MATHEMATICS

EXTENDED TOPOLOGY: DOMAIN FINITENESS.

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Introduction

In previous papers we have established certain particulars and some general background for the extended topological system we are studying. These first papers are sufficient to establish that the level of generality is not excessive and that the use of function notation is a real aid to approaching topological concepts.

In this paper we give a demonstration of the unification achieved for a broad class of mathematical existence theorems, many of which seem to have not been specifically noted before. The basic proof for our theorem is naturally simpler than the detailed ones given in the literature since it rests on the reason they hold –– domain finiteness. In view of the known importance of the applications we have taken the liberty of presenting domain finiteness in greater detail than we otherwise should have done.

We may state one basic theorem for applications roughly as follows:

In an extended topology which has only finite minimal convergent sets, the class comprised of the space and of all maximal closed sets excluding points is the unique minimal intersection basis for the class of all closed sets.

As applications, for example, we have unique minimal intersection bases for classes of: 1. subgroups of a group, 2. subfields of a field, 3. ideals in a ring, 4. convex sets in a linear space. However, these applications depend on closures and we have seen that the more general expansive functions are non-trivially a better operating level for extended topology. Hence we present the full scale generality encouraged by our previous published and unpublished work.

Terminology and Definitions

Since we have already reported on several aspects of extended topology in the journal "Nieuw Archief voor Wiskunde" we may expect that the reader will refer to these for certain results which we merely state here. Moreover we often state a result only in one aspect of the duality always present and leave the reader to carry out the dualization.

Let M be the space and let N be its null set. Let $\mathscr{M}$ be the class of all subsets of M (dually, supersets of N). Let F be the family of all functions

f mapping $\mathscr{M}$ into itself. Then the set algebra and inclusion relations for $\mathscr{M}$ induce, in usual fashion, an algebra and order relation in F. The particular functions f_M, f_N, e, c are the *maximum*, *minimum* (*null*), *identity*, and *complement* functions.

The following subfamilies of F are important here. Some of their properties are discussed in [3, 4, 6].

1. The *isotonic* functions $f : X \subseteqq Y$ implies $fX \subseteqq fY$ (or $f(X \cup Y) \supseteqq fX \cup fY$, or $f(X \cap Y) \subseteqq fX \cap fX \cap fY$ for all X, $Y \in \mathscr{M}$).

2. The *limit* functions $f : f \subseteqq c$.

3. The *primitive* functions $f : fX \equiv \bigcup\{fY \cap cY : Y \subseteqq X\}$.

4. *The domain finite* functions $f : fX \equiv \bigcup\{fY : Y \subseteqq X,\ \|Y\| \text{ finite}\}$ where $\|Y\|$ is the cardinal number of Y.

5. The functions f *domain bounded* with *domain bound* n: There exists a minimal integer $n \geqq 0$ such that $fX \equiv \bigcup \{fY : Y \subseteqq X,\ \|Y\| \leqq n\}$.

6. The *expansive* functions $f : f$ is isotonic and *enlarging* ($f \supseteqq e$).

7. The *contractive* functions $f : f$ is isotonic and *shrinking* ($f \subseteqq e$).

8. The *closure* functions $f : f$ is expansive and *idempotent* ($f^2 = f$).

9. The *interior* functions $f : f$ is contractive and idempotent.

Remarks. In [6] it is shown how to obtain from one function $f \in F$ a limit function, a primitive function, an expansive function, a closure function, a contractive function and an interior function. Since we consider an *extended topology* to be *determined* by M, g where g is an expansive function we confine ourselves to starting with an expansive function g. In the immediately following definitions and statements g is an expansive function:

10. The maximal limit function f contained in g is $f = g \cap c$. Then $gX \equiv X \cup \bigcup \{fY : Y \subseteqq X\}$. This function f is called the *g-limit* function.

11. The minimal isotonic function g' containing the g-limit function f is given by

$$g'X \equiv \bigcup \{fY : Y \subseteqq X\} = \bigcup \{gY \cap cY : Y \subseteqq X\}.$$

The function g' is called the *g-primitive* function. It is a primitive function. A point p is called a *primary g-limit point of a set* X provided $p \in g'X$. Note that $g = e \cup g'$. Moreover if v is any primitive function then $e \cup v$ is an expansive function with v as its primitive function.

12. The unique minimal closure function h containing g is called the *g-closure* function. A set X is g-closed (h-closed) if and only if $gX = X$ ($hX = X$). There exists a unique minimal ordinal $\lambda \geqq 0$ such that $h = g^\lambda$.

The operand for iterated composition at a limit ordinal for *enlarging* functions is defined to be the *union* of all preceding compositions -- e.g. $g^{\omega} = \bigcup \{g^n : n < \omega\}$.

