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Cancellation of projective modules over regular rings with comparability

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Abstract

This paper answers, in the negative, Open Problem 4 in Goodearl's book on von Neumann regular rings: a directly finite regular ring R with s-comparability is constructed which is not unit-regular. It is shown, however, that the behaviour of these rings R is still quite good. They have cancellation of small projective modules and, in particular, their stable range is at most 2.

Keywords: Von Neumann regular ring; s-comparability; Unit-regular ring; Stable range; Hermite ring

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0. Introduction

Generally speaking, if a (von Neumann) regular ring R has some form of comparability for its principal right ideals, then the ring has nice properties. Some model behaviour, for instance, occurs as a consequence of full comparability (or 1-comparability), that is when for any $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$. For then direct finiteness for such R (one-sided inverses are two-sided) implies unit-regularity, equivalently, the class $FP(R)$ of finitely generated projective right R-modules has cancellation: for all $A, B, C \in FP(R)$

$$A \oplus C \cong B \oplus C \Longrightarrow A \cong B.$$

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A weaker form of comparability, but still quite strong, is s-comparability for some positive integer s: either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$ for all $x, y \in R$. Open Problem 4 in [5] asks whether direct finiteness still implies unit-regularity for regular rings with s-comparability. Surprisingly, the answer is "no", as we show with the construction of a counterexample in Section 3. Not only does this example have 2-comparability, it has "almost comparability" in the sense that for any $x, y \in R$, either $xR \lesssim yR \oplus zR$ for all nonzero $z \in R$, or $yR \lesssim xR \oplus zR$ for all nonzero $z \in R$. Yet $FP(R)$ still fails cancellation!.

What, then, can be salvaged in the way of cancellation properties in $FP(R)$ for a directly finite regular ring R with s-comparability? Quite a lot, as it turns out. We show in Section 4 that R has the following cancellation of "small projectives" (Theorem 4.6): for all $A, B, C \in FP(R)$

$$A \oplus C \cong B \oplus C \quad \text{and} \quad C \lesssim nA \text{ for some } n \in \mathbb{N} \Longrightarrow A \cong B.$$

Two consequences of this theorem are worthy of note. Firstly, it implies R is stably finite (all matrix rings $M_n(R)$ are directly finite). Secondly, it implies that the endomorphism ring $\mathrm{End}_R(A)$ of any $A \in FP(R)$ has stable range at most 2 (in fact it is a Hermite ring). Thus, although R need not have stable range 1 (as Example 3.2 shows), R and all its corner rings do have the next best thing. (In the case where R is also simple, it is known [15, Corollary 2] that R does have stable range 1.)

It turns out (Proposition 2.6) that for any directly finite regular ring R with s-comparability, all the factor rings of R are also directly finite. Therefore, Example 3.2 (and indeed any counterexample to Problem 4) also provides a negative answer to the second part of Open Problem 3. For it gives a regular ring which is not unit-regular, but all its factor rings are directly finite.

To help put the s-comparability condition in perspective, some comments on its origin, recent developments, and examples of this condition may be appropriate here. Historically, the s-comparability condition was first formally introduced in the mid-seventies by Handelman [9] and Goodearl and Handelman [7] to characterize uniqueness of rank functions on certain simple regular rings. An account of this can also be found in [5, Ch. 18]. Some of these results have been further refined just recently (1993) by Ara et al. [2]. For instance, [2, Corollary 4.5] shows that, among the directly finite simple regular rings R, those which satisfy s-comparability for some $s \geq 1$ are precisely those for which R has a unique rank function and $FP(R)$ is strictly unperforated (that is, for $A, B \in FP(R)$, if nA is isomorphic to a proper submodule of nB for some $n \in \mathbb{N}$, then A is isomorphic to a proper submodule of B). In the non-simple case, if R is a regular ring with s-comparability for some $s \geq 1$, and if R has some nonzero factor ring which is directly finite, then R has a unique pseudo-rank function. This follows from [2, Theorem 3.5] and Corollary 4.7(1) of our present paper.

Examples of regular rings with 1-comparability are well known, and include the important class of all prime, regular, right self-injective rings [5, Ch. 8]. Goodearl and Handelman observed in the early seventies that regular rings with s-comparability for $s > 1$ also occur naturally, often as ultramatricial algebras over a field F. One instructive example is to take the direct limit $R = \varinjlim R_n$, where $R_n = M_{3^n}(F) \times M_{3^n}(F)$ for $n = 0, 1, 2, \ldots$ and where the maps $R_n \longrightarrow R_{n+1}$ are given by

$$(x,y) \mapsto \left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix} \right).$$

Then R is a simple unit-regular ring with 2-comparability but not 1-comparability. See [5, Examples 8.7 and 18.19]. (We will see later, in Theorem 2.8, that a regular ring with s-comparability for $s > 1$ always has 2-comparability; in fact "$(1+\varepsilon)$-comparability".)

Our paper is in four sections. Section 1 collects some preliminary results, mostly known. Section 2 develops some general properties of regular rings with s-comparability (not necessarily directly finite), including the properties that s-comparability is inherited by finitely generated projective modules and matrix rings, and that direct finiteness is inherited by factor rings. The principal construction of this paper, the counterexample to Open Problem 4, is described in Section 3. Finally, Section 4 establishes the result (Theorem 4.6) on cancellation of small projectives, and examines its consequences.

1. Preliminaries

All rings considered here are associative with 1. All modules will be unital *right* modules, if the contrary is not specified. Also, an ideal will always mean a *two-sided* ideal.

A ring R is *unit-regular* if each $x \in R$ can be written as $x = xux$ for some unit $u \in R$. We refer the reader to [5] for the general theory of von Neumann regular rings. For a ring R, we use $L_2(R)$ to denote the lattice of ideals of R, and $r(a)$ (respectively $\ell(a)$) to denote the right (respectively left) annihilator of an element $a \in R$.

Recall that a ring R satisfies the *n-stable range condition* (for a given positive integer n) if whenever $a_1, \ldots, a_{n+1} \in R$ with $a_1R + \cdots + a_{n+1}R = R$, there exist elements $b_1, \ldots, b_n \in R$ such that

$$(a_1 + a_{n+1}b_1)R + \cdots + (a_n + a_{n+1}b_n)R = R.$$

If n is the least positive integer such that R satisfies the n-stable range condition, then R is said to have *stable range n*, and we write $sr(R) = n$. It is well known that a regular ring has stable range one if and only if it is unit-regular [5, Proposition 4.12]. The reader is referred to [16] for the basic properties of the stable range and to [17, 13, 14] for the connections between cancellation properties of modules and the stable range of their endomorphism rings.

Recall that a ring R is said to be *directly finite* if $xy = 1$ implies $yx = 1$, for $x, y \in R$. We say that R is *stably finite* if $M_n(R)$ is directly finite for all $n \geq 1$.

Let I be an ideal of a ring R. We denote by $FP(I)$ the class of all finitely generated projective R-modules P such that $PI = P$. Given R-modules A and B, we write $A \lesssim B$ (respectively $A \prec B$) to mean that A is isomorphic to a submodule of B (respectively to a *proper* submodule of B). If $A, B \in FP(R)$ and R is regular, then by [5, Theorem 1.11], $A \lesssim B$ (respectively $A \prec B$) if and only if A is isomorphic to a direct summand (respectively proper direct summand) of B.

The following lemma was obtained independently in [10, Lemma 3.1] and [1, Lemma 3.3]. It requires no assumption of comparability of any form–and therein lies its usefulness, because it often enables one to modify arguments that, at first glance, would appear to require full comparability.

Lemma 1.1. *Let A and B be finitely generated projective modules over any regular ring R. If $A \lesssim kB$ for some positive integer k, then there is a decomposition $A = A_1 \oplus \cdots \oplus A_k$ in which $A_1 \lesssim A_2 \lesssim \cdots \lesssim A_k \lesssim B$.*

In 1990, K.R. Goodearl gave (in a private communication) a nice refinement of the above lemma, which is particularly useful when dealing with s-comparability problems. We are grateful for his permission to reproduce the result here. Goodearl's lemma was later discovered independently [18, Lemma 1.9] in the setting of refinement positively ordered monoids (equivalently abelian semigroups with Riesz decomposition). The reader will notice that Goodearl's proof also only uses these properties of $FP(R)$.

