On solutions of backward stochastic differential equations with jumps and applications

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Received November 1995; revised August 1996

Abstract

For backward stochastic differential equation (BSDE) with jumps and with non-Lipschitzian coefficient the existence and uniqueness of an adapted solution is obtained. By generalizing the existence result on partial differential and integral equations (PDIE) and Itô formula to the functions with only first and second Sobolev derivatives the probabilistic interpretation for solutions of PDIE (a new Feynman–Kac formula) by means of solutions of BSDE with jumps is got. With the help of this formula a new existence and uniqueness result for the solution of PDIE with non-Lipschitzian force is obtained. The convergence theorems of solutions to BSDE and PDIE are also derived.

Keywords: BSDE with jumps; Adapted solution; PDIE; Itô formula; Convergence theorem

1. Introduction

The adapted solution for a linear BSDE which appears as the adjoint process for a stochastic control problem was first investigated by Bismut (1973), then by Bensoussan (1982), and others, while the first result for the existence of an adapted solution to a continuous non-linear BSDE with Lipschitzian-coefficient was obtained by Pardoux and Peng (1990). Later Peng and Pardoux developed the theory and applications of continuous BSDEs in a series of papers (1991, 1992, 1993, 1994) under the assumption that the coefficients satisfy the Lipschitzian condition globally, or locally but with some additional condition. Tang and Li (1994) then applied the idea of Peng to get the first result on the existence of an adapted solution to a BSDE with Poisson jumps for a fixed terminal time and with Lipschitzian coefficients. Using this result, he generalized the general maximum principle of Peng (1990) for the continuous SDE

1This work is supported in part by The National Natural Science Foundation of China, The Foundation of Zhongshan University Front Research, and The Foundation of Zhongshan University Advanced Research Centre.
system to the case of SDE with Poisson jumps, which also generalized the result got by Situ Rong (1991).

In this paper we derive an existence result for the adapted solution to BSDE with jumps, with bounded random stopping time as its terminal time and with non-Lipschitzian coefficient. Here the coefficient is assumed to be less than linear growth, jointly continuous and to satisfy some weak "monotone" condition locally. Since the same technique can be applied to the continuous SDE, our result implies Peng's (1993) for continuous SDE with local Lipschitzian coefficient. We also derive the convergence of solutions for such BSDE with jumps. Then by generalizing the existence result of PDIE from Marhno (1976) to some more general case (quasi-linear case) and the Ito formula to the functions having Sobolev derivatives only, we obtain a probabilistic interpretation for the solution of PDIE (a new Feyman–Kac formula) by means of the adapted solution of BSDE with jumps. With the help of this new F–K formula a new existence result for a PDIE with non-Lipschitzian and non-homogeneous term is obtained. Finally, we derive a Krylov type estimate in a weak form for such BSDE with jumps by means of its relation with PDIE.

2. BSDE with jumps and with non-Lipschitzian coefficients

Consider a BSDE in $\mathbb{R}^d$:

$$x_t = X + \int_{t \wedge \tau}^{\tau} b(s, x_s, q_s, \omega) ds - \int_{t \wedge \tau}^{\tau} q_s dw_s - \int_{t \wedge \tau}^{\tau} \int_{Z} p_s(z) (\tilde{N}_k(ds, dz), t \geq 0, \quad (1)$$

where $w_t$ is an $r$-dimensional standard Brownian motion process ($BM$), $k(\cdot)$ is a Poisson point process taking values in a measurable space $(Z, \mathcal{B}(Z))$, $\tilde{N}_k(ds, dz)$ is the Poisson counting measure defined by $k(\cdot)$ with compensator $\Pi(ds)ds$, $\tilde{N}_k(ds, dz)$ is the martingale measure such that

$$\tilde{N}_k(ds, dz) = N_k(ds, dz) - \Pi(ds) ds,$$  

$\Pi(\cdot)$ is a $\sigma$-finite measure on $\mathcal{B}(Z)$, $\tau$ is a bounded $\mathcal{F}_\tau$-stopping time, and $X$ is a $\mathcal{F}_\tau$-measurable and $\mathbb{R}^d$-valued random variable, where $\mathcal{F}_\tau$ is the $\sigma$-algebra generated (and completed) by all $w_s$, $s \leq t$, and $N_k([0, s], U)$, $s \leq t$, $U \in \mathcal{B}(Z)$.

We also use the following notation: $\mathcal{F}^2_{k, (\mathcal{F}_\tau)}([0, \tau]; \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathcal{F}_\tau$-predictable processes $f(t, z, \omega)$ such that

$$E \int_0^\tau \int_Z |f(t, z, w)|^2 \Pi(dz)dt < \infty;$$

$L^2(\mathcal{F}_\tau)([0, \tau]; \mathbb{R}^d)$ is the set of $f(t, \omega)$, which is $\mathcal{F}_\tau$-adapted, jointly measurable and $\mathbb{R}^d$-valued such that

$$E \int_0^\tau |f(t, \omega)|^2 dt < \infty;$$

and $L^2(\mathcal{F}_\tau)([0, \tau]; \mathbb{R}^{d \otimes r})$ is defined similarly.
Denote also $L^2_{\mathcal{F}_t}(\mathbb{R}^d) = \text{The set of } \mathbb{R}^d\text{-valued functions } f(z), z \in Z, \text{ which is } \mathcal{H}(Z)\text{ measurable such that } \|f\| = \left(\int_Z |f(z)|^2 \Pi(dz)\right)^{1/2} < \infty; \langle a, b \rangle = a \cdot b = \text{the inner product of } a, b \in \mathbb{R}^d; \|q\| = \text{the norm of the matrix } q \in \mathbb{R}^d \otimes r.

Introduce now

**Definition 1.** $(x_t, q_t, p_t)$ is said to be a solution of (1), iff

1° $(x_t, q_t, p_t) \in L^2_{\mathcal{F}_t}(\mathbb{R}^d) \times L^2_{\mathcal{F}_t}(\mathbb{R}^d) \times \mathcal{F}_{\mathcal{F}_t}(\mathbb{R}^d),$

2° $(x_t, q_t, p_t)$ satisfies (1).

**Definition 2.** A real function $\rho(u), u \geq 0,$ is said to have the $\bar{\rho}$ property, iff

$\rho(0) = 0, \rho(u) > 0, \text{ as } u > 0, \text{ and } \rho(u) \text{ is increasing, concave and continuous such that }$

$$\int_0^\infty du/\rho(u) = + \infty.$$

We have

**Theorem 1.** Assume that $\tau \leq T$ and

1° $b_t = b_1(t, x_t, q_t, p_t, \omega) + b_2(t, x_t, q_t, p_t, \omega) : [0, T] \times \mathbb{R}^d \times \mathcal{F}_t \times \mathcal{F}_t \times L^2_{\mathcal{F}_t}(\mathbb{R}^d) \times \Omega \to \mathbb{R}^d,$

$i = 1, 2$ are $\mathcal{F}_t$-adapted and measurable processes such that P-a.s.

$$|b_1(t, x_t, q_t, p_t, \omega)| \leq c(t),$$

$$|b_2(t, x_t, q_t, p_t, \omega)| \leq c(t)(1 + |x_t| + \|q_t\| + \|p_t\|),$$

where $c(t) \geq 0$ is real and non-random such that

$$\int_0^T c(t)^2 dt < \infty;$$

2° for all $t \in [0, T]; x, x_t, q_t, q_t \in \mathbb{R}^d; p_t \in L^2_{\mathcal{F}_t}(\mathbb{R}^d), i = 1, 2,$

$$\langle x_1 - x_2, b_1(t, x_1, q_1, p_1, \omega) - b_1(t, x_2, q_2, p_2, \omega) \rangle \leq c(t)(\rho(|x_1 - x_2|^2)

+ |x_1 - x_2|(\|q_1 - q_2\| + \|p_1 - p_2\|)),$$

$$|b_1(t, x, q, p_1, \omega) - b_1(t, x, q, p_2, \omega)| \leq c(t)(\|q_1 - q_2\| + \|p_1 - p_2\|),$$

$$|b_1(t, x, q, p_1, \omega) - b_1(t, x, q, p_2, \omega)| \leq c(t)(\|q_1 - q_2\| + \|p_1 - p_2\|).$$
\[ |b_2(t, x_1, q_1, p_1, \omega) - b_2(t, x_2, q_2, p_2, \omega)| \leq c(t)(|x_1 - x_2| + \|q_1 - q_2\| + \|p_1 - p_2\|), \]

where \( c(t) \) has the same property in 1\(^{\circ}\), and \( \rho(u) \) has the \( \bar{A} \) property;

3\(^{\circ}\) \( b(t, x, q, p, \omega) \) is continuous in \((x, q, p) \in \mathbb{R}^d \times \mathbb{R}^d \times L^2_{\mathbb{F}, \mathcal{F}}(\mathbb{R}^d)\);

4\(^{\circ}\) \( X \) is \( \mathcal{F}_t \)-measurable, and \( E|X|^2 < \infty \).

Then (1) has a unique solution \((x_t, q_t, p_t)\).

**Remark 1.** "Uniqueness" here means that if \((x^1_t, q^1_t, p^1_t), i = 1, 2\) are two solutions of (1), then

\[
E \int_0^t |x^1_t - x^2_t|^2 \, dt = 0, \quad E \int_0^t \|q^1_t - q^2_t\|^2 \, dt = 0, \quad E \int_0^t \|p^1_t - p^2_t\|^2 \, dt = 0.
\]

**Remark 2.** In the one-dimensional space if \( b \) only depends on \((t, x)\) and is decreasing in \( x \in \mathbb{R}^d \), then 2\(^{\circ}\) is satisfied by \( b \). Hence 2\(^{\circ}\) can be referred to as a weak "monotonicity condition".