13. The function $r = cgc$ is the *g-contractive* function. It is the maximal contractive function contained in $cg'c$.

14. The function $i = chc$ is the *g-interior* function. It is the maximal interior function contained in r or $(cg'c)$ moreover, $i = r^{\lambda}$.

15. The function $d = g'h = h'h$ is called the *g-derived* function ($= h$*-derived* function). A point p is a *g-limit point of a set* X provided $p \in dX$. We note that d is isotonic, $e \cup d = h$ and $d^2 \subseteqq d$.

16. A set X is *g-remote* from p provided $p \in cgX$.

17. A set X is an *r-neighborhood* ($=$*g-neighborhood*) of p provided $p \in rX$. Note that if X is an r-neighborhood of p than cX is a g-remote from p. In [6] this use of the term *neighborhood* in relationship to limit point concepts is justified.

18. Let $\mathscr{R}_p = \{X : p \in cgX\}$. Then a subclass $\mathscr{C}_p$ of $\mathscr{R}_p$ is a *base* for $\mathscr{R}_p$ provided every set in $\mathscr{R}_p$ is a subset of a set in $\mathscr{C}_p$.

19. Let $\mathscr{N}_p = \{X : p \in rX\}$. Then a subclass $\mathscr{B}_p$ of $\mathscr{N}_p$ is a *base* for $\mathscr{N}_p$ provided every set in $\mathscr{N}_p$ is a superset of a set in $\mathscr{B}_p$.

Properties of domain finite functions

Let $D(n)$ be the family of all domain bounded functions with domain bound at most n. Let $D(\omega) = \bigcup \{D(n) : n < \omega\}$ be the family of all domain bounded functions. Let $F(\omega)$ be the family of all domain finite functions.

1. Theorem. 1. *If $f_i \in D(n_i)$ for $i = 1, 2, \ldots, k$ then:*
$\bigcup f_i \in D(max\ [n_1, \ldots, n_k])$
$\bigcap f_i \in D(n_1 + \ldots + n_k)$
$f_1 f_2 \ldots f_k \in D(n_1 n_2 \ldots n_k)$.
Hence $D(\omega)$ is closed with respect to finite union, finite intersection, and finite composition.
2. *If $f_a \in D(n)$ for each a then $\bigcup f_a \in D(n)$. Hence $D(n)$ is closed with respect to arbitrary union.*
3. *$F(\omega)$ is closed with respect to arbitrary union, finite intersection and finite composition.*
4. *$F(\omega)$ is closed with respect to arbitrary composition of well-orderings of its member functions, provided the union of all preceding compositions is used to generate operands at limit ordinals. In particular then if $f \in F(\omega)$, $f^{2^{\omega}}$ is the minimal idempotent isotonic function containing f.*
5. *$F(\omega) = \{\bigcup f_n : f_n \in D(n)\ n < \omega\}$ i.e. $F(\omega)$ is the completion of $D(\omega)$ with respect to countable union.*

Proof. The proofs of 1, 2, 3 are essentially the same as those given for expansive functions in [4]. 4. Since $F(\omega)$ is closed with respect to finite composition and arbitrary union from (3) then $F(\omega)$ is closed with respect to arbitrary composition when the operands at limit ordinals are achieved by unions of all preceding compositions. Now if $f \in F(\omega)$ then $f^{\omega} = \bigcup \{f^n : n < \omega\}$ and since f is isotonic $ff^{\omega} = f(\bigcup f^n) \supseteqq \bigcup ff^n = f^{\omega}$. Hence $ff^{\omega} \supseteqq f^{\omega}$ and $f^{n+1}f^{\omega} \supseteqq f^n f^{\omega}$ so that $f^{2\omega} = \bigcup \{f^n f^{\omega} : n < \omega\}$ is a union of an ascending sequence of domain finite functions. We claim $ff^{2\omega} = f^{2\omega}$. Since $f(f^{2\omega}) \supseteqq f^{2\omega}$ suppose $p \in f(f^{2\omega}X)$. Then there is a finite subset Y of $f^{2\omega}X$ such that $p \in fY$ since f is domain finite. But, since $f^n(f^{\omega}X)$ is an ascending sequence of sets and $\|Y\|$ is finite, for some integer m, $f^m(f^{\omega}X) \supseteqq Y$. Hence $p \in f^{m+1}f^{\omega}X \supseteqq fY$ and $p \in f^{2\omega}X$. Hence $ff^{2\omega} = f^{2\omega}$. It follows that $f^{2\omega}(f^{2\omega}) = f^{2\omega}$ and $f^{2\omega}$ is domain finite Let w be an idempotent isotonic function containing f. Then since $w \supseteqq f$, $w = w^2 \supseteqq wf \supseteqq f^2$ and in general $w \supseteqq f^n$. Hence $w \supseteqq f^{\omega}$. Again $w = w^2 \supseteqq wf^{\omega} \supseteqq ff^{\omega}$ and $w \supseteqq f^n f^{\omega}$ or $w \supseteqq f^{2\omega}$. Hence $f^{2\omega}$ is the minimal idempotent isotonic function containing f. (Note that it is possible that f^n for $n \geqq 1$ be idempotent and contain f. In such a case $f^{2\omega} = f^n$.) 5. Since $F(\omega)$ is closed with respect to arbitrary union and $F(\omega) \supseteqq D(\omega)$ we only need to show that $f \in F(\omega)$ implies $f = \bigcup \{f_n : f_n \in D(n)\}$. Define f_n by $f_n X = \bigcup \{fY : Y \subseteqq X,\ \|Y\| = n\}$. Observe that $f_n = f_N$ is possible. Then $f_n \in D(n)$ and $f = \bigcup f_n$ since f is domain finite. Q.E.D.