Lemma 1.2 (K.R. Goodearl). *Let A, B, C be finitely generated projective modules over any regular ring R. If $A \oplus B \cong kC$ for some positive integer k, then there is a decomposition $C = C_0 \oplus C_1 \oplus \cdots \oplus C_k$ such that*

$$A \cong C_1 \oplus 2C_2 \oplus \cdots \oplus kC_k$$

and

$$B \cong kC_0 \oplus (k-1)C_1 \oplus \cdots \oplus C_{k-1}.$$

Proof. We proceed by induction on k, the case $k = 1$ being trivial. Now assume $A \oplus B \cong kC$ for some $k > 1$. Write $A = U \oplus V$ and $B = W \oplus X$ with $U \oplus W \cong (k-1)C$ and $V \oplus X \cong C$. By induction there is a decomposition $C = D_0 \oplus D_1 \oplus \cdots \oplus D_{k-1}$ such that $U \cong D_1 \oplus 2D_2 \oplus \cdots \oplus (k-1)D_{k-1}$ and $W \cong (k-1)D_0 \oplus (k-2)D_1 \oplus \cdots \oplus D_{k-2}$. Then $D_0 \oplus \cdots \oplus D_{k-1} = C \cong V \oplus X$, so we can decompose each D_i as $D_i = V_i \oplus X_i$ with $V_0 \oplus \cdots \oplus V_{k-1} \cong V$ and $X_0 \oplus \cdots \oplus X_{k-1} \cong X$.

Now let $C_0 = X_0$, $C_i = V_{i-1} \oplus X_i$ for $i = 1, \ldots, k-1$, and $C_k = V_{k-1}$. Then,

$$\begin{aligned}
&C_1 \oplus 2C_2 \oplus \cdots \oplus kC_k \\
&\quad = (V_0 \oplus X_1) \oplus 2(V_1 \oplus X_2) \oplus \cdots \oplus (k-1)(V_{k-2} \oplus X_{k-1}) \oplus kV_{k-1} \\
&\quad \cong (X_1 \oplus V_1) \oplus 2(X_2 \oplus V_2) \oplus \cdots \\
&\qquad \oplus (k-1)(X_{k-1} \oplus V_{k-1}) \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_{k-1} \\
&\quad \cong D_1 \oplus 2D_2 \oplus \cdots \oplus (k-1)D_{k-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_{k-1} \\
&\quad \cong U \oplus V \cong A
\end{aligned}$$

and

$$\begin{aligned}
&kC_0 \oplus (k-1)C_1 \oplus \cdots \oplus C_{k-1} \\
&\quad = kX_0 \oplus (k-1)(V_0 \oplus X_1) \oplus \cdots \oplus (V_{k-2} \oplus X_{k-1}) \\
&\quad \cong (k-1)(X_0 \oplus V_0) \oplus (k-2)(X_1 \oplus V_1) \oplus \cdots \\
&\qquad \oplus (X_{k-2} \oplus V_{k-2}) \oplus X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1} \\
&\quad \cong (k-1)D_0 \oplus (k-2)D_1 \oplus \cdots \oplus D_{k-2} \oplus X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1} \\
&\quad \cong W \oplus X \cong B.
\end{aligned}$$

Hence the induction works. □

In the case of *simple* regular rings, it was shown in [15] that Open Problem 4 has indeed a positive answer. For convenience we restate the result here, because we shall need it later. It enables us to deduce that for any directly finite regular ring with s-comparability, its (unique) simple factor is unit-regular (Corollary 2.7). Also, if such a ring has a minimal ideal M, then M is unit-regular (Proposition 2.4(1)).

Theorem 1.3 (O'Meara [15, Corollary 2]). *Let R be a directly finite simple regular ring which satisfies s-comparability for some $s > 0$. Then R is unit-regular.*

An alternative approach to the proof of Theorem 1.3 was given in [1]. In Section 4, we will expand some of the techniques introduced in [1] to get stable range at most 2 in the non-simple case (and even stronger results). We recall the following key concepts introduced in [1]. For $A, B \in FP(R)$, A is *almost subisomorphic* to B, written $A \lesssim_a B$, if $A \prec B \oplus C$ for all nonzero $C \in FP(R)$. If $A \lesssim_a B$ and $B \lesssim_a A$, then A is said to be *almost isomorphic* to B, written $A \cong_a B$. A regular ring R is said to satisfy *almost comparability* if for all $x, y \in R$, either $xR \lesssim_a yR$ or $yR \lesssim_a xR$. The following important connection between s-comparability and almost comparability was established in [2].

Theorem 1.4 (Ara et al. [2, Corollary 4.5]). *For simple regular rings, s-comparability for some $s > 0$ is equivalent to the ring satisfying almost comparability.*

In the non-simple case, almost comparability gives 2-comparability but not conversely; see Example 4.11.

2. General properties of s-comparability

In this section we establish some general properties of a regular ring R which has s-comparability for some $s > 0$. Many of these properties do not require R to be directly finite. Lemma 2.2, Proposition 2.3(a) and Proposition 2.6 were first proved by

K.R. Goodearl (private communication), and we thank him for allowing us to use the results here.

One property that we shall use frequently, and implicitly, is that the lattice $L_2(R)$ of two-sided ideals of R is totally ordered. The proof is the same as for the 1-comparability case [5, Proposition 8.5]: if $J, K \in L_2(R)$ and $J \nsubseteq K$, $K \nsubseteq J$, then choose $x \in J \setminus K$, $y \in K \setminus J$. Then $x \notin RyR$ implies $xR \not\lesssim s(yR)$, while $y \notin RxR$ implies $yR \not\lesssim s(xR)$. This is not possible.

In particular, therefore, a regular ring with s-comparability has a unique maximal ideal.

As with 1-comparability, s-comparability is inherited by finitely generated projective modules and by finite matrix rings:

Proposition 2.1. *Let R be a regular ring with s-comparability. Then:*

(1) The finitely generated projective R-modules also satisfy s- comparability.

(2) For any $A \in FP(R)$, the endomorphism ring $\mathrm{End}_R(A)$ satisfies s-comparability. In particular, all the matrix rings $M_n(R)$ satisfy s-comparability.

Proof. (1) We must show that for $A, B \in FP(R)$, either $A \lesssim sB$ or $B \lesssim sA$. This can be done by induction on n where $A, B \lesssim nR$ (based on the proof in [5, 8.2] for 1-comparability). So assume the result holds for $n-1$, and suppose $A, B \lesssim nR$. Write $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where $A_i, B_i \lesssim (n-1)R$ for $i = 1,2$. By the induction hypothesis either $A_1 \lesssim sB_1$ or $B_1 \lesssim sA_1$, and either $A_2 \lesssim sB_2$ or $B_2 \lesssim sA_2$.

If $A_1 \lesssim sB_1$ and $A_2 \lesssim sB_2$, then $A = A_1 \oplus A_2 \lesssim sB_1 \oplus sB_2 = sB$. Similarly, if $B_1 \lesssim sA_1$ and $B_2 \lesssim sA_2$, then $B \lesssim sA$. Therefore it is sufficient to consider the case where, say, $A_1 \lesssim sB_1$ and $B_2 \lesssim sA_2$. By Lemma 1.1 there is a direct summand V of A_1 such that $V \lesssim B_1$ and $A_1 \lesssim sV$, and there is a direct summand W of B_2 such that $W \lesssim A_2$ and $B_2 \lesssim sW$. Write $B_1 \cong V \oplus C$ and $A_2 \cong W \oplus D$, and notice that $C, D \lesssim (n-1)R$. Using the induction hypothesis once more, we get either $C \lesssim sD$ or $D \lesssim sC$. If $C \lesssim sD$, then

$$\begin{aligned} B = B_1 \oplus B_2 &\cong V \oplus C \oplus B_2 \\ &\lesssim V \oplus sD \oplus sW \cong V \oplus sA_2 \\ &\lesssim sA_1 \oplus sA_2 = sA \end{aligned}$$

so $B \lesssim sA$. Similarly, $D \lesssim sC$ yields $A \lesssim sB$.