By using the conditional expectation method of Peng (1993) and some localization technique one can relax condition 2\(^{\circ}\) to a local condition provided that \( X \) satisfies

5\(^{\circ}\) \( |X|^2 \leq k_0, \) \( \mathbb{P} \)-a.s.

where \( k_0 \geq 0 \) is a constant, and \( X \) is \( \mathcal{F}_t \)-measurable.

**Corollary 1.** Assume that all conditions except 1\(^{\circ}\) and 2\(^{\circ}\) in Theorem 1 hold, and assume that 5\(^{\circ}\) also holds and

1\(^{\prime}\) \( b = b_1 + b_2, \)

\[
|b_1(t, x, q, p, \omega)| \leq c(t)(1 + |x|),
\]

\[
|b_2(t, x, q, p, \omega)| \leq c(t)(1 + |x| + \|q\| + \|p\|),
\]

where \( c(t) \geq 0, \) non-random and

\[
\int_0^T c(t)^2 \, dt < \infty ;
\]

2\(^{\prime}\) For all \( t \in [0, T] ; q, q_t \in \mathbb{R}^d \otimes \mathcal{F}_t ; p, p_t \in L^2_{\mathbb{F}, \mathcal{F}}(\mathbb{R}^d) \)

\[
|b_2(t, x_1, q_1, p_1, \omega) - b_2(t, x_2, q_2, p_2, \omega)| \leq c(t)(|x_1 - x_2| + \|q_1 - q_2\| + \|p_1 - p_2\|); 
\]
\[ \langle x_1 - x_2, b_1(t, x_1, q_1, p_1, \omega) - b_1(t, x_2, q_2, p_2, \omega) \rangle \]
\[ \leq c_N(t) (|x_1 - x_2|^2 + |x_1 - x_2| (\|q_1 - q_2\| + \|p_1 - p_2\|)), \]
\[ |b_1(t, x, q, p_1, \omega) - b_1(t, x, q, p_2, \omega)| \leq c_N(t) \|p_1 - p_2\|, \]
as \[ |x| \leq N, \ |x_i| \leq N, \ i = 1, 2, \ N = 1, 2, \ldots; \] where \( c_N(t) \) and \( \rho_N(u) \) have the same property as \( c(t) \) and \( \rho(u) \) in 2° of Theorem 1 for each \( N \), and no generality is lost to assume that
\[ c_N(t) \leq c_{N+1}(t), \ \rho_N(t) \leq \rho_{N+1}(t), \ \text{for all} \ N. \]

Then (1) has a unique solution \((x_t, q_t, p_t)\).

To show Corollary 1 we need the following interesting

**Proposition 1.** Assume that \( \tau \leq T \), and
\[ |b(t, x, q, p, \omega)| \leq c(t) (1 + |x| + \|q\| + \|p\|), \]
where \( c(t) \) satisfies conditions in 1°, and assume that 5° holds. If \((x_t, q_t, p_t)\) is a solution of (1), then there exists a constant \( N_0 \) depending on \( k_0, T \) and \( c(\cdot) \) only such that
\[ |x_t| \leq N_0, \ \text{for all} \ t \in [0, \tau], \ \text{P-a.s.} \]
i.e. \( x_t \) is uniformly bounded in \( t \in [0, \tau], \ w \in \Omega \).

**Proof.** By the Itô formula
\[ |x_{t \wedge \tau}|^2 = |X|^2 + \int_{t \wedge \tau}^\tau 2x_s b(s, x_s, q_s, p_s, \omega) \, ds - \int_{t \wedge \tau}^\tau \|q_s\|^2 \, ds \]
\[ - \int_{t \wedge \tau}^\tau \|p_s\|^2 \, ds + \int_{t \wedge \tau}^\tau dM_s, \]
where \( M_t \) is a \( F_t \)-martingale. From this as \( 0 \leq r \leq t \leq T \) \((E^\tau_r(x) = E(x|\mathcal{F}_t))\)
\[ E^{\tau_r}\left( |x_{t \wedge \tau}|^2 + \frac{1}{2} \int_{t \wedge \tau}^r \|q_s\|^2 \, ds + \frac{1}{2} \int_{t \wedge \tau}^r \|p_s\|^2 \, ds \right) \]
\[ \leq E^{\tau_r} |X|^2 + T + 5E^{\tau_r} \int_{t \wedge \tau}^r c(s)^2 |x_s|^2 \, ds + 2E^{\tau_r} \int_{t \wedge \tau}^r c(s)|x_s|^2 \, ds \]
\[ \leq T + k_0 + \int_t^T (2c(s) + 5c(s)^2) E^{\tau_s} |x_s|_{F_s}^2 \, ds. \]
Hence by the Cronwall inequality
\[ E^{\tau_r}\left( |x_{t \wedge \tau}|^2 + \frac{1}{2} \int_{t \wedge \tau}^r \|q_s\|^2 \, ds + \frac{1}{2} \int_{t \wedge \tau}^r \|p_s\|^2 \, ds \right) \]
\[ \leq (T + k_0) \exp\left( \int_t^T (2c(s) + 5c(s)^2) \, ds \right) \leq N_0. \]
Let \( r = t \). Then
\[
|x_{t \wedge r}|^2 \leq N_0,
\]
i.e. \( P\)-a.s. for all \( t \in [0, t] \),
\[
|x_r|^2 = |x_{t \wedge r}|^2 \leq N_0.
\]

Now we are in a position to prove Corollary 1.

**Proof.** For each \( N \) let \( W^N(x) \in C^0_\infty(\mathbb{R}^d) \) be such that
\[
W^N(x) = \begin{cases} 
1 & \text{for } |x| \leq N + 2, \\
0 & \text{for } |x| \geq N + 3,
\end{cases}
\]
\[
|W^N(x) - W^N(y)| \leq \bar{k}_0|x - y| \quad \text{for all } x, y \in \mathbb{R}^d,
\]
and \( 0 \leq W^N(x) \leq 1 \), where \( \bar{k}_0 \) is a constant. Define
\[
b^N(t, x, q, p, \omega) = b(t, x, q, p, \omega) \cdot W^N(x).
\]
Then it is easily seen that if one lets
\[
\bar{c}_N(t) = c(t)(1 + N + 3)/\bar{k}_0 + c_{N+3}(t),
\]
\[
\bar{\rho}_N(u) = \rho_{N+3}(u) + u,
\]
then for each \( N \) the global condition 2° in Theorem 1 holds for \( b^N \) with respect to such \( \bar{c}_N(t) \) and \( \bar{\rho}_N(u) \). Hence by Theorem 1 there exists a unique solution \((x^N_t, q^N_t, p^N_t)\) of (1) with such \( b^N \) for each \( N \).

Applying Proposition 1 one sees that there exists a natural number \( N_0 \)
\[
|x^N_t| \leq N_0 + 2 \quad \text{for all } N = 1, 2, \ldots.
\]
Hence \((x^{N_0}_t, q^{N_0}_t, p^{N_0}_t)\) is a solution of (1). The uniqueness will be derived by Lemma 2 below.

Let us give an example to show that under conditions of Theorem 1 or Corollary 1 coefficient \( b \) can be non-Lipschitzian in \( x \).

**Example 1.** \( b = b_1 + b_2 \), where \( b_2 \) satisfies conditions of Theorem 1 and we assume that \( 0 \leq \nu_i(t, \omega), i = 1, 2, 3, \) are \( \mathcal{F}_r \)-progressive processes on \([0, T]\) and uniformly bounded.

Let
\[
b_1 = \sum_{i=1}^3 b_{1i},
\]
where
\[ b_{11}(t, x, q, p, \omega) = -|x|^r - 2 x \cdot v_1(t, \omega) I_{x \neq 0}, \]
\[ b_{12}(t, x, q, p, \omega) = -|x|^r - 2 x \cdot v_2(t, \omega) I_{x \neq 0} \left( \|q\| I_{\|q\| < k_0} + k_0 I_{\|q\| > k_0} \right), \]
\[ b_{13}(t, x, q, p, \omega) = -|x|^r - 2 x \cdot v_3(t, \omega) I_{x \neq 0} \left( \|p\| I_{\|p\| < k_1} + k_1 I_{\|p\| > k_1} \right), \]
and \( k_i \geq 0, i = 0, 1; r_j \in (1, 2), j = 0, 1, 2; \) which are all constants. Then \( b_1 \) satisfies all conditions in 1°, 2° and 3°, but it is not Lipschitzian continuous.

Now we are going to prepare a series of lemmas and theorems, which are necessary for proving Theorem 1, and are of interest on their own.

**Lemma 1.** Assume that \( \tau \leq T, E|X|^2 < \infty, \) and
\[ \langle x, b(t, x, q, p, \omega) \rangle \leq c(t)(1 + |x|^2 + |x|(\|q\| + \|p\|)), \]
where \( c(t) \) has the property stated in 1° of Theorem 1. If \((x_t, q_t, p_t)\) is a solution of (1), then
\[ E\left( \sup_{t \leq \tau} |x_t|^2 + \frac{1}{2} \int_0^\tau \|q_t\|^2 \, dt + \frac{1}{2} \int_0^\tau \|p_t\|^2 \, dt \right) \leq k_T < \infty, \]
where \( k_T \geq 0 \) is a constant depending on \( T, \int_0^T c(t)^2 \, dt \) and \( E|X|^2 \) only.

