Two discrete versions of the Inscribed Square Conjecture and some related problems

Feliú Sagols, Raúl Marín *
Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, 07000 México City, Mexico

ARTICLE INFO

Keywords:
Inscribed Square Conjecture
Jordan Curve Theorem
Digital topology
Simple closed digital curves
4-connectivity
8-connectivity

ABSTRACT

The Inscribed Square Conjecture has been open since 1911. It states that any plane Jordan curve \( J \) contains four points on a non-degenerate square. In this article two different discrete versions of this conjecture are introduced and proved. The first version is in the field of digital topology: it is proved that the conjecture holds for digital simple closed 4-curves, and that it is false for 8-curves. The second one is in the topological graph theory field: it is proved that any cycle of the grid \( \mathbb{Z}^2 \) contains an inscribed square with integer vertices. The proofs are based on a theorem due to Pak. An infinite family of 4-curves in the digital plane containing a single non-degenerate inscribed square is introduced as well as a second infinite family containing one 4-curve with exactly \( n \) inscribed squares for each positive integer value of \( n \). Finally an algorithm with time complexity \( O(n^2) \) is given to find inscribed squares in simple digital curves.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

A Jordan curve (simple closed curve) is described by the set of points \( \omega(t) = (x(t), y(t)) \in \mathbb{R}^2 \), with \( x, y : [0, 1] \rightarrow \mathbb{R} \) continuous such that \( \omega(0) = \omega(1) \) and if \( 0 \leq t_1 < t_2 < 1 \), then \( \omega(t_1) \neq \omega(t_2) \). A polygon \( P \) is inscribed in a set \( S \) if all vertices of \( P \) belong to \( S \).

Toeplitz introduced in 1911 the following conjecture known as the “Inscribed Square Conjecture” (see [1,2]).

Conjecture 1. Every Jordan curve (simple closed curve in the plane) contains four vertices of some non-degenerate square.

“Non-degenerate” means that the square vertices are distinct points. Henceforth “inscribed square” should be understood as synonymous with “non-degenerate inscribed square”.

It is quite simple to understand the Conjecture 1 (see Fig. 1), but it is hard to find solutions for particular Jordan curves, even for simple cases. After 1911 several attempts to solve it have been made but this problem remains open. Nonetheless, several important results have been published since then.

Theorem 1 ([2]). If the Jordan curve \( J \) is symmetric about a point \( c \), and each ray issuing from \( c \) meets \( J \) in a single point, then \( J \) admits an inscribed square.

Theorem 2 ([2]). Each convex Jordan curve\(^1\) admits an inscribed square.

Theorem 3 (Pak [3]). Every simple polygon on the plane has an inscribed square.

But the most remarkable result is due to Stromquist [4]. To introduce it, a preliminary definition is needed.

\(^{*}\) Corresponding author. Tel.: +52 (55) 57 47 38 00x6479; fax: +52 (55) 57 47 38 76.

E-mail addresses: fsagols@math.cinvestav.mx (F. Sagols), rmarin@math.cinvestav.mx, raulmar236@yahoo.com.mx (R. Marín).

\(^{1}\) A Jordan curve is convex if the region bounded by the curve is convex.

0304-3975/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2010.10.004
A simple closed curve $\omega$ in $\mathbb{R}^2$ is \textit{locally monotone} if for every point $p$ on this curve there are a real number $\epsilon > 0$ and a non-zero vector $n(p)$ such that for every pair $p_1$ and $p_2$ of distinct points in $B(p, \epsilon) \cap \omega$ (here $B(p, \epsilon)$ represents the open ball with center at $p$ and radius $\epsilon$), the equality $p_1 - p_2 = \lambda n(p)$ is not satisfied for any $\lambda \in \mathbb{R}$. In other words, no chord of $\omega$ contained in $B(p, \epsilon)$ is parallel with $n(p)$.

This definition was introduced by Stromquist in [4]. Actually it is quite general. Let us explain it with the help of Fig. 2. At the center of the figure appears a shaded region representing the union of lines containing a chord in $\omega \cap B(p, \epsilon)$, named the span of $\omega$ at $p$ and $\epsilon$, which separates the plane into two disjoint regions. Any vector crossing the span may be used as $n(p)$. When $\omega$ is "smooth" near $p$ the span is close to a straight line, and finding $n(p)$ is simple. But if $\omega$ has sudden changes within $B(p, \epsilon)$, then the span covers a greater area as is shown in Fig. 2 (at right). In this situation the span of $\omega'$ at $p'$ and $\epsilon'$ covers the entire plane and there does not exist any possible assignment to $n(p')$. In this particular situation a smaller value of $\epsilon'$ is enough to prove that the curve is locally monotone at $p'$. If the spiral around $p'$ in Fig. 2 (at right) is replaced by another spiral with infinitely many turns preserving the Jordan curve, then the resulting curve will not be locally monotone. In Fig. 3 appears an approximate graphic representation. For Jordan curves containing these pathologies the Inscribed Square Conjecture remains open.

\textbf{Theorem 4 (Stromquist [4])}. If $\omega$ is a locally monotone curve in $\mathbb{R}^2$, then $\omega$ admits an inscribed square.