(2) Let $T = \mathrm{End}_R(A)$. Observe that for $x, y \in T$, if say $xA \lesssim s(yA)$ then $xT \lesssim s(yT)$. Hence (2) follows immediately from (1). □

For any regular ring R with s-comparability, we know that its ideals form a chain, and so any two ideals are comparable with respect to inclusion. The next proposition (Proposition 2.3) shows that if we have strict inclusion of two principal ideals, say $RxR < RyR$, then we get strict subisomorphism of the corresponding principal one-sided ideals. Therefore, the only lack of full comparability of two principal right ideals

xR and yR is when the ideals RxR and RyR are at the same level. In fact the result can be extended to finitely generated projective modules via their trace ideals. For an R-module A, its *trace ideal* is $\operatorname{tr}(A) = \sum f(A)$ where f ranges over all R-homomorphisms from A to R. This is indeed an ideal of R and, in case R is regular and $A \in FP(R)$, there is an idempotent $e \in R$ such that $\operatorname{tr}(A) = ReR$ (because A is isomorphic to a finite direct sum of principal right ideals of R). For a regular ring R, the trace ideal of $A \in FP(R)$ is also characterized as the smallest ideal I of R for which $AI = A$. Moreover, for any $A, B \in FP(R)$, where R is regular,

$$\operatorname{tr}(A) \subseteq \operatorname{tr}(B) \quad \text{iff} \quad A \lesssim kB \text{ for some } k \in \mathbb{N}$$

(see [5, 2.10 and 2.23]). The following lemma will help us make the above connections.

Lemma 2.2 (K.R. Goodearl). *Let R be a regular ring with s-comparability, and let $A, B \in FP(R)$. If $(s+1)A \lesssim 2B$, then $A \lesssim B$.*

Proof. From $(s+1)A \lesssim 2B$, we can obtain $(s+1)A \cong C \oplus D$ for some finitely generated projective R-modules $C, D \subseteq B$. By Lemma 1.2 there is a decomposition $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{s+1}$ such that $C \cong A_1 \oplus 2A_2 \oplus \cdots \oplus (s+1)A_{s+1}$ and $D \cong (s+1)A_0 \oplus sA_1 \oplus \cdots \oplus A_s$. Using s-comparability and Proposition 2.1(1), we have either $A_0 \lesssim sA_{s+1}$ or $A_{s+1} \lesssim sA_0$. If $A_0 \lesssim sA_{s+1}$, then

$$\begin{aligned} A &= A_0 \oplus \cdots \oplus A_{s+1} \lesssim A_1 \oplus A_2 \oplus \cdots \oplus (s+1)A_{s+1} \\ &\lesssim A_1 \oplus 2A_2 \oplus \cdots \oplus (s+1)A_{s+1} \cong C \lesssim B \end{aligned}$$

so $A \lesssim B$. On the other hand, if $A_{s+1} \lesssim sA_0$, then $A \lesssim (s+1)A_0 \oplus A_1 \oplus \cdots \oplus A_s \lesssim D \lesssim B$ and again $A \lesssim B$. □

Proposition 2.3. *Let R be a regular ring satisfying s-comparability.*

(a) *If $x, y \in R$ with $RxR \subsetneq RyR$, then $xR \prec yR$.*

(b) *If $A, B \in FP(R)$ with $\operatorname{tr}(A) \subsetneq \operatorname{tr}(B)$, then $A \prec B$.*

(c) *Let N be a proper ideal of R and let $A, B \in FP(R)$. If $A/AN \prec B/BN$, then $A \prec B$.*

Proof. (a) Since $yR \not\subseteq RxR$, we cannot have $yR \lesssim s(s+1)^s(xR)$. Therefore, because s-comparability holds in $FP(R)$ (see Proposition 2.1(1)), we must have $(s+1)^s(xR) \lesssim s(yR)$ and so $(s+1)^s(xR) \lesssim 2^s(yR)$. By Lemma 2.2, $xR \lesssim yR$, whence $xR \prec yR$ because $yR \not\subseteq RxR$.

(b) Notice that $\operatorname{tr}(B) = \operatorname{tr}(C)$ for some principal right ideal C of R with $C \lesssim B$. Therefore, it suffices to argue by induction on k that whenever $A \lesssim kR$ and $\operatorname{tr}(A) \subsetneq \operatorname{tr}(C)$ for some $C \lesssim R$, then $A \prec C$. Part (a) is precisely the case $k = 1$. For a general k, write $A = A_1 \oplus A_2$ with $A_1 \lesssim R$ and $A_2 \lesssim (k-1)R$. We have $A_1 \lesssim C$ by (a) because $\operatorname{tr}(A_1) \subseteq \operatorname{tr}(A) \subseteq \operatorname{tr}(C)$. Now write $C = C_1 \oplus C_2$ with $A_1 \cong C_1$. Observe that $\operatorname{tr}(A_2) \subsetneq \operatorname{tr}(C_2)$ because $\operatorname{tr}(A) \subsetneq \operatorname{tr}(C)$. By induction $A_2 \prec C_2$. Hence $A = A_1 \oplus A_2 \prec C_1 \oplus C_2 = C$, giving $A \prec C$.

(c) By applying [5, Proposition 2.20], we obtain decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, with $A_1 \cong B_1$ and $A_2 = A_2N$ and $B_2 \neq B_2N$. Since $B_2 \neq B_2N$, its trace ideal cannot be contained in N. By comparability of ideals, $N \subsetneq \mathrm{tr}(B_2)$ and, so, $\mathrm{tr}(A_2) \subseteq N \subsetneq \mathrm{tr}(B_2)$. Thus $A_2 \prec B_2$ by (b) and, so, $A = A_1 \oplus A_2 \prec B_1 \oplus B_2 = B$. □

Proposition 2.4. *Let R be a directly finite regular ring satisfying s-comparability, and assume that R has a minimal ideal M. Then*:

(1) *M is unit-regular (that is, all corners eRe with $e \in M$ are unit-regular). In particular, the class FP(M) has cancellation and every $A \in FP(M)$ is directly finite.*

(2) *For any $A \in FP(M)$ and $B \in FP(R)$, either $A \prec B$ or $B \lesssim_a A$.*

Proof. (1) Let e be a nonzero idempotent of M and let $S = eRe$. Since $ReR = M$ is a minimal ideal of R, the ring S is simple. Hence S is a directly finite simple regular ring, and satisfies s-comparability by Proposition 2.1(2). Therefore S is unit-regular by Theorem 1.3. In particular $FP(S)$ has cancellation and so therefore does the naturally equivalent $FP(M)$.

(2) Consider first the special case where R is also simple. The result is trivial for Artinian rings so we can assume R is not Artinian. By Theorem 1.4, R satisfies almost comparability. Let $A, B \in FP(R)$ be nonzero. Let $X \in FP(R)$, $X \neq 0$. Since R is a prime ring with zero socle, we can write

$$B = B_1 \oplus B_2, \quad X = X_1 \oplus X_2, \quad B_1 \cong X_1, \quad X_1 \neq 0, \quad X_2 \neq 0.$$

By almost comparability, either $A \lesssim_a B_2$ or $B_2 \lesssim_a A$. If $A \lesssim_a B_2$ then clearly $A \prec B_1 \oplus B_2 = B$. Suppose $B_2 \lesssim_a A$. Then $B_2 \prec A \oplus X_2$ implies $B = B_1 \oplus B_2 \prec A \oplus X_1 \oplus X_2 = A \oplus X$. Hence $B \lesssim_a A$. This proves the special case.

In the general case, $M = ReR$ for some idempotent $e \in M$ such that the corner ring eRe is simple. Let $S = eRe$. Then S is a directly finite simple regular ring with s-comparability. For $A \in FP(M)$ and $B \in FP(R)$, if $B \notin FP(M)$ then $\mathrm{tr}(A) \subsetneq \mathrm{tr}(B)$ implies $A \prec B$ by Proposition 2.3(b). Therefore we can assume that both $A, B \in FP(M)$. Since $FP(M)$ is equivalent to $FP(S)$, the above special case shows that either $A \prec B$ or $B \prec A \oplus Y$ for all nonzero $Y \in FP(M)$. Suppose it is the latter. Let $X \in FP(R)$, $X \neq 0$. Then $XM \neq 0$, because R is a prime ring, and so there exists $0 \neq Y \in FP(M)$, $Y \subseteq XM$. Now $B \prec A \oplus Y \lesssim A \oplus X$. This shows that either $A \prec B$ or $B \lesssim_a A$. □

In a regular ring with s-comparability, each proper ideal must contain a countably infinite direct sum of copies of each of its principal right ideals. In fact we have the following strengthened versions of Proposition 2.3(a), (b).