**Proof.** Similarly as for (4), one gets that for all \( t \in [0, T] \)
\[ E\left( |x_t|^2 + \frac{1}{2} \int_0^\tau \|q_s\|^2 \, ds + \frac{1}{2} \int_0^\tau \|p_s\|^2 \, ds \right) \leq k_T, \]
where
\[ k_T = (T + E|X|^2) \exp \left( \int_0^T (2c(t) + 5c(t)^2) \, dt \right). \]
In particular,
\[ E(|x_0|^2 + \frac{1}{2} \int_0^\tau \|q_s\|^2 \, ds + \frac{1}{2} \int_0^\tau \|p_s\|^2 \, ds) \leq k_T. \]
Note that
\[ |x_t|^2 \leq 4^2 \left( |x_0|^2 + \left( \int_0^{t \wedge \tau} b(s, x_s, q_s, p_s, \omega) \, ds \right)^2 \right. \]
\[ \left. + \int_0^{t \wedge \tau} q_s \, dw_s \right)^2 + \left( \int_0^{t \wedge \tau} p_s(z) N_{z}(ds, dz) \right)^2 = 4 \sum_{i=1}^4 I_i, \]
However, by the martingale inequality
\[ E \sup_{t \leq T} \left( \int_0^{\tau \wedge t} Q_s \, dw_s \right)^2 \leq 2E \int_0^{\tau \wedge t} \|q_s\|^2 \, ds \leq 4k_T < \infty. \]

Similarly,
\[ E \sup_{t \leq T} I_4 \leq c_0 k_T < \infty. \]

Moreover, by the Schwarz inequality
\[ E \left( \int_0^{\tau \wedge t} c(s) \|p_s\| \, ds \right)^2 \leq \int_0^{\tau} c(s) \|p_s\|^2 \, ds, \]
\[ E \left( \int_0^{\tau \wedge t} c(s) \|q_s\| \, ds \right)^2 \leq \int_0^{\tau} c(s)^2 \, ds, \]

etc. Hence
\[ E \sup_{t \leq T} |X_{t \wedge t}|^2 \leq 4^2 \sum_{i=1}^4 \sup_{r \leq T} |I_i| \leq k_T < \infty. \]

**Lemma 2.** Assume that all conditions in Lemma 1 hold and
\[ \Gamma_0 \langle x_1 - x_2, b(t, x_1, q_1, p_1, \omega) - b(t, x_2, q_2, p_2, \omega) \rangle \]
\[ \leq c(t)(\rho(|x_1 - x_2|^2) + |x_1 - x_2|(|q_1 - q_2| + \|p_1 - p_2\|)), \]
for all \((x_i, q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^d \times L_2(\mathbb{R}^d), i = 1, 2,\)

where \(c(t)\) has the same property as that in 1° of Theorem 1, and \(\rho(u)\) has the \(\bar{A}\) property.

Then the solution of (1) is unique.

**Proof.** Let
\[ X_t = x_{t}^1 - x_{t}^2, \quad Q_t = q_{t}^1 - q_{t}^2, \quad P_t = p_{t}^1 - p_{t}^2. \]

Then by the Ito formula for \(X_t\) as in (3) one gets that
\[ Z_t = E \left( |X_{t \wedge t}|^2 + \frac{1}{2} \int_{t \wedge t}^T \|Q_s\|^2 \, ds + \frac{1}{2} \int_{t \wedge t}^T \|P_s\|^2 \, ds \right) \]
\[ \leq E \int_{t \wedge t}^T \left( 2\rho(|X_s|^2) + 5|X_s|^2 \right) c_1(s) \, ds \]
\[ \leq \int_t^T \rho_1(Z_s)c_1(s) \, ds, \]

where
\[ \rho_1(u) = 2\rho(u) + 5u, \quad c_1(s) = c(s) + c(s)^2. \]
Hence (see Situ, 1985)

\[ Z_t = 0, \text{ for all } t \in [0, T]. \]

It implies that for all \( 0 \leq t \leq T \)

\[ E|X_{t \wedge r}|^2 = 0, \quad E \int_0^r \left\| Q_s \right\|^2 \, ds = 0, \quad E \int_0^r \left\| p_s \right\|^2 \, ds = 0. \]

**Theorem 2.** If in Theorem 1

\[ b_1 = 0, \]

then (1) has a unique solution \((x_t, q_t, p_t)\).

Theorem 2 will be proved in the next section. Let us prove Theorem 1 now.

**Proof of Theorem 1.** For simplicity we assume that \( b_2 = 0 \). (In case \( b_2 \neq 0 \) we can just smooth out \( b_1 \) and proceed as follows.) Let us smooth out \( b \) to get \( b^n \), i.e. let

\[ J_d(u) = \begin{cases} 
  c_d \exp \left( - (1 - |u|^2)^{-1} \right) & \text{for } |u| < 1, \\
  0 & \text{otherwise, for all } u \in \mathbb{R}^d,
\end{cases} \]

such that constant \( c_d \) satisfies

\[ \int_{\mathbb{R}^d} J_d(u) \, du = 1; \]

and for \( u \in \mathbb{R}^{d \otimes r} \) (it is viewed as a \( d \cdot r \)-dimensional vector), \( J_{d^r}(u) \) is similarly defined. Set for \( x \in \mathbb{R}^d, q \in \mathbb{R}^{d \otimes r} \)

\[ J(x, q) = J_d(x) \cdot J_{d^r}(q). \]

Define

\[ b^n(t, x, q, p, \omega) = \int_{\mathbb{R}^d \times \mathbb{R}^d} b(t, x - n^{-1} \bar{x}, q - n^{-1} \bar{q}, p, \omega) J(\bar{x}, \bar{q}) \, d\bar{x} \, d\bar{q}. \]

Since for each \( n = 1, 2, \ldots \) it is easily seen that

\[ |b^n(t, x_1, q_1, p_1, \omega) - b^n(t, x_2, q_2, p_2, \omega)| \leq k_4 c(t) (|x_1 - x_2| + \|q_1 - q_2\| + \|p_1 - p_2\|), \]

as \((x_i, q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^{d \otimes r} \times L^2_{\text{lip}}(\mathbb{R}^d), i = 1, 2\). Hence by Theorem 2 there exists a unique solution \((x^n_t, q^n_t, p^n_t)\) of the following BSDE (denote \( t^r = t \wedge \tau \) below)

\[ x^n_t = X + \int_{t^r}^r b^n(s, x^n_s, q^n_s, p^n_s, \omega) \, ds - \int_{t^r}^r q^n_s \, dw_s - \int_{t^r}^r \int_z ^r p^n_s N(ds, dz). \]
By the Ito formula

$$
|X^n_t - X^s_t|^2 = 2 \int_t^s (X^n_s - X^s_s) \cdot (b^n(s, X^n_s, Q^n_s, P^n_s, \omega) - b^n(s, X^s_s, Q^s_s, P^s_s, \omega)) \, ds
$$

$$
- 2 \int_t^s (X^n_s - X^s_s) \cdot (Q^n_s - Q^s_s) \, dw_s - \int_t^s \|Q^n_s - Q^s_s\|^2 \, ds
$$

$$
2 \int_t^s \int_Z (X^n_s - X^s_s) \cdot (P^n_s - P^s_s) \tilde{N}(ds, dz)
$$

$$
- \int_t^s \int_Z |P^n_s - P^s_s|^2 N_k(ds, dz) = \sum_{i=1}^5 I_i.
$$

Note that

$$
I_1 = 2 \int_t^s (X^n_s - X^s_s) \cdot \int (b(s, X^n_s - m^{-1} \bar{x}, Q^n_s - m^{-1} \bar{q}, P^n_s, \omega)) J(\bar{x}, \bar{q}) \, d\bar{x} \, d\bar{q} \, ds
$$

$$
\leq 2 \int_t^s ds \int ((\rho(|X^n_s - X^s_s - (m^{-1} - n^{-1})\bar{x}|^2) + |X^n_s - X^s_s - (m^{-1} - n^{-1})\bar{x}|)
$$

$$
\times (\|Q^n_s - Q^s_s - (m^{-1} - n^{-1})\bar{q}\| + \|P^n_s - P^s_s\|) c(s)
$$

$$
+ |m^{-1} - n^{-1}| |\bar{x}| 2c(s)) J(\bar{x}, \bar{q}) \, d\bar{x} \, d\bar{q}.
$$

Since by Lemma 1 for all $n$

$$
E\left(\sup_{t \in [0, T]} |X^n_t|^2 + \int_0^t \|Q^n_s\|^2 \, dt + \int_0^t \|P^n_s\|^2 \, dt\right) \leq k_T < \infty.
$$

Hence

$$
E\left(|X^n_t - X^s_t|^2 + \int_t^s \|Q^n_s - Q^s_s\|^2 \, ds + \int_t^s \|P^n_s - P^s_s\|^2 \, ds\right)
$$

$$
\leq k_T \int_t^T \int (\rho(E|X^n_s - X^s_s - (m^{-1} - n^{-1})\bar{x}|^2) + E|X^n_s - X^s_s|^2)
$$

$$
\times J(\bar{x}, \bar{q}) \, d\bar{x} \, d\bar{q} \, c_1(s)^2 \, ds + k_T (m^{-1} + n^{-1}),
$$

where $c_1(t)^2 = c(t)^2 + c(t)$. Note that

$$
\rho(2E|x^n_t - x^s_t|^2 + 2(m^{-1} - n^{-1})^2|\bar{x}|^2) \leq \rho(2k_T + 2|\bar{x}|^2).
$$