Smooth (infinitely differentiable) curves, convex curves, and most piecewise continuously differentiable curves are locally monotone and satisfy Theorem 4. But in general, it is possible that most curves fail to admit inscribed squares (see [5]). Additional information about the Inscribed Square Conjecture can be found in [6–10].
This work addresses theoretical issues of Conjecture 1 on the digital plane and on the grid $\mathbb{Z}^2$. But it also opens the opportunity to devise new applications in digital topology and image analysis. Only a few applications to the Inscribed Square Conjecture are reported in the literature. One of them is provided by Fenn.

**Theorem 5** (Table Theorem [8]). Let $D$ be a bounded convex set in $\mathbb{R}^2$, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function that is non-negative in the interior of $D$ and zero elsewhere. Let $d$ be a fixed positive integer. Then there is a square of side $d$ whose center is in $D$, such that $f$ takes the same value at all four vertices of the square.

Fenn interprets this theorem as the existence of a square table stand level on a bumpy floor. This theorem is related to the Inscribed Square Conjecture by considering the non-trivial level curves of $f$. In the digital plane, with the results given in this paper, a similar result may be proved useful in digital terrain modeling applications, for example to find locations for facilities requiring square basements.

All simple closed 4-curves in the digital plane are isomorphic to cycles in the infinite grid $\mathbb{Z}^2$ but there are cycles in $\mathbb{Z}^2$ (see Fig. 4) which does not correspond to simple closed 4-curves. For the embedding of these cycles the Inscribed Square Conjecture cannot be established in the digital plane. For this reason the conjecture was reformulated from the topological graph theory point of view.

We have based our study mainly on digital topology because in this way we considered it easier to develop computer applications based on the Inscribed Square Conjecture. However, we could have developed our article using exclusively topological graph theory.

This paper is divided into three parts: solution to the main conjectures, algorithmic constructions, and results derived from the main theorems. In a more specific way, in Section 2 the topological graph theory version of the Inscribed Square Conjecture is introduced as well as the topological graph theory concepts needed in this work. In Section 3 basic concepts on digital topology and the digital plane are developed. Then in Section 4 it is answered positively the digital version of Conjecture 1 under 4-connectivity, and a counter example is provided for 8 connectivity; in addition this section contains the proof of the topological theory version of the inscribed square conjecture. In Section 5 is developed an algorithm with time complexity $O(n^2)$ to find all inscribed squares in 4-curves with $n$ points; this algorithm uses basic properties of digital squares. In Section 6 two infinite families of 4-curves are introduced; the first one contains a single inscribed square per curve and the second contains one 4-curve with exactly $k$ inscribed squares for each positive integer value of $k$. Finally, Section 7 contains the conclusions of this work.

## 2. The Inscribed Square Conjecture from a topological graph theory perspective

Before establishing and solving the digital topology version of the Inscribed Square Conjecture, a topological graph theory version of this problem is introduced in this section. First the preliminary topological graph theory concepts are introduced (see [11]).

Given a simple graph $G$, a **planar graph embedding** of $G$ is a function $\psi_G$ transforming every vertex in $V(G)$ into a point on the plane and every edge in $G$ into a homeomorphic copy of the closed real interval $[0, 1]$. The images under $\psi_G$ of two distinct edges in $E(G)$ only meet at their endpoints and for each edge $vw \in E(G)$ the image $\psi_G(vw)$ coincides with exactly one end point of $\psi_G(vw)$. A **path** in $G$ of length $n$ is a sequence of pairwise different vertices $v_0, \ldots, v_n$, pairwise different except possibly $v_0$ and $v_n$, where $n$ is a non-negative integer, $v_0, v_1, \ldots, v_n \in V(G)$ and $(v_i, v_{i+1}) \in E(G)$ for $i = 0, \ldots, n - 1$. If $v_0 = v_n$ and $n > 2$, then the path is a cycle. In a planar graph embedding $\psi_G$ can be extended to paths making $U_{i=0, \ldots, n-1} \psi_G((v_i, v_{i+1}))$. In a planar graph embedding the image under $\psi_G$ of a cycle is a Jordan curve.

The grid $\mathbb{Z}^2$ is the infinite graph $(V(\mathbb{Z}^2), E(\mathbb{Z}^2))$ where $V(\mathbb{Z}^2)$ are the ordered pairs $(i, j)$ such that $i, j \in \mathbb{Z}$, and $((i_1, j_1), (i_2, j_2))$ are in $E(\mathbb{Z}^2)$ if and only if $(i_1, j_1) - (i_2, j_2) \in \{(1, 0), (0, 1), (−1, 0), (0, −1)\}$. The grid $\mathbb{Z}^2$ is considered as a planar graph embedding $\psi_{\mathbb{Z}^2}$ where $\psi_{\mathbb{Z}^2}((i, j)) = (i, j)$ for each $(i, j) \in V(\mathbb{Z}^2)$ and $\psi_{\mathbb{Z}^2}(((i_1, j_1), (i_2, j_2)))$ is the line segment $t((i_1, j_1)) + (1-t)((i_2, j_2))$ with $t \in [0, 1]$.

We met the Inscribed Square Conjecture through the following variant raised by Tommy Jensen and introduced to us by Enrique García Moreno Esteva in the “International Workshop Combinatorial and Computational Aspects of Optimization, Topology and Algebra” held in México in 2006.