2. Theorem. 1. *If $g \in F(\omega)$ and g is expansive then $h = g^{\omega}$ is the minimal closure function containing g.*
Moreover $h \in F(\omega)$.
2. *The family $D(0)$ is comprised of all constant-valued functions $f_Z : f_Z X \equiv Z$.*
3. *The family $D(1)$ is comprised of all universally additive functions f. i.e. $f \in D(1)$ if and only if $f(\bigcup X_a) = \bigcup (fX_a)$ for each nonempty subclass $\{X_a\}$ of $\mathscr{M}$.*
4. *The family $D(1)$ contains all the additive functions in $F(\omega)$ i.e. if f is additive and $f \in F(\omega)$ then $f \in D(1)$.*

Proof. 1. If $g \in F(\omega)$ and g is expansive then $\{g^n\}$ is an increasing sequence of functions in $F(\omega)$. Hence $gg^{\omega} = g^{\omega}$ is an idempotent expansive function i.e. a closure function containing g. But if w is any idempotent isotonic function containing g then w is also enlarging and hence a closure function. Moreover $w \supseteqq g^{\omega} = h$. Hence h is the g-closure function and, being the union of domain finite functions, h is domain finite.

2. Clearly if f_Z is a constant function $f_Z X \equiv f_Z N$ and $f_Z \in D(0)$. On the other hand if $f \in D(0)$ then $fX \equiv fN$ and f is a constant function.

3. Since $X \equiv N \cup \{\{p\} : p \in X\}$, if $f \in D(1)$ then $fX = fN \cup \bigcup \{f\{p\} : p \in X\}$ and f is universally additive. The converse follows directly.

4. Suppose $f \in F(\omega)$, if such an f were additive then $fY = fN \cup \{f\{p\}$:

$p \in Y\}$ for each finite set Y. But then $f \in D(1)$ and f is universally additive. Hence $D(1)$ contains all the additive functions in $F(\omega)$. Q.E.D.

3. Theorem. *Let $f \in F(\omega)$ and define $u \in F$ by $uX \equiv fX - \bigcup \{fY : Y \subseteqq X, \|Y\| < \|X\|\}$.*
The following statements hold:
1. *$uN = fN$ and $u \subseteqq f$.*
2. *$uX \neq N$ implies $\|X\|$ finite.*
3. *$Y \subset X$ implies $uY \cap uX = N$ and $fY \cap uX = N$.*
4. *$fX \equiv \bigcup \{uY : Y \subseteqq X\}$ i.e. f is the minimal isotonic function containing u.*
5. *If $v \in F$ and $fX \equiv \bigcup\{vY : Y \subseteqq X\}$ then $u \subseteqq v$.*
6. *u is a limit function if and only if f is a primitive function.*

Proof 1. $uN = fN$ and $u \subseteqq f$ directly from the definition.

2. Suppose $\|X\|$ is infinite. Then since $f \in F(\omega)$, $fX = \bigcup \{fY : Y \subseteqq X, \|Y\| < \|X\|\}$ and $uX = N$. Hence $uX \neq N$ implies $\|X\|$ finite.

3. Since $\|X\|$ infinite implies $uX = N$ we may suppose $\|X\|$ finite. Then $Y \subset X$ implies $\|Y\| < \|X\|$ and hence $uX \cap fY = N$. But $uY \subseteqq fY$ from (1) and hence $uY \cap uX = N$.

4. Since f is isotonic and $f \supseteqq u$ we have always $fX \supseteqq \bigcup \{uY : Y \subseteqq X\}$. Suppose $p \in fX$. Then there is a finite subset Y_1 of X such that $p \in fY_1$ and now if $p \in uY_1$ we are finished. Otherwise there is a subset Y_2 of Y_1 such that $\|Y_2\| < \|Y_1\|$ and $p \in fY_2$. Continuing, in a finite number of steps we must obtain a subset Y_n of Y_{n-1} such that $p \in uY_n$ and hence $fX = \bigcup \{uY : Y \subseteqq X\}$.