But by assumption it yields that

$$
\int \rho(2k_T + 2|\bar{x}|^2)J(\bar{x}, \bar{q}) \, d\bar{x} \, d\bar{q} \leq \rho(2K_T + 2) < \infty.
$$
Hence by Lemma 1 and by the Fatou lemma it is easily seen that

\[
\lim_{m,n \to \infty} E|x^n_{t\tau} - x^n_{t\tau}|^2 + \lim_{m,n \to \infty} E \int_0^T \left( |q^n_s - q^n_s|^2 + \|p^n_s - p^n_s\|^2 \right) ds \\
\leq \tilde{k}_T \int_0^T \rho_1 \left( \lim_{m,n \to \infty} 2E|x^n_{t\tau} - x^n_{t\tau}|^2 \right) c_1(s)^2 ds,
\]

where

\[
\rho_1(u) = \rho(u) + u,
\]

since \(\rho_1(u)\) is increasing, therefore (cf. Situ, 1985)

\[
\lim_{m,n \to \infty} E|x^n_{t\tau} - x^n_{t\tau}|^2 = 0, \quad \text{for all } t \in [0, T], \quad t' = t \wedge \tau
\]

and

\[
\lim_{m,n \to \infty} E \int_0^T \left( |q^n_s - q^n_s|^2 + \|p^n_s - p^n_s\|^2 \right) ds = 0.
\]

By Lemma 1

\[
E \sup_{t \leq \tau} |x^n_{t\tau}|^2 \leq K_T < \infty.
\]

Hence by the Lebesgue domination convergence theorem as \(m, n \to \infty\)

\[
E \int_0^T |x^n_{t\tau} - x^n_{t\tau}|^2 dt \leq E \int_0^T |x^n_{t\tau} - x^n_{t\tau}|^2 dt \\
= \int_0^T E|x^n_{t\tau} - x^n_{t\tau}|^2 dt \to 0.
\]

Let now for \((x., q., p.) \in L^2_{(\mathbf{F}_\tau)} ([0, \tau]; \mathbb{R}_r^d) \times L^2_{(\mathbf{F}_\tau)} ([0, \tau]; \mathbb{R}_r^{d \otimes r}) \times \mathcal{S}^2_{k,(\mathbf{F}_\tau)} ([0, \tau]; \mathbb{R}_r^d) = \tilde{B}\)

\[
\| (x., q., p.) \| = \| x. \|_1 + \| q. \|_2 + \| p. \|_3,
\]

where

\[
\| x. \|_1 = E \int_0^T |x_s|^2 ds = \int_{[0, T] \times \Omega} |x_s(\omega)|^2 d\mu(s, \omega),
\]

\[
\| q. \|_2 = E \int_0^T \| q_s \|^2 ds = \int_{[0, T] \times \Omega} \| q_s(\omega) \|^2 d\mu(s, \omega),
\]

\[
\| p. \|_3 = E \int_0^T \| p_s \|^2 ds = \int_{[0, T] \times \Omega} \| p_s(\omega) \|^2 d\mu(s, \omega),
\]

where

\[
d\mu(t, \omega) = I_{(0 < t \leq \tau(\omega))(t, \omega)} dt P(d\omega).
\]
Then \((\tilde{B}, \| \cdot \|)\) is a Banach space. By the completeness of \(\tilde{B}\) there exists a unique \((x_\omega, q_\omega, p_\omega) \in \tilde{B}\) such that as \(n \to \infty\)
\[\| (x^n_\omega, q^n_\omega, p^n_\omega) - (x_\omega, q_\omega, p_\omega) \| \to 0.\]

Therefore there exists a subsequence \(\{n_k\}\) of \(\{n\}\), denote it by \(\{n\}\) again such that as \(n \to \infty\)
\[x^n_\omega(\omega) \to x_\omega(\omega), \quad \text{in } \mathbb{R}^d,\]
\[q^n_\omega(\omega) \to q_\omega(\omega), \quad \text{in } \mathbb{R}^d \otimes \mathcal{F},\]
\[p^n_\omega(\omega) \to p_\omega(\omega), \quad \text{in } L^2(\mathcal{F}_\infty(\mathbb{R}^d), \mu\text{-a.e. } (t, \omega) \in [0, T] \times \Omega.\]

Hence as \(n \to \infty\)
\[b^n(s, x^n_\omega(\omega), q^n_\omega(\omega), p^n_\omega(\omega), \omega) \to b(s, x_\omega(\omega), q_\omega(\omega), p_\omega(\omega), \omega),\]
\(\mu\text{-a.e. } (s, \omega) \in [0, T] \times \Omega.\)

In fact:
\[\int |b^n(s, x^n_\omega(\omega), q^n_\omega(\omega), p^n_\omega(\omega), \omega) - b(s, x_\omega(\omega), q_\omega(\omega), p_\omega(\omega), \omega)| J(\tilde{x}, \tilde{q}) \, d\tilde{x} \, d\tilde{q} = I_n.\]

But by assumption 1° and the continuity of \(b = b_1\) in \((x, q, p)\) as \(n \to \infty\)
\[I_n \to 0.\]

Since now we know that \(|b^n(t, \omega)|\) and \(|b(t, \omega)|\) are smaller than \(c(t)\), by taking the limit in \(L^1(\Omega, \mathcal{F}, P)\) one obtains that \((x_\omega, q_\omega, p_\omega)\) is a solution of (1). The uniqueness of solution for (1) is derived by Lemma 2. \(\square\)

3. Proof of Theorem 2 and convergence theorems of solutions

In this section we are going to prove Theorem 2. For this we first discuss a simpler BSDE as follows:
\[x_t = X + \int_{t \wedge \tau}^\tau f(s, \omega) \, ds - \int_{t \wedge \tau}^\tau q_s \, dw_s - \int_{t \wedge \tau}^\tau \int Z p_s(z) \tilde{N}_s(\omega) \, ds \, dz, \quad t \geq 0, \quad (7)\]
we have

Lemma 3. If \(\tau\) is a bounded stopping time, \(X\) is a \(\mathcal{F}_\tau\)-measurable and \(\mathbb{R}^d\)-valued random variable, \(f(t, \omega)\) is a \(\mathcal{F}_t\)-adapted and \(\mathbb{R}^d\)-valued random process such that
\[E|X|^2 < \infty, \quad E\int_0^\tau |f(s, \omega)|^2 \, ds < \infty,\]
then (7) have a unique solution.
**Proof.** Let

\[ x_t = E \left( X + \int_0^t f(s) \, ds \bigg| \mathcal{F}_{t\wedge \tau} \right). \]

This makes sense, since by assumption

\[ E \left| X + \int_0^t f(s) \, ds \right|^2 < \infty \quad \text{for all } t \geq 0. \]

Hence

\[ x_0 = E \left( X + \int_0^\tau f(s) \, ds \bigg| \mathcal{F}_0 \right). \]

Note that

\[ M_t = E \left( X + \int_0^t f(s) \, ds \bigg| \mathcal{F}_t \right) \]

is a square integrable martingale. Hence by the martingale representation theorem (e.g. see [Tang and Li, Lemma 2.3]) there exists a unique \((q_t, p_t)\) such that

\[ M_t = M_0 + \int_0^t q_s \, dw_s + \int_0^t p_s(z) \tilde{N}_k(ds, dz), \quad t \geq 0, \quad (\ast) \]

and

\[ E \int_0^T (\|q_s\|^2 + \|p_s\|^2) \, ds < \infty \quad \text{as } t < \infty. \]

But

\[ M_{t\wedge \tau} = E \left( X + \int_0^{t\wedge \tau} f(s) \, ds \bigg| \mathcal{F}_{t\wedge \tau} \right) = \int_0^{t\wedge \tau} f(s) \, ds + x_t, \]

and by \((\ast)\)

\[ X + \int_0^t f(s) \, ds = x_0 + \int_0^t q_s \, dw_s + \int_0^t \int_Z p_s(z) \tilde{N}_k(ds, dz). \]

Hence

\[
\begin{align*}
X + \int_{t\wedge \tau}^\tau f(s) \, ds - \int_{t\wedge \tau}^\tau q_s \, dw_s - \int_{t\wedge \tau}^\tau \int_Z p_s(z) \tilde{N}_k(ds, dz) \\
= x_0 + \int_0^{t\wedge \tau} q_s \, dw_s + \int_0^{t\wedge \tau} \int_Z p_s(z) \tilde{N}_k(ds, dz) - \int_0^\tau f(s) \, ds \\
= M_{t\wedge \tau} - \int_0^{t\wedge \tau} f(s) \, ds = x_t.
\end{align*}
\]
It is proved that \((x_t, q_t, p_t)\) satisfies (7). Now by the Jensen inequality
\[
E \int_0^T |x_t|^2 \, dt = E \int_0^T \left| E \left( X + \int_{t \wedge \tau}^t f(s) \, ds \right) \right|^2 \, dt \\
\leq \int_0^T E \left| X + \int_{t \wedge \tau}^t f(s) \, ds \right|^2 \, dt < \infty,
\]
where we assume that \(0 \leq \tau \leq T\). Note also that
\[
E \int_0^T (\|q_s\|^2 + \|p_s\|^2) \, ds \leq E \int_0^T (\|q_s\|^2 + \|p_s\|^2) \, ds < \infty.
\]
Hence \((x_t, q_t, p_t)\) is a solution of (7).

The uniqueness of solution to (7) is derived by the Ito formula. \(\Box\)

Now we are in a position to prove Theorem 2.