**Conjecture 2.** For every cycle $C$ in $\mathbb{Z}^2$, $\psi_{\mathbb{Z}^2}(C)$ always contains an inscribed square with integer coordinates vertices.
The author of this conjecture considered that finding a positive answer could solve Conjecture 1. His idea was to refine infinitely the grid \( \mathbb{Z}^2 \) to approximate any continuous Jordan curve. Certainly, for each refinement a square exists, and a convergent subsequence of squares also exists because of compactness; however, it cannot be ensured that the limit of such a subsequence is non-degenerate.

We do not know whether it is possible to overcome this difficulty and finally whether Conjecture 2 helps to solve Conjecture 1. But the condition on the integral coordinates of the inscribed square claimed in Conjecture 2 suggests a question: Is there a way to state an equivalent digital topology conjecture?

The answer is yes. A key fact is that any inscribed square in a simple closed digital curve necessarily has vertices with integer coordinates.

Conjecture 2 is true; it will be proved as a consequence of Theorem 12 in Section 4.

3. The digital plane

Several digital space models exist (see [12]), here a very simple one is used. The digital plane is the set of points in \( \mathbb{R}^2 \) having integer coordinates (points in \( \mathbb{Z}^2 \)). Two distinct points \( p \) and \( q \) of \( \mathbb{Z}^2 \) are 8-adjacent if both coordinates of one differ from the corresponding coordinate of the other by at most 1. Points \( p \) and \( q \) are 4-adjacent if they are 8-adjacent and differ in exactly one of their coordinates.

Let \( k \) be either of the numbers 4 or 8. A digital \( k \)-path is a finite sequence \( p_0, p_1, \ldots, p_n \) of points in \( \mathbb{Z}^2 \) such that if \( |i − j| = 1 \), then \( p_i \) and \( p_j \) are \( k \)-adjacent. A \( k \)-path is a \( k \)-path such that \( p_0 = p_n \).

Definition 1. A simple closed digital 8-curve \( J \) (or simply, a simple closed 8-curve) is a closed 8-path \( p_0, p_1, \ldots, p_n \) with \( n \geq 4 \) and \( p_i \neq p_j \) for all \( i, j \) such that \( 0 \leq i < j < n \), and each point is 8-adjacent to exactly two other points of \( J \).

Definition 2. A simple closed digital 4-curve \( J \) (or simply, a simple closed 4-curve) is a closed 4-path \( p_0, p_1, \ldots, p_n \) with \( n \geq 8 \) and \( p_i \neq p_j \) for all \( i, j \) such that \( 0 \leq i < j < n \), and each point is 4-adjacent to exactly two other points of \( J \).

A digital set \( C \subset \mathbb{Z}^2 \) is \( k \)-connected (\( k \in \{4, 8\} \)) if for any two points \( q_1, q_2 \in C \) a \( k \)-path \( p_0, p_1, \ldots, p_n \) in \( C \) with \( p_0 = q_1 \) and \( p_n = q_2 \) exists. For a more detailed exposition of digital curves see [13].

The digital plane definition is given as a discrete approximation to the real plane. It is intended that basic properties of the real plane be fulfilled in the digital one. An example is the proof due to Rosenfeld [14] of a digital version of the Jordan Curve Theorem: every Jordan curve divides the plane into two components, the bounded one (the inside of the curve) and the unbounded (the outside of the curve). He proved that any simple closed 8-curve (or 4-curve) separates the digital plane into two disjoint 4-connected (resp. 8-connected) regions. Thus, the combined use of 4- and 8-connectivity reproduces in the digital plane a fundamental property of the real plane. In Section 4 it is proved that the digital version of the Inscribed Square Conjecture is true for simple closed 4-curves, and false for 8-curves.

A square in the digital plane is a set of four distinct points \( p_1 = (a_1, b_1), p_2 = (a_2, b_2), p_3 = (a_3, b_3), \) and \( p_4 = (a_4, b_4) \), that are vertices of a square on the real plane, and ordered clockwise around the square boundary. An alternate definition making no reference to the real plane establishes that if \( (a_2, b_2) = (a_1 + b_1 - b_3, a_4 - a_1 + b_1) \) and \( (a_3, b_3) = (a_2 + b_2 - b_1, a_1 - a_2 + b_2) \), then \( p_1, p_2, p_3 \) and \( p_4 \) are the vertices of a square in the digital plane. That is, a 90° rotation of \( p_4 \) around \( p_1 \) must be equal to \( p_2 \), and a 90° rotation of \( p_3 \) around \( p_2 \) must be equal to \( p_3 \). This implies the following:

Lemma 6. If \( p_1p_2p_3p_4 \) is a square in the real plane, and two consecutive points (clockwise or counter clockwise) have integer coordinates, then all four points have integer coordinates.

4. The digital Inscribed Square Conjecture

Let us introduce the digital version of Conjecture 1 for 8-connectivity.

Conjecture 3. Every simple closed 8-curve contains four points of some non-degenerate square.

Proposition 7. Conjecture 3 is false.