5. Let v be any function as described and suppose contrary to conclusion, $p \in uX$, $p \in cvX$. Then $\|X\|$ is finite from (2) and $p \in fX \supseteqq uX$. Hence for some proper subset Y of X, $p \in vY$. But then $p \in fY$ and from (3) $fY \cap uX = N$ which is a contradiction. Hence $u \subseteqq v$.

6. If u is a limit function then the u-primitive function is f. Hence we need only show that if f is a primitive function u is a limit function. Suppose, contrariwise, $p \in uX \cap X$. Then $\|X\|$ is finite and since $fY \cap uX = N$ if $Y \subset X$ from (3) we must have $p \in fX - X$ from $fX = \bigcup \{fY - Y : Y \subseteqq X\}$. Hence we have $p \in cX$ and $p \in X$ which is a contradiction. Hence u is a limit function if f is a primitive function. Q.E.D.

Remarks. The foregoing Theorem while closely related to our reduction of limit functions [3] differs in that we here start with $f \in F(\omega)$ and hence f is not a limit function unless $f = f_N$. Note that the function u is the minimal function with f as the minimal isotonic function containing it. Part (6) shows that the process to obtain u always gives a minimal limit function generating f when f is a primitive function in $F(\omega)$. The class $\mathscr{C}=\{X : uX\neq N\}$ may be called *the determining* or *generating class* for f. The next theorem is basic in its applicability to subsequent proofs.

4. Theorem. *Let $f\in F(\omega)$ and let $\mathscr{C}$ be any nonempty subclass of $\mathscr{M}$ such that $X_1, X_2\in\mathscr{C}$ implies there is $X_0\in\mathscr{C}$ such that $X_0\supseteqq X_1\cup X_2$. Then $f\{\bigcup X : X\in\mathscr{C}\}=\bigcup\{fX : X\in\mathscr{C}\}$.*

Proof. Since f is isotonic $f(\bigcup X)\supseteqq\bigcup(fX)$. Suppose $p\in f(\bigcup X)$. Then there is a finite set Y, $Y\subseteqq\bigcup X$ such that $p\in fY$. Hence either $Y=N$ or $Y=\{p_1, \ldots, p_k\}$. If $Y=N$ then $p\in fY\subseteqq fX$ for each $X\in\mathscr{C}$ and since $\mathscr{C}$ is not empty we are finished. Otherwise let $X_i\in\mathscr{C}$, $p_i\in X_i$ for $i=1, \ldots, k$. There exists $X_0\in\mathscr{C}$ such that $X_0\supseteqq\bigcup\{X_i : 1\leqq i\leqq k\}\supseteqq Y$. Hence $p\in fX_0\supseteqq fY$ and $f(\bigcup X)=\bigcup(fX)$. Q.E.D.

Remarks. Note that if $\mathscr{C}$ is any non-empty subclass of $\mathscr{M}$ such that $\mathscr{C}$ is closed under finite union the theorem applies and in particular then, if $\mathscr{C}$ is simply ordered by inclusion the condition is satisfied. Of course, if $\mathscr{C}$ contains a maximal set the Theorem is trivially true.

5. Theorem (*converse to Theorem* 4). *Let $f\in F$ such that $f(\bigcup X_\alpha)=$ $=\bigcup(fX_\alpha)$ for $\{X_\alpha\}$ ranging over all non-empty ascending well-orderings of subsets of M. Then $f\in F(\omega)$.*

Proof. First note that $X_2\supseteqq X_1$ implies $fX_2\supseteqq fX_1$ and hence f is isotonic. Next suppose $\|X\|$ is aleph-null. Then $X=\bigcup\{X_n : n<\omega\}$ where $\|X_n\|=n$ and $X_{n+1}\supset X_n$. Hence $fX=\bigcup fX_n$ and f is domain finite *on its countable subsets*. Let $\|X\|$ be now the first cardinal (aleph-one) greater than aleph-null, and let β_1 be the first ordinal of cardinal aleph-one. Then $X=\bigcup\{X_\alpha : \alpha<\beta_1\}$ where $\|X_\alpha\|$ is aleph-null and $X_{\alpha+1}\supset X_\alpha$. Hence $fX=\bigcup fX_\alpha=\bigcup\{fY_\alpha : Y\subseteqq X, \|Y\| \text{ finite}\}$ and f is domain finite on all sets of cardinal at most aleph-one. Thus an induction is indicated which establishes that there exists no first cardinal number such that f is not domain finite on sets of that cardinal number. Hence f is domain finite. Q.E.D.

Remarks. This Theorem is important since the existence theorems which follow depend on Theorem 4 and this shows further restrictions will need to be made to obtain similar results for functions which are domain infinite.