**Proof of Theorem 2.** Similarly to [Tang and Li, 1994] let us use the contractive mapping principle to show this result. Introduce a new norm as follows:

For \((x_i, q_i, p_i) \in \bar{B} (\bar{B} \text{ is defined in (6)}) let
\[
\| (x_i, q_i, p_i) \|^2_M = \sup_{t \in \mathcal{T}} e^{-\lambda(t)} E |x_{t \wedge \tau}|^2 + \sup_{t \in \mathcal{T}} e^{-\lambda(t)} E \int_{t \wedge \tau}^t \|q_s\|^2 \, ds \\
+ \sup_{t \in \mathcal{T}} e^{-\lambda(t)} E \int_{t \wedge \tau}^t \|p_s\|^2 \, ds,
\]
where \(\lambda \geq 0\) is a constant, which will be determined later, and
\[
A(t) = \int_{t \wedge \tau}^T c(s)^2 \, ds.
\]
Denote
\[
H = \{(x_i, q_i, p_i) \in \bar{B} : \| (x_i, q_i, p_i) \|^M < \infty \}.
\]
Then \((H, \| \cdot \|_M)\) is a Banach space.

Now by Lemma 3 for any \((\bar{x}_i, \bar{q}_i, \bar{p}_i) \in H, i = 1, 2\), there exist unique solutions \((x_i, \bar{q}_i, \bar{p}_i), i = 1, 2\), of the following BSDE
\[
x_i = X + \int_{t \wedge \tau}^t b(s, x_s, q_s, p_s, \omega) \, ds - \int_{t \wedge \tau}^t q_s \, dw_s - \int_{t \wedge \tau}^t \int Z p_s(s) \, d\tilde{N}_s (ds, dz),
\]
for all \(t \geq 0, i = 1, 2\);

moreover, by Lemma 1 \((x_i, q_i, p_i) \in H, i = 1, 2\). By the Ito formula
\[
E \left( |x_{t \wedge \tau} - x_{t \wedge \tau}|^2 + \int_{t \wedge \tau}^t \|q_s|^2 \, ds + \int_{t \wedge \tau}^t \|p_s|^2 \, ds \right) \\
\leq \gamma^{-1} E \left( \int_{t \wedge \tau}^t |\bar{x}_s - \bar{x}_s|^2 \, ds + \int_{t \wedge \tau}^t \|\bar{q}_s - \bar{q}_s|^2 \, ds \right)
\]
\[ + \int_{t \wedge \tau}^{\tau} \left\| \tilde{p}_t^1 - \tilde{p}_s^2 \right\|^2 ds + \hat{\gamma} E \int_{t \wedge \tau}^{\tau} c(s)^2 |x_t^1 - x_s^2|^2 ds \]

\[ = \hat{\gamma}^{-1} I_t^1 + \hat{\gamma} E \int_{t \wedge \tau}^{\tau} c(s)^2 |x_t^1 - x_s^2|^2 ds \]

\[ \leq \hat{\gamma}^{-1} I_t^1 + \hat{\gamma} \int_{t}^{T} c(s)^2 E(|x_{t \wedge \tau}^1 - x_{s \wedge \tau}^2|^2) ds \]

\[ \leq \hat{\gamma}^{-1} I_t^1 + \int_{t}^{T} \exp(\hat{\gamma}(A(t) - A(s))) c(s)^2 I_t^1 ds, \]

where we have applied the following result.

**Lemma 4.** (Gronwall inequality). If \( 0 < y_t \leq \hat{\gamma}^{-1} v_t + \hat{\gamma} \int_{t}^{T} c(s)^2 y_s ds, \ t > 0, \) where \( \hat{\gamma} > 0 \) is a constant, then

\[ y_t \leq \hat{\gamma}^{-1} v_t + \int_{t}^{T} \exp \left( \hat{\gamma} \int_{t}^{s} c(r)^2 dr \right) c(s)^2 v_s ds, \ t \geq 0. \]

Denote

\[ u_t = \| \tilde{x}_t^1 - \tilde{x}_t^2 \|^2. \]

Note that \( A(t) \) is decreasing, then

\[ e^{-b \cdot A(t)} E \int_{t \wedge \tau}^{\tau} u_s ds \leq e^{-A(t) b} \int_{t}^{T} E u_{t \wedge \tau} ds \]

\[ \leq T \sup_{s \leq T} e^{-b A(s)} E u_{s \wedge \tau}, \]

\[ e^{-b \cdot A(t)} \int_{t}^{T} e^{\hat{\gamma}(A(t) - A(s))} c(s)^2 E \int_{s \wedge \tau}^{\tau} u_r dr ds \]

\[ \leq \int_{t}^{T} e^{-(b - \hat{\gamma})(A(t) - A(s))} c(s)^2 ds T \sup_{r \leq T} e^{-b A(r)} E u_{r \wedge \tau} \]

\[ \leq T \sup_{r \leq T} e^{-b A(r)} E u_{r \wedge \tau} (b - \hat{\gamma})^{-1}, \quad \text{for } 0 < \hat{\gamma} < b. \]

Denote now \( u_t = E \int_{t \wedge \tau}^{\tau} \| \tilde{q}_t^1 - \tilde{q}_t^2 \|^2 ds \) or \( E \int_{t \wedge \tau}^{\tau} \| \tilde{p}_t^1 - \tilde{p}_s^2 \|^2 ds. \) Then

\[ e^{-b \cdot A(t)} \int_{t}^{T} e^{\hat{\gamma}(A(t) - A(s))} c(s)^2 u_s ds \]

\[ \leq \sup_{s \leq T} e^{-b A(s)} u_s \int_{t}^{T} e^{-(b - \hat{\gamma})(A(t) - A(s))} c(s)^2 ds \]

\[ \leq \sup_{s \leq T} e^{-b A(s)} u_s (b - \hat{\gamma})^{-1}. \]
Hence
\[ \|(x^1. - x^2., q^1. - q^2., p^1. - p^2.)\|^2 \]
\[ \leq \text{Max}(3\hat{\tau}^{-1}(T + 1), 3(b - \hat{\tau})^{-1}(T + 1))\|(\hat{x}^1. - \hat{x}^2., \hat{q}^1. - \hat{q}^2., \hat{p}^1. - \hat{p}^2.)\|^2. \]

After appropriately choosing \( \hat{\tau} \) and \( b \), one can easily complete the proof. \( \square \)

Now we have the following convergence theorem for solutions of BSDE (1).

**Theorem 3.** Assume that \( \tau \) is a \( \mathcal{F}_\tau \)-stopping time for \( n = 0, 1, 2, \ldots \)

1° \( b^n = b^n(t, x, q, p, \omega) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \otimes r} \times L^2_{\mathcal{F}_\tau}(\mathbb{R}^d) \rightarrow \mathbb{R}^d \) are \( \mathcal{F}_\tau \)-adapted such that \( P \)-a.s.

\[ \langle x, b^n(s, x, q, p, \omega) \rangle \leq c(s)(1 + |x|^2 + |x|(|q| + ||p||)), \]

where \( c(t) \) has the property as that of 1° in Theorem 1;

2° \( \langle x_1 - x_2, b^0(t, x_1, q_1, p_1, \omega) - b^0(t, x_2, q_2, p_2, \omega) \rangle \)

\[ \leq c(t)(\rho(|x_1 - x_2|^2) + |x_1 - x_2|(|q_1 - q_2| + ||p_1 - p_2||)), \] \( P \)-a.s.,

for all \( (x_i, q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^{d \otimes r} \times L^2_{\mathcal{F}_\tau}(\mathbb{R}^d), i = 1, 2, \)

where \( \rho(u) \) has the same property stated in 2° of Theorem 1,

3° \( X^n \) is \( \mathcal{F}_{T \land \tau} \)-adapted for all \( n = 0, 1, 2, \ldots \), and

\[ E(X^n - X^0)^2 \rightarrow 0, \] \( n \rightarrow \infty, E|X^n|^2 < \infty; \)

4° \( \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d, q \in \mathbb{R}^{d \otimes r}, p \in L^2_{\mathcal{F}_\tau}(\mathbb{R}^d), \omega \in \Omega} \int_0^T |b^n(t, x, q, p, \omega) - b^0(t, x, q, p, \omega)|^2 \, dt = 0. \)

If \( (x^n_t, q^n_t, p^n_t) \) are solutions of the following BSDEs: as \( 0 \leq s \leq T \)

\[ x^n_{s, \tau} = X^n + \int_s^{T \wedge \tau} b^n(r, x^n_r, q^n_r, p^n_r, \omega) \, dr - \int_s^{T \wedge \tau} q^n_r \, dw_r - \int_s^{T \wedge \tau} p^n_r \tilde{N}_k(dr, dz), \]

\[ n = 0, 1, 2, \ldots, \]

then for all \( 0 \leq s \leq T \)

\[ \lim_{n \rightarrow \infty} E \left( |x^n_{s, \tau} - x^0_{s, \tau}|^2 + \int_s^{T \wedge \tau} (\|q^n_r - q^0_r\|^2 + \|p^n_r - p^0_r\|^2) \, dr \right) = 0. \]