Proposition 2.5. *Let R be a regular ring satisfying s-comparability.*

(1) *If $x, y \in R$ with $RxR \subsetneq RyR$, then $\aleph_0(xR) \lesssim yR$.*

(2) *If $A, B \in FP(R)$ with $\mathrm{tr}(A) \subsetneq \mathrm{tr}(B)$, then $\aleph_0 A \lesssim B$.*

(3) *Let f be a nonzero idempotent of R and let N be the maximal ideal of fRf. If fRf/N is directly infinite, then $\aleph_0(fR) \lesssim fR$.*

Proof. (1) Let $yR = hR$, where $h = h^2$. We shall construct an infinite sequence $g_1, g_2, \dots$ of orthogonal idempotents of hRh such that $g_nR \cong xR$ for all n. Since $RxR \subsetneq RhR$, we have $xR \lesssim hR$ by Proposition 2.3(a). Therefore there is an idempotent $g_1 \in hRh$ with $g_1R \cong xR$.

Now suppose we have constructed $g_1, \dots, g_n$ for some n. Set $g = g_1 + \cdots + g_n$. Then $g \in RxR$ and $RxR \subsetneq R(h-g)R$, so again by Proposition 2.3(a) we have $xR \lesssim (h-g)R$. Hence there is a direct summand $X \subseteq (h-g)R$ with $X \cong xR$. Now there exists an idempotent g_{n+1} in $(h-g)R(h-g)$ such that $X = g_{n+1}R$. Clearly g_{n+1} is orthogonal to $g_1, \dots, g_n$. Since $g_{n+1} \in hRh$ and $g_{n+1}R \cong xR$, the induction works.

(2) This is similar to (a) $\Longrightarrow$ (b) of Proposition 2.3. Assume $A \lesssim kR$, $k > 1$, and $\mathrm{tr}(A) \subsetneq \mathrm{tr}(C)$ for some $C \lesssim R$. By s-comparability and Proposition 2.1(1) we can write $A = A_1 \oplus A_2$ with $A_1, A_2 \lesssim (k-1)R$ and $A_1 \lesssim sA_2$. By induction, since $\mathrm{tr}(A_2) \subseteq \mathrm{tr}(A) \subsetneq \mathrm{tr}(C)$, we have $\aleph_0 A_2 \lesssim C$ which implies

$$\aleph_0 A = \aleph_0 A_1 \oplus \aleph_0 A_2 \lesssim \aleph_0 A_2 \lesssim C.$$

(3) Write $S = fRf$. It suffices to prove that $2S \lesssim S$. Set $\overline{S} = S/N$. Since $\overline{S}$ is simple and directly infinite, $2\overline{S} \prec \overline{S}$. By Proposition 2.3(c), $2S \prec S$. □

Proposition 2.6 (K.R. Goodearl). *If R is a directly finite regular ring with s-comparability for some $s > 0$, then all the factor rings of R are also directly finite.*

Proof. Let I be a proper ideal of R and let $\overline{R} = R/I$. If $\overline{R}$ is directly infinite, then $\overline{R} \prec \overline{R}$ and, so, $R \prec R$ by Proposition 2.3(c), giving a contradiction. □

Corollary 2.7. *Suppose R is a directly finite regular ring satisfying s-comparability. Then the (unique) simple factor ring of R is unit-regular .*

Proof. Notice that s-comparability is always inherited by factor rings. The Corollary now follows directly from Proposition 2.6 and Theorem 1.3. □

Although it is not the case in general that s-comparability implies almost comparability (see Example 4.11), the next result says that, in a sense, s-comparability for some $s > 0$ implies "$(1+\varepsilon)$-comparability".

Theorem 2.8. *The following conditions are equivalent for a regular ring R :*
(a) *R satisfies s-comparability for* some $s > 1$.
(b) *R satisfies $(s:t)$-comparability for* all *s and t with $1 \leq t < s$.*

Proof. Clearly, (b) implies (a). Assume R has s'-comparability for some $s' > 1$, and fix $1 \leq t < s$. Choose $x, y \in R$. Without loss of generality, we can assume that $RxR \subseteq RyR$. If $RxR \subsetneq RyR$ then $xR \prec yR$ by Proposition 2.3(a), and, so, clearly $t(xR) \lesssim s(yR)$. Suppose $RxR = RyR$. Let $e \in I := RxR$ be an idempotent with $x, y \in eRe$. Set $S = eRe$ and let N be the maximal ideal of S. Write $\overline{S} = S/N$. By Proposition 2.1, $\overline{S}$ satisfies s'-comparability. By Theorem 1.4, either $\overline{xS} \lesssim_a \overline{yS}$

or $\overline{yS} \lesssim_a \overline{xS}$. Assume that $\overline{xS} \lesssim_a \overline{yS}$. Then also $t(\overline{xS}) \lesssim_a t(\overline{yS})$. Since $\overline{y} \neq 0$ and $s > t$, we have $t(\overline{xS}) \prec t(\overline{yS}) \oplus (s-t)(\overline{yS}) = s(\overline{yS})$. By Proposition 2.3(c), we obtain $t(xS) \prec s(yS)$. Therefore $t(xR) \prec s(yR)$. Consequently R satisfies $(s:t)$-comparability. □

3. The counterexample to Open Problem 4

For an abelian group X, denote by $B(X)$ the abelian group of row- and column-finite countably infinite matrices over X, and by X_∞ the subgroup of countably infinite matrices over X that have only a finite number of nonzero entries. If A is a ring then $B(A)$ is a ring and A_∞ is a two-sided ideal of $B(A)$. If X is an $S-T$ bimodule then $B(X)$ and X_∞ are $B(S)-B(T)$ bimodules.

We start by recalling the construction by Chuang and Lee [4]. Let F be any countable field. Let $F[t]$ be the ring of polynomials over F and $F(t)$ the quotient field of $F[t]$. Let δ be the valuation on $F(t)$ defined by $\delta r(t) = +\infty$ if $r(t) = 0$ and $\delta r(t) = n$ if $r(t) = t^n f(t)/g(t)$ where t does not divide $f(t)g(t)$. Let V be the valuation ring associated to δ, namely $V = \{r(t) \in F(t) \mid \delta r(t) \geq 0\}$. Choose a basis $v_0, v_1, \ldots$ of V such that $v_0 = 1$ and $\delta v_i = i$ for all i; see [4, p. 18]. Denote by φ the map on V given by multiplication by t, and by ψ the map on V given by multiplication by t^{-1} on tV and with $\psi(v_0) = 0$. Let S be the set of all $x \in \mathrm{End}_F(V)$ such that there exists $a \in F(t)$ and $n \geq 0$ such that $(x-a)t^nV = 0$. Then S is a ring and φ, ψ belong to S. Set $K = F(t)$. There exists a ring epimorphism $\pi : S \longrightarrow K$ such that for all $x \in S$, $(x - \pi(x))t^nV = 0$ for some $n \geq 0$. By the arguments of Chuang and Lee, $S \subseteq B(F)$, where we are thinking of the elements of S as matrices in the basis $v_0, v_1, \ldots$ of V. Also, $M := \ker \pi = F_\infty$. Note that every element $a \in F(t)$ such that $\delta(a) = 0$ lifts to a unit of S, namely the endomorphism on V given by multiplication by a.

Consider the opposite ring of S, S^{o}, and let $x \to \overline{x}$ be a ring antiautomorphism from S onto S^{o}. Consider the embedding α of S^{o} into $B(F)$ given by $\alpha(\overline{x}) = x^{\mathrm{t}}$, where x^{t} denotes the transpose of x.