6. Theorem. *Let $f\in F(\omega)$ and let $w=cfc$.*
1. *To each $X\in\mathscr{M}$ there corresponds a non-empty class of maximal supersets Y of X such that $fY=fX$.*
2. *To each set $Z\in\mathscr{M}$ and to each set X such that $fX\subseteqq Z$ there corresponds a non-empty class of maximal supersets Y of X such that*

$fY \subseteqq Z$. If Y_1, Y_2 are distinct sets in this class then $Y_1 \not\subset Y_2$ and $Y_2 \not\subset Y_1$.
3. *To each $X \in \mathscr{M}$ there corresponds a non-empty class of minimal subsets Y of X such that $wY = wX$.*
4. *To each set $Z \in \mathscr{M}$ and each set X such that $wX \supseteqq Z$ there corresponds a non-empty class of minimal subsets Y of X such that $wY \supseteqq Z$.*

Proof. 1. Let $\{X_\alpha\}$ be a complete ascending well-ordering of supersets of X such that $fX_\alpha = fX$. Then with $Y = \bigcup x_\alpha$ by Theorem 4, $fY = fX$ and clearly Y is maximal. The class $\{X_\alpha\}$ is not empty since X is a possible choice for X_1.

2. This proof is the same as (1) except now the ascending supersets X_α of X are taken so that $fX_\alpha \subseteqq Z$. That a maximal set Y such that $fY \subseteqq Z$ must be a maximal set with value fY is obvious since $f(Y \cup \{p\}) \not\subseteqq Z$ for $p \in cY$ by definition of Y.
3.4. These statements are dual to 1, 2. Q.E.D.

7. Theorem. *Let $f \in F(\omega)$ and let $w = cfc$.*
1. *There exists a unique minimal non-empty subclass $\mathscr{C}$ of $\mathscr{M}$ such that to each $X \in \mathscr{M}$ there is a non-empty subclass $\mathscr{C}(X)$ of $\mathscr{C}$ comprised of all supersets of X in $\mathscr{C}$ and $fX = \bigcap \{fY : Y \in \mathscr{C}(X)\}$. The class $\mathscr{C}$ is comprised of M and of all maximal subsets Y_p of M such that $fY_p \subseteqq c\{p\}$ for each $p \in M$.*
2. *The function $v \in F$ defined by*

$$vX = \bigcap \{Y : Y \in \mathscr{C}(X)\}$$

is a closure function and $fv = f$. If f is expansive then $f \supseteqq v \supseteqq e$. If f is a closure function then $f = v$.
3. *To each $Z \in \mathscr{M}$ there corresponds a unique minimal subclass $\mathscr{C}_Z$ of $\mathscr{M}$ with the following property. If $X \in \mathscr{M}$ and $fX \subseteqq Z$ the subclass $\mathscr{C}_Z(X)$ of $\mathscr{C}_Z$ comprised of all supersets of X in $\mathscr{C}_Z$ implies $fX = \bigcap \{fY : Y \in \mathscr{C}_Z(X)\}$. Moreover if $vX = \bigcap \{Y : Y \in \mathscr{C}_Z(X)\}$ then v is a closure relative to $\{Y : fY \subseteqq Z\}$ and $fv = f$ in this class. If f is expansive $f \supseteqq v$ and if f is a closure then $f = v$ on $\{Y : fY \subseteqq Z\}$.*

Proof. We prove (3) since (1) and (2) are less general. The dualization is left to the reader. Let $\mathscr{C}_Z$ be comprised of all maximal sets Y such that $fY \subseteqq Z$ and all maximal sets Y_p such that $fY_p \subseteqq Z - \{p\}$ for each $p \in Z$. Now suppose $X \in \mathscr{M}$, $fX \subseteqq Z$. Let $\mathscr{C}_Z(X)$ be the class of all supersets of X in $\mathscr{C}_Z$. Then $\mathscr{C}_Z(X)$ is not empty since there exists a maximal superset Y_0 of X such that $fY_0 \subseteqq Z$ (Theorem 6) and hence $Y_0 \in \mathscr{C}_Z$. Moreover if $fY_0 \neq fX$ then let $p \in fY_0 - fX \subseteqq Z$ and let Y_p be a maximal superset of X such that $fY_p \subseteqq Z - \{p\}$. Then $fX = fY_0 \cap \bigcap \{fY_p : p \in fY_0 - fX,\} \supseteqq \bigcap \{fY : Y \in \mathscr{C}_Z(X)\} \supseteqq fX$.

