Global Existence of Solutions for a Class of Nonlinear Differential Equations

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Dedicated to Professor Roberto Conti on the occasion of his 80th birthday.

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Abstract—By making use of a special Lyapunov-type function and applying the comparison method due to Conti, we prove global existence of solutions for a general class of nonlinear second-order differential equations that includes, in particular, van der Pol, Rayleigh, and Liénard equations, widely encountered in applications. Relevant examples are discussed. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we are concerned with global existence of solutions for a general nonlinear differential equation

\[(a(t)u')' + q(t)f(u,u') + g(u) = e(t),\]  

where the functions \(q, f, g, e\) are continuous, and \(a\) is continuously differentiable and positive on \(\mathbb{R}\). Equation (1) can be classified as a generalized Liénard equation with external perturbation (forcing term) and its numerous particular cases arise in various applied problems of radio and electrotechnics.
This research has been motivated by the recent papers by Constantin [1] and Souplet [2] dealing with global existence of solutions for the second-order nonlinear differential equations
\[ u'' + f(u') + g(u) = e(t), \]  
and
\[ u'' - f(u') + g(u) = 0, \]
where the functions \( e, f, g : \mathbb{R} \to \mathbb{R} \) are continuous. The reader can find surveys of results on global existence of solutions of the second-order nonlinear differential equations and extensive lists of references in the papers by Constantin [1] and by the present authors [3].

For the convenience of the reader, we state here two results that motivated this research.

**Theorem 1.** (See [1, Theorem 2.4, p. 246].) Suppose that
(i) there exists a function \( w : \mathbb{R}_+ \to \mathbb{R}_+ \), continuous, nondecreasing on \( \mathbb{R}_+ \) such that
\[ |xf(x)| \leq w(x^2), \quad x \in \mathbb{R}, \]
where \( w(r) > 0 \) for \( r \geq \delta > 0 \) and \( \int_{\delta}^{\infty} w(s)^{-1} \, ds = +\infty; \)
(ii) \( 0 \leq xg(x) \) for \( |x| \) large enough.

Then solutions of equation (2) are defined on \( \mathbb{R} \).

Constantin [1] also studied existence in the future of the solutions of equation
\[ (a(t)u')' + q(t)h(u)l(u') = e(t), \]
where \( q(t) > 0 \) is differentiable, and \( l(v) > 0 \) for \( v \in \mathbb{R} \) requiring the function \( H(x) = \int_{0}^{x} h(s) \, ds \) be bounded from below for \( x \in \mathbb{R} \), that is, there exists a positive constant \( C \) such that
\[ H(x) \geq -C, \quad \text{for all } x \in \mathbb{R}. \]

The latter assumption appears to be somewhat restrictive and, as the following result due to Souplet [2] prompts, can be eliminated.

**Theorem 2.** (See [2, Theorem 1.1, p. 187].) Assume that there exists a constant \( K > 0 \) such that
(i) \( f(x)x \geq Kx^2 \) for \( |x| \) large enough;
(ii) \( |g(x)| \leq (K^2/4)|x| \) for \( |x| \) large enough.

Then, for every \( t_0 \in \mathbb{R} \) and every \( u_0 \in \mathbb{R} \), there exists at least a solution of equation (3), global on \( \mathbb{R} \), such that \( u(t_0) = u_0 \).

As it will become clear in the sequel, a global existence result for equation (3) that complements Theorem 2 in the case where \( f(u') \) satisfies the "confinement inequality"
\[ Kx^2 \leq f(x)x \leq Nx^2, \quad \text{for } |x| \text{ large enough}, \]
where \( N > 0 \) and \( K \) are real constants, can be easily deduced from Theorem 3.

### 2. MAIN RESULT

**Theorem 3.** Suppose that
(i) there exist a function \( v : \mathbb{R}_+ \to \mathbb{R}_+ \), continuously differentiable, with bounded on \( \mathbb{R}_+ \) by a constant \( M > 0 \) derivative \( v' \), and a constant \( P > 0 \) such that \( X(x) = v(x^2) + P \int_{0}^{x} g(s) \, ds \geq 0 \) for all \( x \) in \( \mathbb{R} \);
(ii) there exists a function \( w : \mathbb{R}_+ \to \mathbb{R}_+ \), continuous, nondecreasing on \( \mathbb{R}_+ \) such that
\[ |yf(x,y)| \leq X(x) + w(y^2), \quad x, y \in \mathbb{R}, \]
where \( w(r) > 0 \) for \( r \geq \delta > 0 \) and \( \int_{\delta}^{\infty} w(s)^{-1} \, ds = +\infty; \)
(iii) there exists a constant \( Q > 0 \) such that
\[ \frac{a'(t)}{|a(t)|^2} \geq -Q, \quad t \geq 0 \quad \text{and} \quad \frac{a'(t)}{|a(t)|^2} \leq Q, \quad t \leq 0. \]

Then solutions of equation (1) are defined on \( \mathbb{R} \).
PROOF. Noting that equation (1) is equivalent to the system

\[
\begin{align*}
\frac{dx}{dt} &= \frac{1}{a(t)} y, \\
\frac{dy}{dt} &= -q(t)f \left( x, \frac{1}{a(t)} y \right) - g(x) + e(t),
\end{align*}
\]

we shall prove that solutions of system (7) are defined in the future.