Proof. Let \( C \) be the simple closed 8-curve: \( (0, 1), (1, 2), (2, 3), (3, 3), (4, 2), (5, 1), (4, 0), (3, 0), (2, 0), (1, 0), (0, 1) \). No subset of four pairwise different points in \( C \) forms a square in the digital plane, hence the conjecture is false. In Fig. 5 a graphical representation of \( C \) is illustrated. \( \square \)

To prove the main result in this paper, let us introduce some preliminary definitions. Let \( J \) be a simple closed 4-curve in the digital plane. As a sequence of points, it is a cycle in the infinite grid \( \mathbb{Z}^2 \) (see Section 2), and the image of \( J \) under \( \psi_{\mathbb{Z}^2} \) is by definition the real plane embedding of \( J \).

Let \( J \) be a simple closed 4-curve and let \( p \) be a point in \( \psi_{\mathbb{Z}^2}(J) \). Then \( p \) has type \( h \) about \( \psi_{\mathbb{Z}^2}(J) \) if \( p \) has integer coordinates or if \( p \) belongs to a line segment with end points \( (i, j), (i + 1, j) \); the notation type\( (p) = h \) is used. It is said \( p \) has type \( v \) about \( \psi_{\mathbb{Z}^2}(J) \) if \( p \) does not have type \( h \) and it belongs to a line segment with end points \( (i, j), (i, j + 1) \); the notation type\( (p) = v \) is used. If \( p_1p_2p_3p_4 \) is a square, ordered clockwise, with vertices in \( \psi_{\mathbb{Z}^2}(J) \), then the type of the square about \( \psi_{\mathbb{Z}^2}(J) \) is the sequence type\( (p_1p_2p_3p_4) \) := type\( (p_1) \), type\( (p_2) \), type\( (p_3) \), type\( (p_4) \). Note that the same square can have different types depending upon the cyclic order of its vertices. When only one curve \( J \) is under discussion the expression “about \( \psi_{\mathbb{Z}^2}(J) \)” is omitted.
Four preliminary results are needed to prove the Theorem 12. First, in Lemmas 8 and 11 it is proved that if a non-degenerate square of type $hhhh$ and $hvhu$ is inscribed in the plane embedding of a simple closed 4-curve $J$, then $J$ itself contains an inscribed square. Then in Lemmas 9 and 10 it is proved that squares of types $hvhv$ and $hhuv$ are never possible. With these results the proof of Theorem 12 is straightforward.

Henceforth, $p_1p_2p_3p_4$ will denote a square in the real plane ordered clockwise. Let $p$ and $q$ be points in the real plane, $L(p, q)$ (resp. $R(p, q)$) denotes the point obtained when the point $p$ is rotated 90° (resp. $-90°$) around $q$. Since the vertices of $p_1p_2p_3p_4$ are ordered clockwise the identities $R(p_2, p_1) = p_4$, $R(p_1, p_4) = p_3$, and $L(p_1, p_2) = p_3$ are satisfied.

**Lemma 8.** Let $J$ be a simple closed 4-curve. If $ψ_{22}(J)$ contains an inscribed square $p_1p_2p_3p_4$ of type $hhhh$, then $J$ also contains an inscribed square. If one of $p_1$, $p_2$, $p_3$ or $p_4$ is not integer then $J$ contains at least two inscribed squares.

**Proof.** From the definition of type, $p_1$, $p_2$, $p_3$ and $p_4$ have, respectively, coordinates $(a_1 + x_1, b_1), (a_2 + x_2, b_2), (a_3 + x_3, b_3)$ and $(a_4 + x_4, b_4)$, for integer values $a_1, b_1, \ldots, a_4, b_4$ and real values $x_1, x_2, x_3, x_4 \in [0, 1)$. Without loss of generality it may be assumed that $a_1 = b_1 = 0$, and so $p_1 = (x_1, 0)$.

Since $R(p_2, p_1) = p_4$ it follows that $(b_2 + x_1, -a_2 - x_2 + x_1) = (a_4 + x_4, b_4)$ (see Fig. 6). This means that both $x_1 − x_4$ and $x_1 − x_2$ have integer values. But $x_1 − x_4$ cannot be greater than zero because otherwise $x_1$ would be greater than one and that is impossible. For a similar reason, $x_1 − x_4$ would not be less than zero. The only alternative for $x_1 − x_4$ to be an integer is $x_1 = x_2$. For the same argument $x_1 = x_2$.

It follows from $R(p_2, p_4) = p_1$ that $x_1 = x_2$, and therefore $x_1 = x_2 = x_3 = x_4$. In this way, a translation of $p_1p_2p_3p_4$ by $x_1$ units in the direction $(-1, 0)$ yields a non-degenerate square with vertices $(a_1, b_1), (a_2, b_2), (a_3, b_3)$, and $(a_4, b_4)$ which belong to $ψ_{22}(J)$ and have integer coordinates as required. This means that the vertices of this new square belong to $J$.

Note that if one of $p_1, p_2, p_3$, and $p_4$ is not in $\mathbb{Z}^2$, then there are two distinct non-degenerate squares inscribed in $J$. One is obtained from the translation in the previous paragraph and the other is produced translating $p_1p_2p_3p_4$ by $1 − x_1$ units along the direction $(1, 0)$ (see Fig. 6). This observation will be useful to prove Corollary 13.