Now if $vX = \bigcap \{Y : Y \in \mathscr{C}_Z(X)\}$ then $vX \supseteqq X$, $fvX \supseteqq fX$ but $fvX \subseteqq$

$\subseteqq \bigcap \{fY : Y \in \mathscr{C}_Z(X)\} = fX$. Hence $fv = f$. Clearly $\mathscr{C}_Z(vX) = \mathscr{C}_Z(X)$ and hence $v^2 = v$. Moreover if $X_1 \subseteqq X_2$ and $fX_2 \subseteqq Z$ then $fX_1 \subseteqq Z$ and, since $\mathscr{C}_Z(X_1) \supseteqq \mathscr{C}_Z(X_2)$, we have $vX_1 \subseteqq vX_2$ or v is isotonic. Hence v is a closure relative to $\{Y : fY \subseteqq Z\}$. Now if f is also expansive then $f \supseteqq e$ whence $fX = \bigcap \{fY : Y \in \mathscr{C}_Z(X)\} \supseteqq \bigcap \{Y : Y \in \mathscr{C}_Z(X)\} = vX$ and $f \supseteqq v$. If f is a closure then the sets $Y \in \mathscr{C}_Z$ are all closed sets, i.e., $fY = Y$ and hence $f = v$ in $\{Y : fY \subseteqq Z\}$.

The class $\mathscr{C}_Z$ is minimal. Suppose Y is a maximal set such that $fY \subseteqq Z$ then $\mathscr{C}_Z(Y) = \{Y\}$ necessarily. If Y is not maximal such that $fY \subseteqq Z$ but Y is maximal such $fY \subseteqq Z - \{p\}$ for some $p \in Z$ then $Y_1 \supset Y$, $fY \subseteqq Z$ implies $p \in fY_1$ and hence fY_1 is not the intersection set of any subclass of its supersets from $\{Y : fY \subseteqq Z\}$ unless Y_1 is in that subclass. Hence $\mathscr{C}_Z$ is minimal. Q.E.D.

8. Corollary. *Let g be a domain finite expansive function, let $r = cgc$, $h = g^\omega$ and $i = chc$.*

1. *There exists for each $p \in M$ a unique minimal neighborhood base $\mathscr{B}_p$ for the class $\mathscr{N}_p$ of all r-neighborhoods of p.*
2. *The class $\mathscr{C}_p = \{cX : X \in \mathscr{B}_p\}$ is the unique minimal base for the class $\mathscr{R}_p$ of all sets g-remote from p.*
3. *There exists for each $p \in M$ a unique minimal i-neighborhood base $\mathscr{B}^*_p$ which is comprised of g-open sets.*
4. *The class $\mathscr{C}^*_p = \{cX : X \in \mathscr{B}^*_p\}$ is the class of maximal g-closed sets excluding p.*
5. *Let $\mathscr{B} = \{N\} \cup \{\mathscr{B}_p : p \in M\}$. Let $\mathscr{B}(X)$ be the class of all subsets of X in $\mathscr{B}$. Then $rX = \bigcup \{fY : Y \in \mathscr{B}(X)\}$ and if $r_0X = \bigcup \{Y : Y \in \mathscr{B}(X)\}$ then r_0 is an interior function $rr_0 = r$, and $r \subseteqq r_0$.*
6. *Let $\mathscr{C} = \{M\} \cup \{\mathscr{C}_p : p \in M\}$. Let $\mathscr{C}(X)$ be the class of all supersets of X in $\mathscr{C}$. Then $gX = \bigcap \{gY : Y \in \mathscr{C}(X)\}$ and if $g_0X = \bigcap \{Y : Y \in \mathscr{C}(X)\}$ then g_0 is a closure function, $gg_0 = g$, and $g \supseteqq g_0$.*
7. *Let $\mathscr{B}^* = \{N\} \cup \{\mathscr{B}^*_p : p \in M\}$. Then $\mathscr{B}^*$ is the minimal union basis for the class of all g-open subsets of M.*
8. *Let $\mathscr{C}^* = \{N\} \cup \{\mathscr{C}^*_p : p \in M\}$. Then $\mathscr{C}^*$ is the minimal intersection basis for the class of all g-closed subsets of M.*

Remarks. This corollary is a corollary of the preceding two theorems. For simplicity we did not include the statements relative to subsets Z of M but because of the importance in applications and in revealing the meaning of the topological terminology we have stated both sides of the duality. While parts 3, 4 and 8 may be considered the most striking in giving existence proofs en masse for separately proved theorems in the literature of analysis and algebra, it is the case that the parts 1, 2 and 5, 6 are more general.

9. Corollary. *There exists a unique minimal intersection basis for each of the following classes:*

1. *The subgroups of a group,*
2. *The subsemigroups of a group or semigroup,*
3. *The ideals in a ring (left, right or two-sided),*
4. *The normal subgroups in a group,*
5. *The convex sets in a linear space,*
6. *The linear varieties in a linear space,*
7. *The convex subsets of any subset of a linear space; in particular, the convex subsets of $c\{\phi\}$ ($\phi =$ origin),*
8. *The subfields of a field,*
9. *The subrings of a ring,*
10. *The Boolean subalgebras of a Boolean algebra,*
11. *The sets closed under any singulary, binary, ternary, ..., n-ary operations or under any union of such operations,*
12. *Subloops of a loop,*
13. *Subalgebras of an algebra.*

In addition, if h is the closure function in each of these cases, there exists at least one maximal closed set contained in any subset Z of M if $hN \subseteqq Z$ and a unique minimal intersection basis for the class of all closed subsets of Z.