Let L be any unit-regular F-algebra. Set

$$A = \begin{pmatrix} B(L) & L_\infty \\ L_\infty & B(L) \end{pmatrix}.$$

The natural map from F into the centre of L enables us to define an embedding

$$j : B(F) \times B(F) \longrightarrow \begin{pmatrix} B(L) & 0 \\ 0 & B(L) \end{pmatrix} \subseteq A.$$

Set $\beta = j(\mathrm{id} \times \alpha) : S \times S^{\mathrm{o}} \longrightarrow A$. We will identify S with its image under the embedding $S \subset B(F) \longrightarrow B(L)$. Analogously, S^{o} will be identified with his image under $S^{\mathrm{o}} \xrightarrow{\alpha} B(F) \longrightarrow B(L)$. We will denote by $\overline{x}$ the transpose of a matrix $x \in B(L)$. (Of course the map $x \to \overline{x}$ will not be an antiautomorphism in general.)

Observe that $M \subset L_\infty$. Write $D = S + L_\infty$. Then L_∞ is a unit-regular ideal of D and $D/L_\infty \cong K$. Denote by π' the canonical projection from D onto K with kernel L_∞. Note that $\overline{D} = \{\overline{d} \mid d \in D\} = \overline{S} + \overline{L}_\infty = \overline{S} + L_\infty$ is a subring of $B(L)$ (although we are not claiming that $\overline{D}$ is isomorphic to the opposite ring of D).

Now let $T = \{(x, \overline{y}) \in S \times S^{\mathrm{o}} \mid \pi(x) = \pi(y)\}$. T is a directly finite regular ring [13, Lemma 13] with a unique maximal ideal $M \times M^{\mathrm{o}}$ such that $T/(M \times M^{\mathrm{o}}) \cong K$. Set $R = \beta(T) + N$, where $N = M_2(L_\infty)$. A clever way to look at R is the following:

$$R = \left\{ \begin{pmatrix} a & b \\ c & \overline{d} \end{pmatrix} \in \begin{pmatrix} D & L_\infty \\ L_\infty & \overline{D} \end{pmatrix} \mid \pi'(a) = \pi'(d) \right\}.$$

Observe that N is a unit-regular ideal of R and $R/N \cong K$. In particular R is a regular ring. We denote by π'' the canonical projection from R onto K with kernel N. Note that

$$\pi''\left(\begin{pmatrix} a & b \\ c & \overline{d} \end{pmatrix} \right) = \pi'(a) = \pi'(d).$$

Denote by I the identity in $B(L)$. For an idempotent p of L define $p_\infty = \mathrm{diag}(p, p, \ldots,)$, and observe that we have $I - p_\infty = \mathrm{diag}(1-p, 1-p, \ldots,)$. For nonzero idempotents $p, q \in L$ define

$$k(p,q) = \begin{pmatrix} p_\infty & 0 \\ 0 & q_\infty \end{pmatrix} \in A$$

and set $R(p,q) = k(p,q)Rk(p,q)$. Note that $R(p,q)$ is a ring for all nonzero $p, q \in L$, since $xk(p,q) = k(p,q)x$ for all $x \in \beta(T)$. Set $N(p,q) = k(p,q)Nk(p,q)$ and note that $N(p,q)$ is a unit-regular ideal of $R(p,q)$ such that $R(p,q)/N(p,q) \cong K$. In particular $R(p,q)$ is a regular ring for all nonzero idempotents $p, q \in L$.

Lemma 3.1. *Let L be a unit-regular F-algebra and let p and q be two nonzero idempotents in L. Then $R(p,q)$ is unit-regular if and only if $p \sim q$.*

Proof. Assume first that p and q are two equivalent idempotents in L. Since $N(p,q)$ is a unit-regular ideal of $R(p,q)$ and $R(p,q)/N(p,q) \cong K$ is unit-regular, we need only to prove that every unit in K lifts to a unit in $R(p,q)$ (see, for example, [3, Lemma 3.5]).

Every element $a \in K$ such that $\delta(a) = 0$ lifts to a unit of S, and so to a unit in $R(p,q)$. Consequently, it suffices to show that t lifts to a unit of $R(p,q)$. Since $p \sim q$, there exist $a_0 \in pLq$ and $b_0 \in qLp$ such that $a_0 b_0 = p$ and $b_0 a_0 = q$. Denote by a the matrix in $p_\infty L_\infty q_\infty$ which has a_0 in the left upper corner, and 0's elsewhere. Analogously, $b \in q_\infty L_\infty p_\infty$ is the matrix having b_0 in the left upper corner and 0's elsewhere. Put

$$X = \begin{pmatrix} \varphi p_\infty & a \\ 0 & \overline{\varphi} q_\infty \end{pmatrix}, \qquad Y = \begin{pmatrix} \psi p_\infty & 0 \\ b & \overline{\psi} q_\infty \end{pmatrix}.$$

Then $X, Y \in R(p,q)$ and $XY = YX = 1$. Moreover, $\pi''(X) = t$, so that t lifts to a unit of $R(p,q)$ and consequently $R(p,q)$ is unit-regular.

Assume now that $R(p,q)$ is unit-regular. Denote by $\beta_{p,q}$ the map $k(p,q)\beta : T \longrightarrow R(p,q)$, which is an injective homomorphism. Consider the idempotents $g, h \in T$ given by $g = (\varphi\psi, 1)$ and $h = (1, \overline{\varphi\psi})$. Since $g = (\varphi, \overline{\varphi})(\psi, \overline{\psi})$ and $h = (\psi, \overline{\psi})(\varphi, \overline{\varphi})$, we see that g and h are equivalent in T and, consequently, $\beta_{p,q}(g)$ and $\beta_{p,q}(h)$ are equivalent in $R(p,q)$. Since $R(p,q)$ is unit-regular, the idempotents $k(p,q) - \beta_{p,q}(g) = \beta_{p,q}(1-g)$ and $k(p,q) - \beta_{p,q}(h) = \beta_{p,q}(1-h)$ must be equivalent in $R(p,q)$. Noting that $1 - g = (\text{diag}(1,0,0,\ldots,),0)$ and $1 - h = (0, \text{diag}(1,0,0,\ldots,))$, we conclude that $p \sim q$, as desired. □

Now we are ready to give our example.

Example 3.2. There exists a regular ring U such that:
(a) U satisfies almost comparability. In particular, U satisfies 2-comparability.
(b) U is stably finite.
(c) U is not unit-regular.

Proof. Let F be a countable field and let S be the ring constructed in [4], as given before. Let L be any simple, unit-regular F-algebra satisfying almost comparability but not comparability (for example, by Theorem 1.4 we can take the algebra constructed in [5, Example 18.19] and described in the Introduction). Then L has a unique rank function P and (by Proposition 2.4(2)), for idempotents $e, f \in L$, $P(e) = P(f)$ if and only if eL is almost isomorphic to fL. Let p and q be two idempotents in L such that $P(p) = P(q)$ but p is not equivalent to q, and consider the corresponding ring $U = R(p,q)$. Then U is a regular ring which is not unit-regular by Lemma 3.1.

By construction U has a unique nontrivial ideal $V = N(p,q)$ and $U/V \cong F(t)$. Moreover V is a unit-regular ideal satisfying almost comparability, and for idempotents $e, f \in V$, we have $eU \lesssim (1-f)U$. To show that U satisfies almost comparability, take two idempotents $e, f \in U$. If $e, f \in V$ then either $eU \lesssim_{\rm a} fU$ or $fU \lesssim_{\rm a} eU$ because V satisfies almost comparability. If $e \in V$ and $f \notin V$ then $1 - f \in V$, so that $eU \lesssim fU$, by the above observation. Finally, if both e and f are not in V then either $(1-e)U \lesssim_{\rm a} (1-f)U$ or $(1-f)U \lesssim_{\rm a} (1-e)U$. Assume that the former possibility holds. Choose an idempotent $g \in V$ such that $1-e, 1-f \in gVg$. Since gVg is a simple unit-regular ring, it follows from [1, Theorem 1.9] that $(g-(1-f))U \lesssim_{\rm a} (g-(1-e))U$. Adding to both parts of this relation $(1-g)U$ we obtain that $fU \lesssim_{\rm a} eU$. Therefore U satisfies almost comparability.