**Lemma 9.** Let $J$ be a simple closed 4-curve. If $ψ_{22}(J)$ contains an inscribed square $p_1p_2p_3p_4$, then type$(p_1p_2p_3p_4)$ cannot be $hhhu$.

**Proof.** Suppose that type$(p_1p_2p_3p_4) = hhhv$. Then vertices $p_1, p_2, p_3$, and $p_4$ have coordinates $(a_1 + x_1, b_1), (a_2 + x_2, b_2), (a_3 + x_3, b_3)$, and $(a_4, b_4 + y_4)$, respectively. Here $x_1, x_2, x_3 \in [0, 1), y_4 \in (0, 1)$, and $a_1, b_1, \ldots, a_4, b_4 \in \mathbb{Z}$. Without loss of generality it may be assumed that $a_1 = b_1 = 0$, and so $p_1 = (x_1, 0)$.

It follows from $R(p_2, p_1) = p_4$ that $(b_2 + x_1, -a_2 - x_2 + x_1) = (a_4, b_4 + y_4)$ and thus $x_1 = 0$ and $y_4 + x_2 − x_1$ is an integer. Similarly $L(p_1, p_2) = p_3$, that is $(b_2 + a_2 + x_2, x_1 − a_2 − x_2 + b_2) = (a_1 + x_3, b_3)$. Then $x_2 = x_3$ and $x_1 − x_2$ is an integer, as $x_1 = 0$ the equality $x_2 = 0$ is hold. Therefore $x_1 = x_2 = x_3 = 0$, and since $y_4 + x_2 − x_1$ must be an integer, $y_4$ must be an integer too, but the latter condition is impossible because $y_4 \in (0, 1)$. In other words, no inscribed square can have type $hhhu$. □

**Lemma 10.** Let $J$ be a simple closed 4-curve. If $ψ_{22}(J)$ contains an inscribed square $p_1p_2p_3p_4$, then type$(p_1p_2p_3p_4)$ cannot be $hvhv$. □

---

*Fig. 5.* A simple closed 8-curve without inscribed squares.

*Fig. 6.* Square of type $hhhh$ (left), and squares obtained from Lemma 8 (right).
Lemma 11. Let $J$ be a simple closed 4-curve. If $\psi_2(J)$ contains an inscribed square $p_1p_2p_3p_4$ of type hvuv, then $J$ contains at least two inscribed squares.

Proof. Here $p_1, p_2, p_3$ and $p_4$ have, respectively, coordinates $(a_1 + x_1, b_1), (a_2 + x_2, b_2), (a_3 + x_3, b_3)$ and $(a_4 + x_4, b_4)$, for integer values $a_1, b_1, \ldots, a_4, b_4$ and real values $x_1, x_2 \in (0, 1)$ and $y_3, y_4 \in (0, 1)$. Without loss of generality it may be assumed that $a_1 = b_1 = 0$, and so $p_1 = (x_1, 0)$.

Since $L(p_1, p_2) = (a_2 + x_2 + x_3 - a_2 - x_2 + b_2) = (a_3 + b_3 + y_3)$ the equation $a_2 + x_2 + x_3 = a_3$ holds and in consequence $x_2$ must be an integer, this entails $x_2 = 0$ and therefore $y_3$ is an integer. But this is impossible because $y_3 \in (0, 1)$. The conclusion is that $p_1p_2p_3p_4$ cannot have type hhuv.

Theorem 12. Every simple closed 4-curve $J$ contains four vertices of some non-degenerate square.

Proof. Since $\psi_2(J)$ is a simple polygon, Theorem 3 guarantees the existence of an inscribed square $S$.

Let $p_1p_2p_3p_4$ be vertices of $S$, recall that vertices are ordered clockwise. The sequence type$(p_1p_2p_3p_4)$ belongs to $A := \{c_1c_2c_3c_4|c_1, c_2, c_3, c_4 \in \{h, v\}\}$. An equivalence relation $R$ on $A$ can be defined: two elements in $A$ are $R$ related if and only if they represent the same cyclic sequence. Note that any cyclic order of $p_1p_2p_3p_4$ represents $S$; in the same way, any cyclic order of type$(p_1)$ type$(p_2)$ type$(p_3)$ type$(p_4)$ could be one type for $S$, and thus only a representative of each equivalence class of $R$ must be analyzed to locate the inscribed square in $\psi_2(J)$ with integer coordinates. The equivalences classes of $R$ are

1. $\{hhhh\}$
2. $\{hhuv, hvhh, vhhh, hhvh\}$
3. $\{hhuv, hvuh, uhvh, uuvh\}$
4. \{hvhv, vhvh\}
5. \{vuvv\}
6. \{huvu, vuhv, vhvu, vvuh\}.

A square of type \(vuvv\) is transformed into a square of type \(hhhh\) by rotating the real plane \(90^\circ\). The same rotation transforms a square of type \(huvu\) into a square of type \(vhvh\) if the first vertex does not have integer coordinates; otherwise it is transformed into an \(hhhh\) type square. For this reason the existence of inscribed squares in \(J\) should be proved for types \(hhhh, hhvu, hvvu, hvhv\). But from Lemmas 9 and 10 the types \(hhvu\) and \(hvvu\) are impossible for \(p_1p_2p_3p_4\). For the types \(hhhh\) and \(hvhu\) the result follows from Lemmas 8 and 11.