Assume, for the sake of contradiction, that (7) has a solution \((x(t), y(t))\), starting from \((x(0), y(0))\), which blows up in finite time; that is, there exists a \(T > 0\) such that

\[
\lim_{t \to T^{-}} \left| y(t) \right| = +\infty.
\]

Introducing a positive definite function \(V(t, x, y)\) by the formula

\[
V(t, x, y) = \frac{x^2}{a(t)} + \frac{2}{P} \int_{0}^{x} g(s) \, ds + x^2 + 1,
\]

we observe that \(V(t, x, y) \to \infty\) as \(x \to \infty\) uniformly in \(t\) for \(t\) in any compact subset. Let us compute the derivative of \(V(t, x, y)\) along the solution \((x(t), y(t))\) of system (7),

\[
\frac{d}{dt} \left[ V(t, x(t), y(t)) \right] = -g \left( x(t), \frac{1}{a(t)} y(t) \right) \frac{1}{a(t)} y(t) \\
+ 2x \frac{1}{a(t)} y(t) - 2 \frac{1}{a(t)} y(t) f \left( x(t), \frac{1}{a(t)} y(t) \right) + 2 \frac{1}{a(t)} y(t) e(t(t)).
\]

Using assumption (6), Condition (ii) of the theorem, and elementary inequalities, we obtain

\[
\begin{align*}
\frac{d}{dt} \left[ V(t, x(t), y(t)) \right] \leq & Q \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x(t), y(t)) + \frac{2}{P} M + 1 \left( 1 + \frac{1}{a(t)} \right) V(t, x(t), y(t)) \\
+ & 2|q(t)| \left[ w \left( \frac{y^2}{a(t)} \right) + \frac{1}{a(t)} y(t) + P \int_{0}^{x} g(s) \, ds \right] + \frac{y^2 + 1}{a(t)} |e(t)| \\
\leq & Q \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x(t), y(t)) + \frac{2}{P} M + 1 \left( 1 + \frac{1}{a(t)} \right) V(t, x(t), y(t)) \\
+ & 2|q(t)| w \left( \frac{1}{a(t)} V(t, x(t), y(t)) + \frac{1}{2} V(t, x(t), y(t)) \right) \\
+ & |e(t)| \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x(t), y(t)) \\
\leq & Q \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x(t), y(t)) + \frac{2}{P} M + 1 \left( 1 + \frac{1}{a(t)} \right) V(t, x(t), y(t)) \\
+ & 2|q(t)| w \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x(t), y(t)) \\
+ & \left( Q + \frac{2}{P} M + 1 + P|q(t)| + |e(t)| \right) \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x(t), y(t)).
\end{align*}
\]
Evaluating the derivative of the function \((1 + a(t) + 1/a(t))V(t, x(t), y(t))\) along the solution \((x(t), y(t))\) of system (7), we obtain

\[
\frac{d}{dt} \left[ (1 + a(t) + \frac{1}{a(t)}) V(t, x, y) \right] = a'(t) \left( 1 + \frac{1}{a^2(t)} \right) V(t, x, y) + \left( 1 + a(t) + \frac{1}{a(t)} \right) \frac{d}{dt} [V(t, x, y)]
\]

\[
\leq |a'(t)| \left| 1 + \frac{1}{a^2(t)} \right| V(t, x, y) + \left( 1 + a(t) + \frac{1}{a(t)} \right) \frac{d}{dt} [V(t, x, y)]
\]

\[
+ 2 \left( 1 + a(t) + \frac{1}{a(t)} \right) |q(t)| \omega \left( \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x, y) \right)
\]

\[
+ \left( \frac{2}{P} M + 1 + P|q(t)| + |e(t)| \right) \left( \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x, y) \right)
\]

\[
\leq 2 \left( 1 + a(t) + \frac{1}{a(t)} \right) |q(t)| \omega \left( \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x, y) \right)
\]

\[
+ \left( 1 + a(t) + \frac{1}{a(t)} \right) \left( \frac{Q + 2}{P} M + 1 + P|q(t)| + |e(t)| \right)
\]

\[
+ |a'(t)| \left| 1 + \frac{1}{a^2(t)} \right| \left( \left( 1 + a(t) + \frac{1}{a(t)} \right)^{-2} \right) \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x, y)
\]

\[
= \Phi(t) \omega \left( \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x, y) \right)
\]

\[
+ \Psi(t) \left[ \left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x, y) \right],
\]

where the functions \(\Phi(t), \Psi(t)\) are continuous and nonnegative on \(\mathbb{R}_+\).