**Proof.** By the Ito formula (3) one has

\[ E \left( |x^n_{s, \tau} - x^0_{s, \tau}|^2 + \int_s^{T \wedge \tau} (\|q^n_r - q^0_r\|^2 + \|p^n_r - p^0_r\|^2) \, dr \right) \]

\[ \leq k_0 \left( \int_s^T c_1(r)^2 \rho_1(E|x^n_{r, \tau} - x^0_{r, \tau}|^2) \, dr + E|X^n - X^0|^2 \right) \]

\[ + k_0 E \int_s^T |b^n(r, x^n_r, q^n_r, p^n_r, \omega) - b^0(r, x^n_r, q^n_r, p^n_r, \omega)|^2 \, dr, \]
where
\[ c_1(r)^2 = c(r)^2 + c(r) + 1, \quad \rho_1(u) = \rho(u) + u. \]
Hence
\[ \lim_{n \to \infty} E|x^n_{\tau} - x^0_{\tau}|^2 \leq k_0 \int T \ C_1(r)^2 \rho_1 \left( \lim_{n \to \infty} E|x^n_{\tau} - x^0_{\tau}|^2 \right) \text{d}r. \]
Therefore (cf. Situ, 1985)
\[ \lim_{n \to \infty} E|x^n_{\tau} - x^0_{\tau}|^2 = 0. \]
The other conclusion now is easily derived. \( \square \)

We also have some other useful convergence theorem as follows:

**Theorem 4.** Assume that for all \((x, q, p), (y, \bar{q}, \bar{p}) \in \mathbb{R}^d \times \mathbb{R}^{d^2} \times L_{\text{fin}}^2(\mathbb{R}^d)\)
\[ 1^\circ \ |b^n(t, x, q, p, \omega)| \leq k_0 (1 + |x| + \|q\| + \|p\|), \quad n = 0, 1, \ldots, \text{where} \quad k_0 \geq 0 \]
is a constant;
\[ 2^\circ \ \langle x - y, b^n(t, x, q, p, \omega) - b^n(t, y, \bar{q}, \bar{p}, \omega) \rangle \leq c(t)(\rho(|x - y|^2)) \]
\[ + |x - y|((\|q - \bar{q}\| + \|p - \bar{p}\|)), \text{P-a.s.} \quad n = 1, 2, \ldots \]
where \(c(t)\) and \(\rho(u)\) have the same property as that in \(2^\circ\) of Theorem 3;
\[ 3^\circ \ \text{The same as} \ 3^\circ \text{ in Theorem 3}; \]
\[ 4^\circ \ \lim_{n \to \infty} b^n(t, x, q, p, \omega) = b^0(t, x, q, p, \omega), \text{P-a.s.} \]
Then the conclusion of Theorem 3 still holds.

The proof can be completed similarly as that of Theorem 3.

### 4. Application to PDIE

Applying the idea of Peng (1991) to explain the solution of partial differential equations (PDE) by the solution of BSDE one can have a probabilistic interpretation for the solution of partial differential and integral equations (PDIE) by means of the solution of BSDE with jumps.

For this we need some preparation on PDIE.

Let \(D\) be a bounded domain in \(\mathbb{R}^d\), \(\partial D\) be the boundary of \(D\), \(D^c = \mathbb{R}^d - D\).

Consider the following first boundary problem (8):
\[
\mathcal{L}_{k, \sigma, \epsilon} u(t, x) \equiv \partial u(t, x)/\partial t + \sum_{i=1}^d b_i(t, x) \partial u(t, x)/\partial x_i \\
+ \frac{1}{\epsilon} \sum_{i,j=1}^d a_{ij}(t, x) \partial^2 u(t, x)/\partial x_i \partial x_j
\]
\[ + \int_{\mathbb{Z}} \left( u(t, x + c(t, x, z)) - u(t, x) - \sum_{i=1}^{d} c_i(t, x, z) \frac{\partial u(t, x)}{\partial x_i} \right) \Pi(dz) \]

\[ = f(t, x, u(t, x), u'(t, x) \cdot \sigma(t, x), u(t, x + c(t, x, \cdot)) - u(t, x)), \]

\[ t \in [0, T), \quad x \in D; \quad (8)_1 \]

\[ u(T, x) = \varphi(x), \quad u(t, x)|_{D'} = \psi(t, x), \quad \psi(T, x) = \varphi(x)|_{D'}, \quad (8)_2 \]

\[ u \in W_p^{1,2}([0, T] \times \mathbb{R}^d) = W_p^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^m), \]

where for coefficients \( b, \sigma, c \) one makes the following assumption (A):

A.1. \( b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \)

is bounded measurable;

A.2. \( \sigma: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \)

is bounded, measurable; continuous in \( x \) (uniformly with respect to \( t \)), and continuous in \( t \) such that there exist constants \( \delta_0 \) and \( k_0 > 0 \) satisfying (here we denote \( a = \sigma \sigma^* = (a_{ij}) \))

\[ \delta_0|\lambda|^2 \leq \langle \lambda, a\lambda \rangle \leq k_0|\lambda|^2 \quad \text{for all} \quad \lambda \in \mathbb{R}^d; \]

A.3. for arbitrary \( G \subset \mathbb{R}^d \)-bounded domain there exists a measure \( \Pi(\cdot) \) such that as \( (s, x) \in [0, T] \times G \)

\[ \Pi(s, x, A) = \int I_A(c(s, x, z)) \Pi(dz) \leq \tilde{\Pi}(A), \]

\[ \int |y|^2 \Pi(dy) < \infty, \quad \int |y| \Pi(dy) < \infty; \]

and

\[ \lim_{x \rightarrow x'} \int_{\mathbb{Z}} (|c(s, x, z)|^2 - |c(s, x', z)|^2) \Pi(dz) = 0; \]

for functions \( f, \varphi, \psi \) one makes the following assumption (B):

B.1. \( f: [0, T] \times D \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow L_p^2(\mathbb{R}^m) \rightarrow \mathbb{R}^m \)

is measurable such that for all \( u \in W_p^{1,2}([0, T] \times \mathbb{R}^d) \) satisfying (8)_2

\[ g^u \in L_p([0, T] \times D), \]

where

\[ g^u(t, x) = f(t, x, u(t, x), u'(t, x) \cdot \sigma(t, x), u(t, x + c(t, x, \cdot)) - u(t, x)); \]

and for arbitrary \( r_1, r_2 \in \mathbb{R}^m; q_1, q_2 \in \mathbb{R}^m \otimes \mathbb{R}; p_1, p_2 \in L_p^2(\mathbb{R}^m) \)

\[ |f(t, x, r_1, q_1, p_1(\cdot)) - f(t, x, r_2, q_2, p_2(\cdot))| \]

\[ \leq k_0(|r_1 - r_2| + \|q_1 - q_2\| + \|p_1 - p_2\|), \]
Theorem 5. Under the assumptions (A) and (B) if \( p > d + 2 \), then (8) has a unique solution \( u(t, x) \in W_p^{1,2}([0, T] \times \mathbb{R}^d) \) such that

\[
\| u \| \leq C_0 (\| g^\sigma \|_{L_p([0, T] \times D)} + \| \psi \|_{W_p^{1,2}([0, T] \times D^e)} + \| \varphi \|_{W_p^{1,1}\cap L_p(\mathbb{R}^d)}),
\]

where \( C_0 \) is a constant.

Remark 1. In case condition B.1 is strengthened to be

B.1' \( f: [0, T] \times D \to \mathbb{R}^m \) such that

\[
f \in L_p([0, T] \times D);
\]

then Theorem 5 becomes Theorem 2 of Marhno (1976) in the case of \( u(t, x) \)-valued in \( m \)-dimensional space. Hence Theorem 5 is a generalization of it.

Proof of Theorem 5. For each \( u_i \in W_p^{1,2}([0, T] \times \mathbb{R}^d) \) satisfying (8)_2 by Theorem 2 of Marhno (1976) there exists a unique \( \tilde{u}_i \in W_p^{1,2}([T-h, T] \times \mathbb{R}^d) \) such that

\[
\mathcal{F}_{b,a,c} \tilde{u}_i = f(t, x, u_i(t, x), u_{1x}(t, x) \cdot \sigma(t, x), u_i(t, x + c(t, x, \cdot)) - u_i(t, x)),
\]

for all \( t \in [T-h, T] \), \( x \in D \),

\[
\tilde{u}_i(T, x) = \varphi(x), \quad \tilde{u}_i(t, x)|_{D^e} = \psi(t, x), \quad \psi(T, x) = \varphi(x)|_{D^e}.
\]

Moreover,

\[
\| \tilde{u}_1 - \tilde{u}_2 \|_{W_p^{2,2}([T-h, T] \times D)} \leq C_0 (\| g^{u_1} - g^{u_2} \|_{L_p([T-h, T] \times D)} + \| u_1 - u_2 \|_{L_p([T-h, T] \times D^e)} + \| u_{1x} - u_{2x} \|_{L_p([T-h, T] \times D^e)} + \| u_{1x} - u_{2x} \|_{L_p([T-h, T] \times D^e)} + \| u_{1x} - u_{2x} \|_{L_p([T-h, T] \times D^e)} + \| u_{1x} - u_{2x} \|_{L_p([T-h, T] \times D^e)} + \| u_{1x} - u_{2x} \|_{L_p([T-h, T] \times D^e)} + \| u_{1x} - u_{2x} \|_{L_p([T-h, T] \times D^e)})
\]

\[
= \sum_{i=1}^3 I_i,
\]

where \( \| u_{ix} \| \) is the norm of \( m \times r \)-matrix.
By the Sobolev imbedding theorem (e.g. see Ladyzenskaja et al., 1968, Lemma 4.5, p. 305)

\[ I_1^{1/p} \leq c h^1 \| u_1 - u_2 \|_{W^{1,2}_p([T-h, T] \times D)} . \]

Similarly,

\[ I_2^{1/p} \leq c h^{1/2} \| u_1 - u_2 \|_{W^{1,2}_p([T-h, T] \times D)} . \]

Moreover,

\[ I_3 \leq c \sup_{t \in [T-h, T], x \in D} |u'_1 - u'_2| |p(t, x) \text{ meas}(D)| \int |y|^2 \overline{\Pi}(dy)|^{p/2} h \]

\[ \leq \overline{k}_0 \| u_1 - u_2 \|_{W^{1,2}_p([T-h, T] \times D)} \cdot h^{1+\lambda_p}, \]

where \( \overline{k}_0 \) is independent of \( \lambda \) and \( h \), and

\[ 0 < 2\lambda = 1 - (d + 2)/p < 1, \]

and we have applied Lemmas II.3.3 and IV.4.5 of [4]. (See Ladyzenskaja et al., 1968, p. 80 and p. 305).