The following result will be used in Section 6.

**Corollary 13.** Let \(J\) be a simple closed 4-curve. If \(p_1p_2p_3p_4\) is an inscribed square in \(\psi_2(J)\) and \(p_i \notin \mathbb{Z}^2\) for some \(i \in \{1, 2, 3, 4\}\), then \(J\) contains at least two different inscribed squares with integer vertices.

**Proof.** It is immediate from Lemmas 8 and 11.

For 4-cycles this corollary is not true. For instance, the embedding of the 4-cycle \((0, 0), (0, 1), (1, 1), (1, 0)\) has \((0, 0.5), (0.5, 1), (1, 0.5), (0.5, 0)\) as inscribed square, but it only has an inscribed square with integer vertices. This is because the contradiction in the last paragraph of the proof of Lemma 11 cannot be reached for all 4-cycles.

In the results of this section, if “simple closed 4-curve” is replaced by “4-cycle”, then the modified writing will still be true. The only exception is on the last paragraph in the proof of Lemma 11, where only one inscribed square can be guaranteed for 4-cycles. Thus any 4-cycle has an inscribed square with integer vertices, and Conjecture 2 is true.

**Theorem 14.** For every cycle \(C\) in \(\mathbb{Z}^2\), \(\psi_2(C)\) always contains an inscribed square with integer coordinates vertices.

5. **Algorithm to find all inscribed squares in 4-curves**

Algorithmic issues about locating inscribed squares in 4-curves are explored in this section. Let \(C\) be a 4-curve in the digital plane and let \(n\) be a positive integer such that \(|C| = n\).

A first approach consists in analyzing all 4-sets of points on the curve to determine whether they form an inscribed square. So the total number of 4-sets is \(\binom{n}{4}\). Then each 4-set must be tested to decide whether it is a square. This can be done in constant time and the total time complexity is \(\theta(n^4)\). The details are given in Algorithm 1.

**Algorithm 1.**

**Input:** Array of points of \(C\).

**Output:** Number of inscribed squares.

1. \(s \leftarrow 0\) \Comment{counter for the number of inscribed squares}
2. for each set of four points \(a, b, c, d\) of \(C\)
3. \hspace{1em} do if \(a, b, c, d\) are vertices of a square
4. \hspace{2em} then \(s \leftarrow s + 1\)
5. return \(s\)

In a second approach this complexity is reduced by considering the \(\binom{n}{2}\) 2-sets of points on the curve instead of the 4-sets. Each 2-set represents the diagonal of a square, and the other square corners can be found in constant time. After this, the algorithm must decide whether these corners belong to \(C\). It could be done by storing in an array the points in \(C\) sorted lexicographically by its coordinates. Since the membership of a point in the array can be decided in time \(\theta(\log n)\), the total complexity of this approach is \(\theta(n^2 \log n)\). A preprocessing time of order \(\theta(n \log n)\) is needed to sort the array. This second approach finds each inscribed square twice because each square has two diagonals. To avoid this consider only 2-sets which correspond to horizontal diagonals or to diagonals with a positive slope. The details are given in Algorithm 2.

**Algorithm 2.**

**Input:** Array of points of \(C\).

**Output:** Number of inscribed squares.

1. \(s \leftarrow 0\) \Comment{counter for the number of inscribed squares}
2. sort the input array lexicographically by the coordinates of its points
3. for each pair \(a, c\) of distinct points of \(C\)
4. \hspace{1em} do if the segment \(ac\) is horizontal or has positive slope
5. \hspace{2em} then find \(b\) and \(d\) such that \(abcd\) is a square
6. \hspace{2em} do a binary search of \(b\) and \(d\) in the input array
7. \hspace{3em} if \(b\) and \(d\) are in \(C\)
8. \hspace{4em} then \(s \leftarrow s + 1\)
9. return \(s\)

If instead of the array a binary 2-dimensional matrix \(M\) of appropriate dimensions is used such that \(M_{ij} = 1\) if and only if \((i, j)\) belongs to \(C\), then the membership of a point in \(C\) can be decided in constant time and the total time complexity of
the algorithm is $\theta(n^2)$, but extra memory space of size $\theta(m_1 \times m_2)$ is required. Here $m_1$ and $m_2$ are the side lengths of the rectangle with minimal area in the digital plane containing $C$. A preprocessing time of order $\theta(m_1 m_2)$ is required. Since $m_1$ and $m_2$ are lower or equal than $n$, the space and preprocessing time complexity functions are $\theta(n^2)$. The details are given in Algorithm 3.

Algorithm 3.

Input: Array of points of $C$.

Output: Number of inscribed squares.