Remarks. To what extent this corollary contains explicit results which are new we have not attempted to determine. That certain of the results have been obtained explicitly for special cases will be obvious to the reader. Thus, that there exists a maximal subgroup including a proper subgroup and excluding an element is stated as a theorem in [8]. We stated the minimal intersection basis property for convex sets in [7] and J. W. Ellis [11] proved the existence of maximal convex cones. The point here of course is not only that these results and many more are now available but the underlying reason they hold has been uncovered. We hope to study other algebraic systems such as Noetherian rings for their transferable topological content.

10. Theorem. *Let h be a domain finite closure function and let $i = chc$.*

1. *Let Z be a closed subset of M. Let X be a maximal closed subset of $Z - \{p\}$ for $p \in Z$. If Y is any maximal closed superset of X contained in $c\{p\}$, then $X = Y \cap Z$. Hence if $\mathscr{C}$ is the minimal intersection basis for the class of all closed sets, $\mathscr{C}(Z) = \{Z \cap Y : Y \in \mathscr{C}\}$ is the minimal intersection basis for the class of all closed subsets of Z.*
2. *Let Z be an open subset of M and let X be a minimal open superset of $Z \cup \{p\}$ for $p \in cZ$. Then if Y is a minimal open subset of X containing p, $X = Y \cup Z$.*

Proof 1. We have $Z - \{p\} \supseteqq Y \cap Z \supseteqq X$, $h(Y \cap Z) = Y \cap Z \subseteqq Z - \{p\}$. Hence $X = Y \cap Z$ since X is maximal.

2. The proof is dual to the above. Q.E.D.

Remarks. Let f_Z be the constant-valued function with value $Z \in \mathscr{M}$. Then the *restriction* of an expansive function g or its closure function h to Z is determined by $g_0 = f_Z \cap g$ and $h_1 = f_Z \cap h$, respectively, when the domains of g_0 and h_1 are restricted to subsets of Z.

11. Theorem. *If g is an expansive function and h is its closure function then the restrictions g_0 and h_1 of g and h to Z are respectively expansive and closure functions in Z. Moreover if h_0 is the closure function of g_0 in Z then $h_0 \subseteqq h_1$. The function $h_0 = h_1$ if Z is g-closed.*

Proof. The function $g_0 = f_Z \cap g$ in F is isotonic since f_Z and g are and g_0 is enlarging when restricted to subsets of Z. Hence g_0 is expansive in Z. Similarly h_1 is also expansive when restricted to Z and we need only prove it is idempotent. We have $h_1^2 = (f_Z \cap h)\,(f_Z \cap h) = f_Z(f_Z \cap h) \cap \cap\, h(f_Z \cap h) = f_Z \cap h(f_Z \cap h)$. Now $h(f_Z \cap h) \subseteqq hf_Z \cap h^2 = hf_Z \cap h$ but $f_Z \cap hf_Z = f_Z$ and hence $h_1^2 \subseteqq h_1$. However h_1 is enlarging in Z and hence $h_1^2 \supseteqq h_1$ and, finally $h_1^2 = h_1$. Since $h \supseteqq g$, $h_1 \supseteqq g_0$ in Z and $h_1 \supseteqq g_0{}^{\lambda} = h_0$. Now if Z is g-closed and $X \subseteqq Z$ then $g^{\lambda}X \subseteqq Z$ for all $\lambda \geqq 0$ and $g_0{}^{\lambda}X = g^{\lambda}X$. Hence $h_0 = h_1$. Q.E.D.

12. Theorem. *Let g be a domain finite expansive function and let $h = g^{\omega}$. Let g_0 and h_1 be the restrictions of g and h to $Z \in \mathscr{M}$.*
1. *If $X \subseteqq Z$ and X is a maximal set g_0-remote from $p \in Z$ then $X = Z \cap Y$ for each maximal superset Y of X such that Y is g-remote from p.*
2. *If $X \subseteqq Z$ and X is a maximal set h_1-remote from $p \in Z$ then $X = Z \cap Y$ for each maximal superset Y of X h-remote from p.*

Proof. 1. We have $Y \cap Z \supseteqq X$ and $g_0(Y \cap Z) \subseteqq Z \cap gY \subseteqq Z - \{p\}$. Hence $X = Y \cap Z$ since X is maximal such that $g_0X \subseteqq Z - \{p\}$.