Finally, we will prove (b). Let $\overline{L}$ be the P-completion of L. Then $\overline{L}$ is a simple, unit-regular, right and left self-injective F-algebra [5, Theorems 19.7 and 19.14]. Moreover L embeds in $\overline{L}$ and P extends to a rank function $\overline{P}$, which is the unique rank function on $\overline{L}$. Since $\overline{L}$ satisfies comparability, $\overline{P}(e) = \overline{P}(f)$ if and only if $e \sim f$, for idempotents $e, f \in \overline{L}$. Now let $\overline{R}(p,q)$ be the ring constructed as before but using $\overline{L}$ instead of L. Since $\overline{P}(p) = \overline{P}(q)$, p and q are equivalent in $\overline{L}$, and, so, Lemma 3.1 gives us that $\overline{R}(p,q)$ is unit-regular. Clearly, U embeds in $\overline{R}(p,q)$ and, thus, U is stably finite. □

4. Cancellation of small projectives

Let R be a directly finite regular ring with s-comparability. Example 3.2 shows that R need not have stable range 1, and so R need not have cancellation of finitely generated projective modules. However, the situation is not nearly as bad as it first looks. For we shall show in this section that R always has "cancellation of small projectives", in the sense of the following definition, and in particular the stable range of R is at most 2.

Definition 4.1. Let R be a regular ring. We say that R has cancellation of small projectives if for all finitely generated projective right R-modules A, B, C

$$A \oplus C \cong B \oplus C \quad \text{and} \quad C \lesssim nA \text{ for some } n \in \mathbb{N} \Longrightarrow A \cong B.$$

Cancellation of small projectives can also be characterized in other ways, as in our next proposition. We recall [12, p. 465] that a ring R is (right) *Hermite* if every 1×2 matrix $A = (a_{11} \ a_{12})$ admits diagonal reduction, that is, PAQ is diagonal $(* \ 0)$ for some units $P \in R$ and $Q \in M_2(R)$. By [13, Theorem 9], a regular Hermite ring R actually has the property that every $m \times n$ matrix over R admits diagonal reduction. Also, by [13, Theorem 9], a regular ring R is Hermite if and only if

$$2R \oplus B \cong R \oplus C \Longrightarrow R \oplus B \cong C$$

for all $B, C \in FP(R)$. Notice that a Hermite ring has stable range at most 2 by [13, Proposition 8(i)].

Proposition 4.2. *For any regular ring R, the following are equivalent:*

(i) *R has cancellation of small projectives.*

(ii) *For all principal right ideals A, B, C of R*

$$A \oplus C \cong B \oplus C \lesssim R \quad \text{and} \quad C \lesssim nA \text{ for some } n \in \mathbb{N} \Longrightarrow A \cong B.$$

(iii) *For all finitely generated projective R-modules A and B,*

$$A \oplus A \cong A \oplus B \Longrightarrow A \cong B.$$

(iv) *R and all its corner rings eRe $(e = e^2 \in R)$ are Hermite.*

(v) *The condition*

$$Rr(a) = R(1-a)R \qquad (**)$$

implies the element $a \in R$ is unit-regular.

Proof. (i) $\Longrightarrow$ (ii). This is trivial.

(ii) $\Longrightarrow$ (i). The proof we shall give is taken from an argument due to K.R. Goodearl (unpublished). Suppose $A, B, C \in FP(R)$ satisfy $A \oplus C \cong B \oplus C$ and $C \lesssim nA$ for some n. We wish to show $A \cong B$. Since $C \lesssim kR$ for some $k \in \mathbb{N}$, we can write $C =$

$C_1 \oplus \cdots \oplus C_k$ with each $C_i \lesssim R$. It is enough, therefore, to handle the case $k = 1$ because in the general case we can cancel one C_i at a time. Similarly we can reduce to the case $n = 1$. Thus we can assume that $C \lesssim R$ and $C \lesssim A$.

By [1, Lemma 1.10] there are decompositions

$$A = A' \oplus A'', \qquad B = B' \oplus B'', \qquad C = C' \oplus C''$$

such that $A' \cong B'$, $A'' \oplus C'' \cong B'' \oplus C''$ and $2A'' \lesssim C'$. Since $C'' \lesssim C \lesssim A = A' \oplus A''$, we can write

$$C'' = C_1 \oplus C_2 \quad \text{with} \quad C_1 \lesssim A' \quad \text{and} \quad C_2 \lesssim A''.$$

From $A'' \oplus C'' \cong B'' \oplus C''$ we have

$$(A'' \oplus C_1) \oplus C_2 \cong (B'' \oplus C_1) \oplus C_2. \tag{1}$$

The two isomorphic projective modules in (1) are isomorphic to a principal right ideal of R because

$$\begin{aligned} (A'' \oplus C_1) \oplus C_2 &\lesssim A'' \oplus C_1 \oplus A'' && \text{since } C_2 \lesssim A'' \\ &\lesssim 2A'' \oplus C'' && \text{since } C_1 \leq C'' \\ &\lesssim C' \oplus C'' && \text{since } 2A'' \lesssim C' \\ &= C \lesssim R. \end{aligned}$$

We can now apply the cancellation property (ii) to (1) and cancel C_2 because $C_2 \lesssim A'' \lesssim A'' \oplus C_1$. This yields $A'' \oplus C_1 \cong B'' \oplus C_1$. Since $C_1 \lesssim A' \cong B'$, we can write $A' \cong X \oplus C_1 \cong B'$ for some $X \in FP(R)$. Finally

$$A = A' \oplus A'' \cong X \oplus C_1 \oplus A'' \cong X \oplus C_1 \oplus B'' \cong B' \oplus B'' = B$$

gives $A \cong B$, as required.

(i) $\Longrightarrow$ (iii). This is immediate.

(iii) $\Longrightarrow$ (i). Suppose $A, B, C \in FP(R)$ with $A \oplus C \cong B \oplus C$ and $C \lesssim nA$ for some n. Again, to show $A \cong B$, it suffices to handle the case $n = 1$. Then, $A \cong C \oplus D$ for some D, whence

$$\begin{aligned} A \oplus C \cong B \oplus C &\Longrightarrow A \oplus C \oplus D \cong B \oplus C \oplus D \\ &\Longrightarrow A \oplus A \cong B \oplus A \end{aligned}$$

and so $A \cong B$ by (iii).

(i) $\Longrightarrow$ (iv). Let e be an idempotent of R and let $A = eR$. If $B, C \in FP(R)$ are such that $2A \oplus B \cong A \oplus C$, then $A \oplus B \cong C$ by (i). By [13, Theorem 9] this shows $\mathrm{End}_R(A)$ is Hermite, whence eRe is Hermite.

(iv) $\Longrightarrow$ (i). By the argument at the beginning of (ii) $\Longrightarrow$ (i) it suffices to show that for $A, B, C \in FP(R)$ with $A \oplus C \cong B \oplus C$ and $C \lesssim R$ and $C \lesssim A$, then $A \cong B$. Write $A \cong C \oplus D$. Since $C \lesssim R$, we have $\mathrm{End}_R(C)$ is Hermite by (iv), whence by [13, Theorem 9]

$$B \oplus C \cong A \oplus C \cong D \oplus 2C \Longrightarrow B \cong D \oplus C \cong A.$$

