ldots, f_{n-1}(x)$, and P be as in* (1). *Let P be as in* (1). *Then there exist $g_1(x), g_2(x), \ldots, g_r(x) \in P$ such that the $r \times r$ matrix $(\operatorname{sgn} g_i(\lambda j))$ is nonsingular.*

Proof. The proof follows easily from Lemmas 7 and 8. ■

Proof of the theorem. Suppose (S_1, T_1) is congruent to (S_2, T_2). Then it follows easily that $S_1^{-1}T_1$ is similar to $S_2^{-1}T_2$. Let $f(x)$, $g(x)$ be as in the theorem. Then (by Lemma 3) $S_1 f(S_1^{-1}T_1)^k g(S_1^{-1}T_1)$ is congruent to $S_2 f(S_2^{-1}T_2)^k g(S_2^{-1}T_2)$ for $k = 0, 1, 2, \ldots, m-1$ and for all $g(x) \in P$. Hence (*) holds.

Conversely, if $S_1^{-1}T_1$ and $S_2^{-1}T_2$ are similar and (*) is satisfied for $k = 0, 1, 2, \ldots, m-1$ and all $g(x) \in P$, then $S_1^{-1}T_1$ and $S_2^{-1}T_2$ have the same real-Jordan canonical form [3, p. 248, Theorem 36.2]:

$$\left(\bigoplus_{i=1}^{r} \bigoplus_{q=1}^{m} \bigoplus_{j=1}^{k_{qi}} \lambda_i I_q + J_q \right) \oplus M,$$

where the λ_i's are the real roots of $S_1^{-1}T_1$ and $S_2^{-1}T_2$, M is the real-Jordan form which corresponds to the nonreal roots, and $\oplus$ denotes direct sum. Therefore (S_1, T_1) is congruent over R to (A, B), where

$$A = \left(\bigoplus_{i=1}^{r} \bigoplus_{q=1}^{m} \bigoplus_{j=1}^{k_{qi}} \varepsilon_{iqj} E_q \right) \oplus H, \tag{9}$$

$$B = \left(\bigoplus_{i=1}^{r} \bigoplus_{q=1}^{m} \bigoplus_{j=1}^{k_{qi}} \varepsilon_{iqj} E_q (\lambda_i I_q + J_q) \right) \oplus G, \tag{10}$$

and where $H^{-1}G$ has only nonreal roots; (S_2, T_2) is congruent to a pair of matrices (9) and (10) with ε_{iqj} replaced by ε'_{iqj}, where ε_{iqj} and ε'_{iqj} are equal to ± 1 for all i, q, j [5]. Let $g_1(x), g_2(x), \ldots, g_r(x)$ be in P and such that the $r \times r$ matrix $(sgn\, g_i(\lambda_j))$ is nonsingular (as in Lemma 9). Then

$$\operatorname{sig} S_1 f(S_1^{-1}T_1)^{m-1} g_s(S_1^{-1}T_1) \tag{11}$$

$$= \operatorname{sig}\left(\bigoplus_{i=1}^{r} \bigoplus_{q=1}^{m} \bigoplus_{j=1}^{k_{qi}} \varepsilon_{iqj} E_q f(\lambda_i I_q + J_q)^{m-1} g_s(\lambda_i I_q + J_q) \right) \tag{12}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{k_{mi}} \varepsilon_{imj} \operatorname{sgn}\left[f'(\lambda_i)^{m-1} g_s(\lambda_i) \right]. \tag{13}$$

(12) follows from Corollary 5, and (13) follows from Lemma 2; in particular, the terms for $q < m$ are zero by Lemma 2. Similarly, (11) with S_1 replaced by S_2 and T_1 replaced by T_2 is equal to (13) with ε_{imj} replaced by ε'_{imj} for $s = 1, 2, \ldots, r$. Since the $r \times r$ matrix $(\operatorname{sgn} g_i(\lambda_j))$ is nonsingular and $f'(\lambda_i) \neq 0$, we have

$$\sum_{i=1}^{r} \sum_{j=1}^{k_{mi}} \varepsilon_{imj} = \sum_{i=1}^{r} \sum_{j=1}^{k_{mi}} \varepsilon'_{imj}, \qquad i = 1, 2, \ldots, r. \tag{14}$$

To show that (14) holds with m replaced by $m-1$, we first observe that (11) with $m-1$ replaced by $m-2$ is equal to

$$\sum_{i=1}^{r} \sum_{j=1}^{k_{m-1,i}} \varepsilon_{i,m-1,j} \operatorname{sgn} f'(\lambda_i)^{m-2} g_s(\lambda_i)$$

$$+ \sum_{i=1}^{r} \sum_{j=1}^{k_{mi}} \varepsilon_{imj} \operatorname{sig} E_m f(\lambda_i I_m + J_m)^{m-2} g_s(\lambda I_m + J_m). \tag{15}$$

Similarly, (11) with S_1, T_1 replaced by S_2, T_2 and m by $m-1$ is equal to (15) with ε_{iqj} replaced by ε'_{iqj}. Therefore (14) is satisfied with m replaced by $m-1$. Continuing in this way, it follows that (14) holds with m replaced by $m-1, m-2, \ldots, 2, 1$. Therefore (S_1, T_1) is congruent to (S_2, T_2). ■

REMARK *(the singular case)*. If (S, T) were an $m \times m$ real symmetric singular pair, then we would consider the pencil of matrices $\lambda S + \mu T$, where λ, μ are real indeterminates. Without loss of generality, we may assume that the determinant of $\lambda S + \mu T$ is identically equal to zero. Then $\lambda S + \mu T$ is congruent to $A \oplus B$ (the congruence operations are rational), where A is the matrix which corresponds to the minimal indices of the pencil $\lambda S+\mu T$, and B is nonsingular and is uniquely determined (up to congruence) [4,6]. In this way, we reduce the singular case to the nonsingular case and then consider only the nonsingular core B.

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