2. This is a corollary of 1. Q.E.D.

Remarks. The reader may phrase the above theorem in terms of the dual neighborhoods. This theorem is important since it establishes the minimal intersection generators of values of g_0 and h_1 through the comparable intersection bases for g and h. It will be observed that the relativization g_0 with respect to g does not generally lead to the same "relative" closure h_0 as the relativization h_1 with respect to h. While this feature may be considered awkward it is actually a part of the greater detail possible using expansive functions.

Applications and examples

The use of domain finite functions which are not expansive occurs in analysis of the primitive functions and the derived functions. If g is any expansive function, h is its closure and g' is the g-primitive function then

$d=g'h$ is the g-derived function. Concerning such derived functions d we have the following theorem:

13. Theorem.
1. *If g is any expansive function and h its closure function then $d=h'$ if and only if d is a primitive function.*
2. *If g is also domain finite then d is domain finite and if v is the closure function of Theorem 7 part (2) such that $dv=d$ then $v \supseteqq h$ and $hv=v$.*

Proof. 1. Since h', the h-primitive function, is a primitive function it is only necessary to show that if d is a primitive function then $d=h'$. But, $e \cup d=h$ and hence $d \cap c=h \cap c$. Hence d is a primitive function $dX \equiv \bigcup \{dY \cap cY : Y \subseteqq X\} \equiv h'X$.
2. Since g' and h are domain finite if g is domain finite then $d=g'h$ is domain finite. By definition vX is the intersection of a subclass of maximal supersets Y such that either $Y=M$ or $dY \subseteqq c\{p\}$. But since $hY \supseteqq Y$ and $dhY=dY$ it follows vX is the intersection of a class of g-closed sets and hence vX is g-closed. Hence $v \supseteqq h$ and $hv=v$. Q.E.D.

Example A. Convex hull closure.
If M is a linear space and h is the convex hull closure then $d=h'h=h'$. This follows since the minimal function u of h is a limit function, i.e., $uX \neq N$ implies X is an $m+1$ point set for some $m \geqq 1$ and X is comprised of the vertices of an m-dimensional simplex. Then uX is the relative interior of that simplex and hence $u \subseteqq c$.

Now $v \neq h$ in this case for suppose that Y is the triangular area without vertices. Then $hY=Y$ since Y is convex but vY contains Y *and* its vertices since $dvY=dY=Y$ in this case. In general vY is the convex hull of Y with such "extreme points" adjoined which give no additional *h'-limit* points. Thus the closure function v is not domain finite in this case and it is a rather subtle function, which seems not to have been studied before. Nevertheless, as required by the theory vX is always a convex set. It is the maximal convex set such that $dvX=dX$. The maximal sets Y such that $dY \subseteqq c\{p\}$ for a point p are the complements of semispaces at p and hence vX is the intersection of all such which contain X. When M is 1-dimensional then vX is the *closed* convex hull of X and dvX is the interior of the closed convex hull of X.

Universally additive closures

The universally additive expansive and closure functions (i.e. functions in $D(1)$) have special properties. These functions are important since they include all closures under singulary operations such as symmetry closures, and other orbital closures. While they form topological spaces (which are not T_1-spaces) they have generally been ignored.

14. Theorem. *Let g be a universally additive expansive function and let $h = g^{\omega}$, $r = cgc$, and $i = chc$.*
1. *If $p \in rM$ there exists a unique minimal r-neighborhood of p.*
2. *If $p \in rM$ there exists a unique maximal set g-remote from p. It is the complement of the minimal r-neighborhood of p.*
3. *If $p \in iM$ there exists a unique minimal i-neighborhood of p. It contains the unique minimal r-neighborhood of p.*
4. *The complement of the minimal i-neighborhood of $p \in iM$ is the unique maximal g-closed set excluding p.*

Proof. We note that if g is universally additive then h is also and r and i are universally intersective. Hence the intersection set of all r-neighborhoods of p is an r-neighborhood of p and it is minimal and unique. Likewise a unique minimal i-neighborhood of each $p \in iM$ exists. Since $i \subseteqq r$ if $p \in iM$ and X is the minimal i-neighborhood it follows that $p \in rX = iX = X$ and hence the minimal r-neighborhood of p is contained in X. Q.E.D.

15. Theorem. *Let g be a universally additive expansive function, let $h = g^{\omega}$, $r = cgc$, and $i = chc$.*
1. *If $p \in rM$ then the minimal r-neighborhood of p is $\{q : p \in g\{q\}\}$.*
2. *If $p \in iM$ then the minimal i-neighborhood of p is $\{q : p \in h\{q\}\}$.*

Concluding remarks

We have here begun a study of domain finite systems. The possibilities of detailed development are numerous. Much of current mathematical theory is imbeddable in extended topology. For example, the theory of ideals in a ring has many directly topological features. We propose to make a survey of certain algebraic theories from this standpoint. Since we are developing the theory abstractly it is the case, for example, that an abstract group may be considered as an extended topology. The unification indicated in this paper is but a clue to forthcoming developments.

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