(ii) $\Longrightarrow$ (v). Suppose $a \in R$ satisfies $Rr(a) = R(1-a)R$. Let $J = R(1-a)R$. There exists an idempotent $g \in J$ such that $1 - a \in gRg$. Note that $J = RgR$. Note also that $a = ag + (1-g)$ so that $r(a) \subseteq gR$ and $agR \subseteq gR$. Let $A = r(a)$ and choose principal right ideals B and C such that

$$gR = A \oplus C = B \oplus agR.$$

Then $agR = aC \cong C$, so

$$A \oplus C \cong B \oplus C \lesssim R. \tag{2}$$

Now $R(1-a)R = Rr(a)$ implies $gR \subseteq RA$, and so $C \cong agR$ implies $C \subseteq RA$. Hence $C \lesssim nA$ for some $n \in \mathbb{N}$. Hence by (ii) we can cancel C in (2) to obtain $A \cong B$. Now we see a is unit-regular in R because

$$R/aR = R/(agR \oplus (1-g)R) \cong gR/agR \cong B \cong A = r(a).$$

(v) $\Longrightarrow$ (ii). Suppose A, B, C are principal right ideals of R with $A \oplus C \cong B \oplus C \lesssim R$ and $C \lesssim nA$ for some n. Write

$$R = A_1 \oplus C_1 \oplus D = B_1 \oplus C_2 \oplus D,$$

where $A_1 \cong A$, $B_1 \cong B$ and $C_1 \cong C_2 \cong C$. Let $a \in R$ induce (by left multiplication) a map of R which is 0 on A_1, an isomorphism from C_1 onto C_2, and the identity on D. Then

$$(1-a)R \lesssim A_1 \oplus C_1 \lesssim (n+1)A_1 = (n+1)r(a)$$

implies $(1-a)R \lesssim (n+1)r(a)$ and so $R(1-a)R \subseteq Rr(a)$. The reverse containment always holds, whence $Rr(a) = R(1-a)R$. By (v), a is unit-regular and so $r(a) \cong R/aR$. Thus $A_1 = r(a) \cong R/(C_2 \oplus D) \cong B_1$ and consequently $A \cong B$, as required. □

Remark 4.3. (a) The proof of (i) $\Longrightarrow$ (iv) shows in fact that $\mathrm{End}_R(A)$ is Hermite for all finitely generated projective R-modules A.

(b) In [11] the condition

$$Rr(a) = \ell(a)R = R(1-a)R \tag{$*$}$$

was studied for regular rings. For several large classes of regular rings (including all unit-regular rings), it was shown that $(*)$ characterizes when an element $a \in R$ is a product of idempotents. If all the factor rings of a regular ring R are directly finite (which is equivalent to $Rr(a) = \ell(a)R$ for all $a \in R$), then $(*)$ is the same as $(**)$. In particular, therefore, if R is a regular ring which has a $(*)$ characterization for products of idempotents, and all factor rings of R are directly finite, then R has cancellation of small projectives (by (v) $\Longrightarrow$ (i) and the fact that products of idempotents are always unit-regular).

(c) Since there exist directly finite regular rings with infinite stable range (such as the free regular ring–see [8, p. 416]), by Proposition 4.2 not even all directly finite regular rings have cancellation of small projectives.

(d) In view of [13, Example 3], a corner ring of a regular Hermite ring need not be Hermite–it can have infinite stable range. Therefore it is not enough in Proposition 4.2(iv) to require only that R be Hermite.

Lemma 4.4. *Let R be a directly finite regular ring satisfying s-comparability for some $s > 0$, and assume that R has a minimal ideal M. Then $R \oplus T$ is directly finite for every $T \in FP(M)$.*

Proof. Let $T \in FP(M)$ and assume that $R \oplus T \oplus T' \cong R \oplus T$ for some $T' \in FP(R)$. By proposition 2.4(1), $\mathrm{End}_R(T)$ is unit-regular, so it has stable range 1, and so T cancels from direct sums. Thus, $R \oplus T \oplus T' \cong R \oplus T$ implies $R \oplus T' \cong R$ and so $T' = 0$ since R is directly finite. □

Proposition 4.5. *Let R be a directly finite regular ring satisfying s-comparability for some $s > 0$. If $B, C_1, C_2 \in FP(R)$ and $R \oplus B \lesssim R \oplus C_i$ for $i = 1, 2$, then $B \lesssim C_1 \oplus C_2$.*

Proof. Obviously, we can assume that $B \neq 0$. Let $M = \mathrm{tr}(B)$ and let N be the maximal ideal of M. For $A \in FP(R)$ we denote by $\overline{A}$ the R/N-module A/AN. By Proposition 2.6, $\overline{R} := R/N$ is a directly finite regular ring satisfying s-comparability, and has a minimal ideal $\overline{M} := M/N$. Therefore, from $\overline{R} \oplus \overline{B} \lesssim \overline{R} \oplus \overline{C}_i$, we can deduce that $\overline{C}_i \neq 0$ for $i = 1, 2$. If $C_i \neq C_i M$ for some i, then we are done by Proposition 2.3(b) because $\mathrm{tr}(B) \subsetneq \mathrm{tr}(C_i)$ implies $B \prec C_i$. So we can assume that $C_i = C_i M$ for $i = 1, 2$ and so $\overline{C}_i \in FP(\overline{M})$ for $i = 1, 2$. If $\overline{C}_i \prec \overline{B}$ for some i, then $\overline{R} \oplus \overline{C}_i \prec \overline{R} \oplus \overline{B} \lesssim \overline{R} \oplus \overline{C}_i$, contradicting the fact that $\overline{R} \oplus \overline{C}_i$ is directly finite (Lemma 4.4). By Proposition 2.4(2), $\overline{B} \lesssim_a \overline{C}_i$ for $i = 1, 2$, and since $\overline{C}_i \neq 0$, we have $\overline{B} \prec \overline{C}_1 \oplus \overline{C}_2$. By Proposition 2.3(c), $B \prec C_1 \oplus C_2$, as desired. □

Theorem 4.6. *Let R be a directly finite regular ring satisfying s-comparability for some positive integer s. Then R has cancellation of small projectives.*

Proof. By Proposition 4.5 and [1, Proof of Theorem 1.7], all corner rings of R have stable range less than or equal to 2. By [17, Theorem 1.9], $sr(\mathrm{End}_R(D)) \leq 2$ for all $D \in FP(R)$.

Assume that

$$A \oplus A \cong A \oplus B$$

for some $A, B \in FP(R)$. Let $M = \mathrm{tr}(A)$ and let N be the maximal ideal of M. We consider two cases:

(1) There exists a decomposition $A = A_1 \oplus A_2$ with $A_1 \neq A_1 N$ and $A_2 \neq A_2 N$.

(2) If $A = A_1 \oplus A_2$ then $A_1 = A_1 N$ or $A_2 = A_2 N$.

Assume (1) holds and that $A = A_1 \oplus A_2$ with $A_1 \neq A_1 N$ and $A_2 \neq A_2 N$. Then $N \subset \mathrm{tr}(A_i) \subseteq M$ implies $\mathrm{tr}(A_1) = \mathrm{tr}(A_2)$ by maximality of N. Therefore $A_1 \lesssim nA_2$ for some $n \in \mathbb{N}$ and we can write $A_1 = X_1 \oplus \cdots \oplus X_n$ with each $X_j \lesssim A_2$. Since $A_1 \neq A_1 N$, there is an X_j with $X_j \neq X_jN$. Let $A' = X_j$. Then $2A' \lesssim A$. Also since $A'N \neq A'$, we see that $N \subset \mathrm{tr}(A') \subseteq M$. Therefore $\mathrm{tr}(A') = M$ by maximality of N. Now $\mathrm{tr}(A) \subseteq \mathrm{tr}(A')$ implies $A \lesssim kA'$ for some $k \in \mathbb{N}$. Write $A = A_1 \oplus \cdots \oplus A_k$ with $A_i \lesssim A'$ for all i. Then we have

$$(A \oplus A_1 \oplus \cdots \oplus A_{k-1}) \oplus A_k \cong (B \oplus A_1 \oplus \cdots \oplus A_{k-1}) \oplus A_k$$

and, since $2A_k \lesssim 2A' \lesssim A$ and $sr(\mathrm{End}_R(A_k)) \leq 2$, we deduce from [17, Theorem 1.2] that $A \oplus A_1 \oplus \cdots \oplus A_{k-1} \cong B \oplus A_1 \oplus \cdots \oplus A_{k-1}$. It follows (by induction on k) that $A \cong B$, as desired.

Assume now that (2) holds. Write $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $A_1 \oplus B_1 \cong A_2 \oplus B_2 \cong A$. Assume that $A_1N = A_1$. Then $A_2 \neq A_2N$ and so Proposition 2.3(b) tells us that $2A_1 \lesssim A_2$. So we can apply [17, Theorem 1.2] to the relation

$$A_1 \oplus A_2 \cong A_1 \oplus B_1$$

to obtain $B_1 \cong A_2$. Hence

$$A \cong A_2 \oplus B_2 \cong B_1 \oplus B_2 = B.$$

If $A_2 = A_2N$, a similar argument shows also that $A \cong B$.