1. $s \leftarrow 0$  // counter for the number of inscribed squares
2. find the side lengths $m$ and $n$ of the minimum rectangle $R$ containing $C$
3. translate the points of $C$ to move the bottom left corner of $R$ to $(0, 0)$
4. initialize an $m \times n$ array $M$ to all zeros
5. for each point $(a_x, a_y)$ of $C$
6. do $M[a_x][a_y] \leftarrow 1$
7. for each set of two points $a, c$ in $C$
8. do if the segment $ac$ is horizontal or has a positive slope
9. find $b = (b_x, b_y)$ and $d = (d_x, d_y)$ such that $abcd$ is a square
10. if $M[b_x][b_y] = 1$ and $M[d_x][d_y] = 1$
11. then $s \leftarrow s + 1$
12. return $s$

Algorithm 3 finds the inscribed squares in a digital curve in time of order $\theta(n^2)$, but we do not know if it could be improved. One attempt is to use Voronoi diagrams. If a Voronoi diagram of a digital curve is considered as an embedding of a plane graph, a Voronoi vertex represents a point equidistant from three or more points of the curve. So, the center of an inscribed square must correspond to a Voronoi vertex, and a better algorithm would consist in analyzing the Voronoi vertices which are the centers of inscribed squares in time $O(n \log n)$. Unfortunately some intersections of Voronoi edges are hidden by Voronoi faces. It means that a center of an inscribed square could be hidden by a Voronoi region whose points are closer to some part of the curve than to the vertices of the inscribed square. Fig. 8 illustrates this situation.

Let us conclude this section with a further improvement which reduces the running time of Algorithms 2 and 3 by more than half by using a basic property of the digital plane.

It is well known that the integer two-dimensional grid is bipartite. So it is possible to separate the points in the digital plane into two chromatic classes $R$ and $N$. For each pair of 4-adjacent points in $Z^2$ one of them is in $R$ and the other one is in $N$.

Lemma 15. Let $v_1v_2v_3v_4$ be the vertices of a square in the digital plane. Then $v_1$, $v_3$ are in the same chromatic class and the same is true for $v_2$ and $v_4$. The chromatic class of $v_1$ and $v_3$ is not necessarily different from the chromatic class of $v_2$ and $v_4$.

Proof. Let us assume that $v_1 = (a_1, b_1)$, $v_2 = (a_2, b_2)$, $v_3 = (a_3, b_3)$, and $v_4 = (a_4, b_4)$. It follows from the definition of digital square that $v_2 = (a_2, b_2) = (a_1 + b_1 - a_4, a_4 - a_1 + b_1)$. The Manhattan distance $d(v_1, v_2) = |a_2 - a_1| + |b_2 - b_1| = |b_1 - b_4| + |a_4 - a_1| = d(v_1, v_4)$. The points $v_2$ and $v_4$ belong to the same chromatic class because they are equidistant to $v_1$. It can be proved similarly that $v_1$ and $v_3$ belong to the same chromatic class too.

This lemma is applied to generate an additional algorithmic approach. Separate the points in $C$ into two sets: $C_N = N \cap C$ and $C_R = R \cap C$. Both sets have cardinality $\frac{n}{2}$. Then use Algorithm 2 or 3 on $C_N$ and on $C_R$. The total number of diagonals to be tested now is $2^{\binom{n}{2}/2}$. That is a reduction of more than half of them.

6. Curves with a single inscribed square

In the real plane there are infinite families of Jordan curves having more than one inscribed square; take for instance the family of circles. At the opposite extreme, there are infinite families of Jordan curves having a single inscribed square; ellipses with eccentricity greater than zero are examples of curves of this type. For digital topology, the number of inscribed squares contained in a simple closed 4-curve is always finite and it makes sense to ask whether some of these curves have
Fig. 9. Example of a 4-curve containing a single inscribed square (provided by Enrique García-Moreno).

Fig. 10. Example of 4-curves and its immersions in $\mathbb{R}^2$ having each a single inscribed square.

Fig. 11. Examples of curves $J(n)$ defined in Proposition 17. From left to right $n = 2, 3, 4$.

Exactly one inscribed square. The answer is yes; some examples of curves with this property appear in Figs. 9 and 10. The curve at the upper-left corner of Fig. 10 is the minimal simple closed digital 4-curve satisfying this property; it was found with a computer program. Theorem 16 guarantees that the immersions in $\mathbb{R}^2$ of the digital curves of Fig. 10 have a single square too.

**Theorem 16.** If $J$ is a simple closed 4-curve with a single inscribed square, then $\psi_{\mathbb{Z}^2}(J)$ is a Jordan curve in the real plane with a single inscribed square.

**Proof.** Let $S$ be the single inscribed square in $J$. Then $S$ is also an inscribed square in $\psi_{\mathbb{Z}^2}(J)$. Let us assume that $\psi_{\mathbb{Z}^2}(J)$ contains a second inscribed square $T$ different from $S$. Necessarily one of the vertices of $T$ is not in $\mathbb{Z}^2$. From Corollary 13 $J$ must contain two inscribed different squares, but that is impossible and so $\psi_{\mathbb{Z}^2}(J)$ has a single non-degenerate square inscribed. $\square$

Some infinite families of simple closed 4-curves with a single inscribed square have been found.