According to Conti's theorem (see [4,5]), solutions of the differential equation

\[
r' = r(t) u(r) + \psi(t), \quad t \geq 0,
\]

are defined in the future. Let \(r_{\max}(t)\) be the maximal solution of (8) satisfying the initial condition

\[
r(0) = \left( 1 + a(0) + \frac{1}{a(0)} \right) V(0, x(0), y(0)).
\]

Since \(r_{\max}(t)\) exists for all \(t \geq 0\), it is bounded from above on \([0,T]\) by a positive constant \(K\). Then, by [6, Theorem 5.10, p. 80], it follows that

\[
\left( 1 + a(t) + \frac{1}{a(t)} \right) V(t, x(t), y(t)) \leq r_{\max}(t) \leq K, \quad t \in [0,T].
\]

Using the fact that \((a(t))^{-1}y^2 + x^2 \leq V(t, x, y)\) and elementary inequality \(1 + a(t) + (a(t))^{-1} \geq 3\), inequality (9) implies that

\[
|y(t)| \leq \sqrt{\frac{Ka(t)}{3}} \quad \text{and} \quad |x(t)| \leq \sqrt{\frac{K}{3}}, \quad \text{for } t \in [0,T],
\]

which contradicts our assumption since \(a(t)\) is bounded on \([0,T]\). Therefore, solutions of (1) are defined in the future.

In order to prove that solutions of (1) are also defined in the past, we reverse the time in order to reduce this problem to the problem of existence in the future of the solutions to the system

\[
\frac{dx}{ds} = \frac{1}{a(-s)} y,
\]

\[
\frac{dy}{ds} = g(-s) f (x, \frac{1}{a(-s)} y) - g(x) + e(-s),
\]

(10)
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where \( s = -t \). With a similar argument, one can show that solutions of system (10) are defined in the future, and consequently, solutions of system (1) are defined in the past. This completes the proof.

3. DISCUSSION

REMARK 1. We note first that Theorem 3 complements Souplet's result (Theorem 2) if one makes use of (5) and considers equation (1) with \( a(t) = q(t) = 1, f(x, y) = -f(y), e(t) = 0, v(r) = r, w(r) = Nr, \) and \( P = 8/K^2 \).

REMARK 2. Clearly, (5) implies that

\[
K|x| \leq |f(x)| \leq N|x|, \quad \text{for } |x| \text{ large enough. (11)}
\]

At first glance, this inequality seems very restrictive. However, considering two differential equations

\[
\begin{align*}
\frac{d^2 y}{dz^2} - y &= 0 \quad \text{(12)} \\
\frac{d^2 y}{dz^2} - (\frac{dy}{dz})^{1+\varepsilon} &= 0, \quad \text{(13)}
\end{align*}
\]

where \( \varepsilon \) is a small positive number, we observe that equation (12) has a two-parametric family of solutions

\[
y(z) = C_1 + C_2 e^z
\]

defined for all real \( z \), while equation (13) has a two-parametric family of solutions

\[
y(z) = C_1 - \frac{\varepsilon^2}{\varepsilon - 1} (C_2 - x)^{(\varepsilon - 1)/\varepsilon}
\]

blowing up in finite time. This example prompts how sharp condition (11) is.

To conclude the paper, we comment on the generality of Hypotheses (i)-(iii) in Theorem 3 and give two relevant examples. Hypothesis (i) of Theorem 3 allows relaxation of the standard sign property of \( g(u) \), that is, Assumption (ii) of Theorem 1.

Hypothesis (iii) of Theorem 3 allows the derivative \( u'(t) \) to take on negative values which is excluded, for instance, by the conditions of the theorem due to [1, Theorem 2.8, p. 256].

EXAMPLE 4. Consider the differential equation

\[
\frac{d^2 u}{dt^2} + 2u' + |u| = 0, \quad t \in \mathbb{R}. \quad \text{(14)}
\]

Here \( a(t) = q(t) = 1, f(x, y) = y, g(z) = |z|, e(t) = 0, v(r) = w(r) = r, \) and \( P = 1 \). Applying Theorem 3, we conclude that solutions of equation (14) are defined on \( \mathbb{R} \). We stress that results due to Constantin [1, Theorems 2.5-2.7] do not apply to this equation since they require the function \( G(z) = \int_0^z g(s) \, ds \) to satisfy assumption (4).

EXAMPLE 5. Consider the differential equation

\[

t^2 + 1) u' - \inf(u, 0) \sin u' + u^3 = 0, \quad t \in \mathbb{R}. \quad \text{(15)}
\]

Here \( a(t) = t^2 + 1, q(t) = 1, f(x, y) = -\inf(x, 0) \sin y, g(x) = x^2, v(r) = w(r) = r, \) and \( P = Q = 1 \). Application of Theorem 3 yields global existence of solutions, while theorems obtained by Constantin [1, Theorems 2.8, 2.9] cannot be applied to equation (15) since they require the function \( II(u) = \int_0^u h(s) \, ds \), where \( h(x) = -\inf(x, 0) \), to satisfy assumption (4).
REFERENCES