It yields that as \( 0 < h < 1 \)

\[ \| \tilde{u}_1 - \tilde{u}_2 \|_{W^{1,2}_p([T-h, T] \times D)} \leq \overline{k}_0 \| u_1 - u_2 \|_{W^{1,2}_p([T-h, T] \times D)} \cdot h. \]

On the other hand,

\[ \| \tilde{u}_1 - \tilde{u}_2 \|_{W^{1,2}_p([T-h, T] \times \mathbb{R}^d)} = 0. \]

Therefore

\[ \| \tilde{u}_1 - \tilde{u}_2 \|_{W^{1,2}_p([T-h, T] \times \mathbb{R}^d)} \leq k_0 \| u_1 - u_2 \|_{W^{1,2}_p([T-h, T] \times \mathbb{R}^d)} \cdot h. \]

Now take \( h > 0 \) small enough, then there exists a unique

\[ u(t, x) \in W^{1,2}_p([T-h, T] \times \mathbb{R}^d) \]

satisfying (8) as \( t \in [T-h, T], x \in D \), and satisfying (8) on \( [T-h, T] \times \mathbb{R}^d \). From this it is not difficult to derive the conclusion. \( \square \)

Consider now the forward SDE (FSDE) with Poisson jumps in \( \mathbb{R}^d \) for any given \( (t, x) \in [0, T] \times D \)

\[ y_s = x + \int_t^s b(r, y_r) \, dr + \int_t^s \sigma(r, y_r) \, dw_r + \int_t^s \int_Z c(r, y_r, z) \overline{N}_k(dr, dz), \]

\[ t \leq s \leq T, \]

(9)
where $\tilde{N}_k$ is as in (2). By Theorem 1 of [Marhno, 1976] under assumption (A) (but the following condition is not necessary:

$$\sigma(s, x) \text{ is continuous in } s$$

$$\int |y| \tilde{N}(dy) < \infty,$$

see [Marhno, 1976]), (9) has a weak solution unique in law. Denote now

$$\mathcal{F}_{t,s} = \sigma(w(r) - w(t), N_k(t, r), U) \quad \text{for all } t \leq r \leq s, \ U \in \mathcal{B}(Z).$$

Then (9) has a pathwise unique strong solution $y_{s,x}^t$ ("strong" means that it is $\mathcal{F}_{t,s}$-adapted) under additional assumption (A):

$$(A)' \quad 2 \langle y_1 - y_2, b(s, y_1) - b(s, y_2) \rangle + \| \sigma(s, y_1) - \sigma(s, y_2) \|^2 + \| c(s, y_1) - c(s, y_2) \|^2 \leq k_N(s) \rho_N(|y_1 - y_2|^2) \quad \text{as } |y_1|, |y_2| \leq N,$$

for each $N = 1, 2, \ldots$,

where for each $N \rho_N(u)$ has the $\tilde{A}$ property (see Definition 2) and $k_N(s) \geq 0$ such that

$$\int_0^T k_N(s) \, ds < \infty \quad \text{for each } 0 \leq T < \infty \quad \text{(see [Situ, 1985]).}$$

For arbitrary $(t, x) \in [0, T] \times D$ let

$$\tau = \tau_x = \inf \{s > t : y_{s,x}^t \notin D\}, \quad \text{and } \tau = \tau_x = T, \quad \text{for } \inf \{\phi\},$$

and make the additional assumption (C):

C.1. $|f(s, x, r, q, p)| \leq k_0(1 + |r| + \| q \| + \| p \|).$

By Theorem 2 we have

Lemma 5. Under the assumptions (A), (A)', (B) and (C) there exists a unique

$$(x_s, q_s, p_s) \in L_{\mathcal{F},d}^2([t, \tau]; \mathbb{R}^m) \times L_{\mathcal{F},d}^2([t, \tau]; \mathbb{R}^m) \times L_{\mathcal{F},d}^2([t, \tau]; \mathbb{R}^m)$$

satisfying (from now on we denote $y_s = y_{s,x}^t$ for simplicity)

$$x_s = I_{t<\tau} \psi(\tau, y_r) + I_{t=\tau} \Phi(y_T) - \int_{s\wedge \tau}^t f(r, y_r, x_r, q_r, p_r) \, dr$$

$$- \int_{s\wedge \tau}^t q_r \, dw_r - \int_{s\wedge \tau}^t \int Z p_s(x(z)) \tilde{N}_k(dr, dz) \quad \text{as } t \leq s \leq T. \quad (10)$$

Proof. Actually, it is seen that $X = I_{t<\tau} \psi(\tau, y_r) + I_{t=\tau} (y_T)$ is $\mathcal{F}_{t,s}$ measurable and $E|X|^2 < \infty$. Therefore Theorem 2 is applied. \hfill \Box

Now we have the following

Theorem 6 (Ito formula). Under assumption (A), but the following condition is not necessary:

$$\sigma(s, x) \text{ is continuous in } s,$$
if $y_s$ satisfies (9), then for all $u(s, x) \in W^{1,2}_p([0, T] \times \mathbb{R}^d)$ as $p > 2d + 4$ one has that for $t \leq s \leq T$

$$u(s \land \tau_G, y_{s \land \tau_G}) - u(t, y_t) = \int_t^{s \land \tau_G} \mathcal{F}_{b, \sigma, c} u(r, y_r) \, dr$$

$$+ \int_t^{s \land \tau_G} u_x(r, y_r) \cdot \sigma(r, y_r) \, dw_r + \int_t^{s \land \tau_G} \int_Z (u(r, y_{r-} + c(r, y_{r-}, z))$$

$$- u(r, y_{r-})) \, \overline{N}_k(dr, dz),$$

where $G \subset \mathbb{R}^d$ is a bounded domain, and

$$\tau_G = \inf(s > t : y_s \notin G), \quad \tau_G = T \quad \text{for } \inf\{\phi\}.$$

The Ito formula for functions with Sobolev derivatives in the continuous SDE case was obtained by Krylov (1980), in SDE with Poisson jumps but for formula with mathematical expectation by Marhno (1976). Here our Ito formula is without the mathematical expectation. Hence we need to assume that $p > 2d + 4$ (but not $p > d + 2$ as in Marhno, 1976).

**Proof.** By the imbedding theorem and so on one can take $u_n \in C^{1,2}([0, T] \times \mathbb{R}^d)$ such that as $n \to \infty$

$$\sup_{t \in (0, T), x \in G} |u_n(t, x) - u(t, x)| \to 0,$$

and

$$\|u_n - u\|_{W^{1,2}_p([0, T] \times \mathbb{R}^d)} \to 0,$$

since (11) holds for $u_n$. Note that the following estimate holds: for $f \in L_p([0, T] \times G)$, $p > d + 2$, $G \subset \mathbb{R}^d$ is a bounded domain, and $\int |y| \Pi(dy) < \infty$, one has

$$E_{t,x} \int_t^{s \land \tau_G} |f(s, y_s)| \, ds \leq c_{p, T, d} \|f\|_{L_p([t, T] \times G)},$$

where we denote $E_{t,x} F(y, \omega) = EF(y, \omega) = EF(y^{t,x}, \omega)$. (See Marhno, 1976, Theorem 1). Hence it is not difficult to derive the formula for $u$ by approximation, e.g. let us show that as $n \to \infty$

$$I_n = \left| \int_t s \land \tau_G \int_Z ((u_n - u)(r, y_{r-} + c(r, y_{r-}, z)) - (u_n - u)(r, y_{r-})) \, \overline{N}_k(dr, dz) \right|$$

$$\to 0$$

(13)

Indeed, since by (12)

$$E \int_t^{s \land \tau_G} \int_Z |((u_n - u)(r, y_{r-} + c(r, y_{r-}, z)) - (u_n - u)(r, y_{r-}))|^2 \Pi(dz)dr$$

$$\leq E \int_t^{s \land \tau_G} dr \left( \int |y|^2 \Pi(dy) \int_0^1 |(u_n - u)'(r, y_{r+} + \alpha y)|^2 \, d\alpha \right)$$
\[
\begin{align*}
&\leq c_{p/2,T,d} \left( \int_{I} \int_{G} \left| y \right|^2 \Pi(dy)^{p/2} \int_{0}^{1} \left| (u_n - u)'_y (r, x + ay) \right|^p \, dx \, dr \right)^{2/p} \\
&\leq c_{p/2,T,d} \int \left| y \right|^2 \Pi(dy) \left\| u_n - u \right\|_{W^{p/2}([t,T] \times \mathbb{R}^d)}^2 \to 0 \quad \text{as} \ n \to + \infty.
\end{align*}
\]

Hence (13) holds. \[\square\]

Now we are in a position to show

**Theorem 7.** Under assumptions (A), (B), (C) and (A)' if \( p > 2d + 4 \) then \( x_t \) solves (8), i.e.

\[
x_t = E_{t,x} x_t = u(t, x),
\]

where \( u(t, x) \) satisfies (8), and \( x_t \) is the unique solution of BSDE (10).