**Proposition 17.** The simple closed 4-curve $J(n) = (0, 0), (0, 1), \ldots, (0, n), (1, n), (1, n + 1), (2, n + 1), \ldots, (n + 2, n + 1), (n + 2, n), (n + 3, n), (n + 3, n − 1), \ldots, (n + 3, 0), (n + 2, 0), \ldots, (1, 0), (0, 0)$ has a single inscribed square for all $n > 1$. Fig. 11 illustrates $J(n)$ for $n = 2, 3, 4$. 
**Proof** (Sketch of Proof). An inscribed square is $S(n) = (1, 0)(1, n + 1)(n + 2, n + 1)(n + 2, 0)$, it must be proved that it is the only one.

Curve $J(n)$ is constituted by the following 4-paths (see Fig. 12):

- $P_1(n) = (0, 0), (0, 1), \ldots, (0, n)$
- $P_2(n) = (1, n)$
- $P_3(n) = (1, n + 1), (2, n + 1), \ldots, (n + 2, n + 1)$
- $P_4(n) = (n + 2, n)$
- $P_5(n) = (n + 3, n), (n + 3, n - 1), \ldots, (n + 3, 0)$
- $P_6(n) = (n + 2, 0), (n + 1, 0), \ldots, (1, 0)$.

To show that there is no other inscribed square, for every pair of different points $q_1, q_2$ in $\mathbb{Z}^2$, there are exactly two squares in the digital plane containing $q_1q_2$ as an edge: $q_1q_2q_3q_4$ and $q_1q_2q'_3q'_4$ with $q_3 = L(q_1, q_2), q_4 = R(q_2, q_1), q'_3 = R(q_1, q_2)$, and $q'_4 = L(q_2, q_1)$. In Section 4 was introduced the notation $L(p, q)$ (resp. $R(p, q)$) for the point obtained when $p$ is rotated $90^\circ$ (resp. $-90^\circ$) around $q$. Similarly, $L(P, q)$ (resp. $R(P, q)$) will denote the $90^\circ$ (resp. $-90^\circ$) rotations of a $k$-path around a point $q$.

For simplicity $J$, $P$, and $S$ are used instead of $J(n)$, $P_i(n)$ and $S(n)$ respectively.

The proof is long because for every possible pair $q_1, q_2$ where $q_1$ is a point of $P_i$ and $q_2$ is a point of $P_j$, $1 \leq i, j \leq 6$, it should be verified that $q_1q_2q_3q_4$ and $q_1q_2q'_3q'_4$ are not inscribed in $J$, with the only exception of $S$. In other words, it must be proved that the logical condition

$$q_1 \in J \wedge q_2 \in J \wedge [q_1, q_2] \in J \wedge \{q_3 \in J \wedge q_4 \in J \wedge (q_3 \in J \wedge q_4 \not\in J)\}$$

is satisfied only when $q_1q_2$ is an edge of $S$.

As an example, let us consider the case when $q_1$ is in $P_2$ and $q_2$ is in $P_1$.

The point $q_1$ is equal to the point $(1, n)$ and since $q_2$ is a point of $P_1$, then $q_4$ must be in $R(P_1, q_1) = (1 - n, n + 1), (2 - n, n + 1), \ldots, (1, n + 1)$. Only the point $(1, n + 1) = R((0, n), q_1)$ is in $J$ (see Fig. 13). Thus, if $q_4 = (1, n + 1)$, then $q_3 = (0, 0)$ therefore $q_3 = (0, n + 1) \not\in J$.

Now, $q_4$ must be in $L(P_1, q_1) = (n + 1, -n), (n + 1, n + 1), \ldots, (1, n + 1)$. In this case no point is in $J$ (see Fig. 13), that is $q_4 \not\in J$.

The remaining cases can be verified in the same way to conclude that $(1, n + 1)(n + 2, n + 1)(n + 2, 0)(1, 0)$ is the only inscribed square of $J$. □

The following result is an immediate consequence of this proposition and Theorem 16.

**Corollary 18.** There exist infinitely many Jordan curves with a single inscribed square.

**Proof.** Take the embeddings on $\mathbb{R}^2$ of the infinite family of digital curves in Proposition 17. □
The Inscribed Square Conjecture has been open for a century. It is a difficult problem, and it seems that no elementary proof is possible. In this paper, two discrete approaches of this conjecture were developed: one in digital topology and the other in topological graph theory. A complete solution was provided for both approaches.

The existence of two infinite families of 4-digital curves was proved. One with a single inscribed square for each curve, the other one with exactly one curve with n inscribed squares for each positive integer number n. These families were transformed into new examples of Jordan curves in the plane with a given number of inscribed squares.

Finally, an algorithm to find all inscribed squares in a 4-curve with time complexity $O(n^2)$ was developed.

Several questions remain open. We consider relevant the following:

- Are the results in this article useful to solve the original inscribed square conjecture?
- Is it possible to give a full characterization of 4-curves with a single inscribed square?
- Is it possible to find a better algorithm to find inscribed squares in 4-curves?

Acknowledgements

The authors are grateful to the anonymous referees of this article and of the article [16], in which it is based, for their careful reading and valuable comments, to Shalom Eliahou for suggesting that the Inscribed Square Conjecture could be interpreted from a digital topology point of view, and to Enrique García-Moreno for his comments about the Inscribed Square Conjecture and his example of a 4-curve with a single inscribed square.

The first author was partially supported by the Mexican SNI (contract 7008), and the second author was partially supported by Conacyt (Mexico, Scholarship 160959).


References