By Proposition 4.2, R satisfies cancellation of small projectives. □

Corollary 4.7. *Let R be a directly finite regular ring satisfying s-comparability for some positive integer s. Then:*

(1) *R and all its factor rings are stably finite.*

(2) *All corner rings of R have stable range at most 2.*

Proof. (1) Assume that $nR \oplus T \cong nR$ for some $n \geq 1$. By applying Theorem 4.6 to the relation

$$(n-1)R \oplus (R \oplus T) \cong (n-1)R \oplus R,$$

we get $R \oplus T \cong R$, so $T = 0$ because R is directly finite. This shows R is stably finite. By Proposition 2.6, so are its factor rings.

(2) This was observed in the proof of Theorem 4.6. (Or alternatively we can use Theorem 4.6 and (i) $\Longrightarrow$ (iv) of Proposition 4.2.) □

Remark 4.8. Suppose that R is a directly finite regular ring with s-comparability. The proof of [15, Theorem 1] can be adapted (through the use of Lemma 2.2) to show that R has a $(*)$ characterization for its products of idempotents. This leads to an alternative route for the proof of Theorem 4.6, via Proposition 2.6 and the implication (v) $\Longrightarrow$ (i) of Proposition 4.2 (see Remark 4.3(b)). Actually the conclusion reached is possibly stronger than Theorem 4.6, because in Proposition 4.2(v), knowing that a is a product of idempotents tells us more than a being just unit-regular; for instance, in a homomorphic image, a will always lift to a unit-regular element. Using this last observation, one can show that if S is any regular ring which contains a unit-regular ideal I such that S/I has a $(*)$ characterization for products of idempotents, and S/I has all its factor rings directly finite, then S too has cancellation of small projectives. (Here S need not be directly finite.)

The proof of Lemma 3.1 showed that the ring $R(p,q)$ is unit-regular if and only if the element t in the homomorphic image K lifts to a unit of $R(p,q)$. We now show that this is part of a more general phenomenon.

Proposition 4.9. *A regular ring R is unit-regular if and only if all the following hold:*

(1) *Every factor ring of R is directly finite.*

(2) *R has a $(*)$ characterization for products of idempotents, that is, the condition $Rr(a) = \ell(a)R = R(1-a)R$ implies the element a is a product of idempotents.*

(3) *Units can be lifted in R/I for all ideals I.*

Proof. The necessity of (1) and (3) is trivial, while (2) follows from [11, Theorem 2.9]. To establish sufficiency, assume (1), (2) and (3) and let $a \in R$. Let $I = Rr(a)$. In the factor ring $\overline{R} = R/I$, observe that $r_{\overline{R}}(\overline{a}) = \overline{r_R(a)} = 0$ so $\overline{a}$ must be a unit of $\overline{R}$ by (1). By (3), there is a unit $u \in R$ with $\overline{a} = \overline{u}$. Let $b = u^{-1}a$. Then $\overline{b} = \overline{1}$ implies $1-b \in I = Rr(a) = Rr(b)$. Hence $R(1-b)R \subseteq Rr(b)$ and so $R(1-b)R = Rr(b)$. Since $R/Rr(b)$ is directly finite by (1), we have $\ell(b)R = Rr(b)$. Therefore $Rr(b) = \ell(b)R = R(1-b)R$, which implies b is unit-regular by (2). Hence $a = ub$ is also unit-regular. Therefore R is unit-regular. □

Corollary 4.10. *Let R be a directly finite regular ring satisfying s-comparability. Then R is unit-regular if and only if units can be lifted in R/I for all ideals I of R.*

Proof. R satisfies (1) and (2) of Proposition 4.9 by Proposition 2.6 and Remark 4.8 respectively. Thus R is unit-regular if and only if (3) holds. (For the purposes of this proof, "products of idempotents" in (2) could be replaced by "unit-regular", in which case the reference to Remark 4.8 should be replaced by one to Proposition 4.2 and Theorem 4.6.) □

We close with a construction that enables us to give some counter-examples, in the non-simple case, to some known behaviour [2, Corollaries 4.4 and 4.5] of simple unit-regular rings with s-comparability.

Example 4.11. (a) *There exists a unit-regular ring R satisfying 2-comparability but not almost comparability.*

(b) *There exists a unit-regular ring R with 2-comparability such that $FP(R)$ is not strictly unperforated.*

Proof. Let S be any simple non-Artinian regular ring satisfying 2-comparability, and denote by F the centre of S. Embed S in the ring T of all linear transformations on a suitable F-vector space V. Let M be the socle of T and note that $S \cap M = 0$. (The elements of M are those linear transformations x on V such that $x(V)$ is finite-dimensional.) Set $R = S + M$. If f is a nonzero idempotent in S then $\dim_F f(V) = \infty$ and so $eR \prec fR$ for every idempotent $e \in M$. Now assume that we have idempotents $e, e' \in S$ and $g \in eRe \cap M$ such that $e'S \prec eS$. We claim that $e'R \prec (e - g)R$. For, let e'' be a nonzero idempotent of S such that $e'S \oplus e''S \cong eS$. By the above remark, $gR \prec e''R$ and so we have $e'R \oplus gR \prec e'R \oplus e''R \cong eR = (e - g)R \oplus gR$. Since $g \in M$, the stable range of gRg is one, and so gR cancels from direct sums. So $e'R \oplus gR \prec (e - g)R \oplus gR$ implies $e'R \prec (e - g)R$, proving our claim.

Now let A and B be two principal right ideals of R. Then $A \cong (e - g)R \oplus g'R$ and $B \cong (f - h)R \oplus h'R$, for some idempotents $e, f \in S$, $g \in eRe \cap M$, $h \in fRf \cap M$, $g' \in (1 - e)R(1 - e) \cap M$ and $h' \in (1 - f)R(1 - f) \cap M$. If $e = f = 0$ then either $A \lesssim B$ or $B \lesssim A$ since M has comparability. Assume that $e \neq 0$ or $f \neq 0$. By Theorem 1.4, either $eS \lesssim_a fS$ or $fS \lesssim_a eS$. We can assume that $eS \lesssim_a fS$ and that $f \neq 0$. Now write $fS = f_1S \oplus f_2S$, where f_1 and f_2 are nonzero idempotents of S and $2(f_1S) \prec fS$. Then

$$\begin{aligned} A &\cong (e - g)R \oplus g'R \lesssim eR \oplus g'R \\ &\lesssim fR \oplus f_1R \oplus g'R = f_1R \oplus f_2R \oplus f_1R \oplus g'R \\ &= 2(f_1R) \oplus (f_2R \oplus g'R) \lesssim 2(f - h)R \lesssim 2B. \end{aligned}$$

So R satisfies 2-comparability. Since the units of $R/M \cong S$ lift obviously to units of R, and M is unit-regular, the ring R is unit-regular if and only if so is S [3, Lemma 3.5].

(a) If S is a simple unit-regular ring satisfying 2-comparability but not comparability, then $R = S + M$ is a unit-regular ring satisfying 2-comparability but not almost comparability. Indeed, if eS and fS are not comparable in S, then eR and fR are not almost comparable in R.

(b) By [6, Theorem 5.1], there exists a simple countable unit-regular ring S such that $K_0(S)$ is strictly unperforated but has nonzero torsion. Also it is observed after [6, Proposition 4.2] that S has a unique rank function, so that S satisfies 2-comparability by [2, Corollary 4.5]. Then $R = S + M$ has 2-comparability but $FP(R)$ is not strictly unperforated. For, assume that $K_0(S)$ has a nonzero element of order n, where n is a positive integer. Then there exists non-isomorphic $A, B \in FP(S)$ such that $nA \cong nB$. Set $C = A \otimes_S R$ and $D = B \otimes_S R$, and note that $nC \cong nD$. Choose a nonzero idempotent $e \in M$ and set $E = D \oplus eR$. Then $nC \prec nE$ but $C \not\prec E$ because $C/CM \cong A \not\lesssim B \cong E/EM$. So $FP(R)$ is not strictly unperforated.

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