**Proof.** Applying the Ito formula (Theorem 6) to \( u(r, y, \cdot) \) on \( r \in [s \wedge \tau, \tau] \), and using Lemma 5 the conclusion is derived. \[\square\]

**Remark.** By the proof of above and the imbedding theorem it yields that in the solution \((x_s, q_s, p_s)\) of (10) \( q_s \) has a cadlag version \( u'_d(s, y, \cdot) a(s, Y_s) \)

\[
\text{as } t \leq s \leq T, \quad \text{provided that} \quad a(s, x) \text{ is jointly continuous.}
\]

Applying Theorems 3 and 6 one can get a convergence theorem on the solutions of PDIEs, which seems to be not easy to derive directly from the PDE theory under such a weak condition.

Assume that \( u^n(t, x) \in W^{1,2}_p([0, T] \times \mathbb{R}^d), \ n = 0, 1, 2, \ldots, \) satisfy the following PDIEs

\[
\mathcal{F}_{b,a,c}(t, x) = f^n(t, x, u^n(t, x), u^\alpha(t, x) \cdot a(t, x), u^n(t, x + c(t, x, \cdot)) - u^n(t, x))
\]

\[
as t \in [0, T], \ x \in D,
\]

\[
u^n(T, x) = \varphi^n(x), \ u^n(t, x)|_{D'} = \psi^n(t, x), \psi^n(T, x) = \varphi^n(x)|_{D'},
\]

where \( D \) is a bounded domain, \( \mathcal{F}_{b,a,c} \) is defined in (8)\(_1\). Let

\[
b^n, f(t, x, q, p) = -f^n(t, y, x, q, p).
\]

We have

**Theorem 8.** Assume that all conditions in Theorem 6 hold, and assumption 1°, 2° and 4° in Theorem 3 for \( b^n, f(s, x, q, p), \ n = 0, 1, \ldots, \) are satisfied uniformly with respect to \( y \) (e.g.

\[
1° \quad \langle x, b^n, f(s, x, q, p) \rangle \leq c(s)(1 + |x|^2 + |x|(|q| + ||p||)),
\]

uniformly for \( y \in \mathbb{R}^d, \ n = 0, 1, 2, \ldots \), etc.}
Moreover, assume that for all \((t, x) \in [0, T] \times \mathbb{R}^d\)
\[\Gamma^0 I_D(x)|\psi^n(t, x)| + |\varphi^n(x)| \leq k_0, n = 0, 1, \ldots\]
and
\[\Gamma^1 |\psi^n(t, x) - \psi^0(t, x)|I_D(x) + |\varphi^n(x) - \varphi^0(x)| \to 0, \text{ as } n \to \infty. \]
Then for all \((t, x) \in [0, T] \times \mathbb{R}^d\)
\[|u^n(t, x) - u^0(t, x)|^2 \to 0, \text{ as } n \to \infty. \]

Proof. For any given \((t, x) \in [0, T] \times D\) by Theorem 6 as \(t \leq s \leq T\) (9) has a weak solution \(y_s\), which is unique in law, and the Ito formula can be applied:

\[u^n(t, y_\tau) - u^n(s \wedge \tau, y_{s \wedge \tau}) = \int_{s \wedge \tau}^t f^n(r, y_r, u^n(r, y_r), u^n_\sigma(r, y_r) \cdot \sigma(r, y_r), \]

\[u^n(r, y_r + c(r, y_r)) - u^n(r, y_r) \) dr + \int_{s \wedge \tau}^t u^n_\sigma(r, y_r) \cdot \sigma(r, y_r) dw_r \]

\[+ \int_{s \wedge \tau}^t \int_Z (u^n(r, y_r + c(r, y_r, z)) - u^n(r, y_r)) N_\kappa(dr, dz) \]

\[n = 0, 1, 2, \ldots, \]
where \(\tau = \tau_x\) has been defined before. By virtue of Theorem 3 as \(n \to \infty\)
\[E_{t, x}[u^n(s \wedge \tau, y_{s \wedge \tau}) - u^0(s \wedge \tau, y_{s \wedge \tau})]^2 \to 0 \text{ for all } t \leq s \leq T. \]

Especially, let \(s = t\), one has
\[|u^n(t, x) - u^0(t, x)|^2 \to 0, \text{ as } n \to \infty. \]

Applying Theorems 4 and 6 we can also get another convergence theorem on solutions of PDIE as follows:

**Theorem 9.** Assume that all conditions in Theorem 6 hold, and assumptions 1°', 2°', and 4° in Theorem 4 are satisfied, where for \(n = 0, 1, 2, \ldots \) in 1°', and for \(n = 1, 2, \ldots \) in 2°', both uniformly with respect to \(y \in \mathbb{R}^d\). Moreover, assume that conditions \(\Gamma^0\) and \(\Gamma^0\) of Theorem 8 holds. Then the conclusion of Theorem 8 still holds.

Since the proof is similar, we omit it here.

The expression of solution to PDIE (8) by (14) (we may call it a new Feynman–Kac formula) is a useful tool in the study of PDIE. We are now going to use it to derive a new existence theorem for the solution of (8) with non-Lipschitzian force \(f\) (the function in the right side of (8)), which seems to be difficult to derive it directly from the PDIE theory.
Theorem 10. Under the assumption of Theorem 5 except B.1 but assuming that
B.1': in the assumption of B.1 the Lipschitzian condition for $f$ in $(r, q, p)$ is relaxed to
that $f$ is continuous in $(r, q, p)$ and

$$-\langle r_1 - r_2, f(t, x, r_1, q_1, p_1) - f(t, x, r_2, q_2, p_2) \rangle$$

$$\leq k_0(\rho(|r_1 - r_2|^2) + |r_1 - r_2|(|q_1 - q_2| + ||p_1 - p_2||),$$

where $\rho(u)$ has the same property as that in $2^o$ of Theorem 1, and

$$|f(t, x, r, q_1, p_1) - f(t, x, r, q_2, p_2)| \leq k_0(|q_1 - q_2| + ||p_1 - p_2||).$$

in addition, assume that

$$|f(t, x, r, q, p)| \leq k_0;$$

then the conclusion of Theorem 5 still holds.

Proof. Denote the smoothness of $f$ with respect to the variable $r$ by $f^n$. Using Theorem
5, one gets a unique solution $u^n(t, x) \in W_p^{1,2}(= W_p^{1,2}([0, T] \times \mathbb{R}^d))$ satisfying (8) with
respect to the given $(f^n, \varphi, \psi)$, and one can select a subsequence of $\{u^n\}$ (denote it by
$\{u^n\}$ again) such that as $n \to \infty$

$$u^n \to u^0,$$ weakly in $W_p^{1,2}.$

Applying Lemma 5 and Theorem 7 for each $n = 0, 1, 2, \ldots$, there exists a unique
solution $x^n$ for BSDE (10) with respect to the given $(\varphi, \psi, f^n)$ such that $u^n(t, x) = x^n$, $n = 1, 2, \ldots$. Since by Theorem 4 it yields that $|x^n - x^0|^2 \to 0$, as $n \to \infty.$ Hence it is
not difficult to derive that $u^n \to u^0$, in $W_p^{1/2}$, as $n \to \infty$. Moreover, $u^0(t, x) = x^0$. Therefore, from the limit one obtains that (8) has a solution $u^0(t, x) \in W_p^{1,2}$ for the
given $(f, \varphi, \psi)$.

Uniqueness. Since by assumption on $f^0 = f$ for each $(t, x) \in [0, T] \times D$ the solution
$(x^0, \varphi^0, p^0) \in L_p^2([t, \tau]; \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ must be unique. Hence the solution of (8) $u(t, x) = x^0$ is also unique. \square

We can also get another existence theorem on (8) with non-Lipschitzian $f$ but
stronger assumption on $a_{ij}$ and without Lipschitzian condition on $f$ to $q$, which can be
proved without the help of BSDE. But the new Feynman–Kac formula still holds.

Theorem 11. Under the assumption of Theorem 5 except B.1 but assuming that B.1" in
the assumption B.1 the Lipschitzian condition for $f$ in $(r, q, p)$ is relaxed to that $f$ is continuous in $(r, q, p)$ and

$$-\langle r_1 - r_2, f(t, x, r_1, q_1, p_1) - f(t, x, r_2, q_2, p_2) \rangle$$

$$\leq k_0(\rho(|r_1 - r_2|^2) + |r_1 - r_2|(|q_1 - q_2| + ||p_1 - p_2||),$$

where $\rho(u)$ has the same property as that in Theorem 10, and

$$|f(t, x, r, q, p_1) - f(t, x, r, q, p_2)| \leq k_0||p_1 - p_2||;$$

(15)
in addition, assume that \( \partial a_{ij}/\partial x_k, i, j, k = 1, 2, \ldots, d \) exist and are bounded,
\[
|a'_{ik}| \leq k_0;
\]
moreover,
\[
|f(t, x, r, q, p)| \leq k_0,
\]
then the conclusion of Theorem 10 still holds.

Remark 1. Conditions in Theorem 11 except (15) are stronger than that in Theorem 10. But (15) is weaker than the condition that \( f \) is Lipschitzian with respect to \( q \) and \( p \).

Since the proof of Theorem 11 is standard, we omit it here.

It is well known that the Krylov estimate (Krylov, 1980) is a powerful tool in the stochastic analysis and its applications. The crucial assumption in it is that the diffusion coefficient in the related SDE should be uniformly non-degenerate. Since now \( q \), in our BSDE can be degenerate, hence it seems to be impossible to get a Krylov type estimate related to the BSDE. Anyway, by means of the relation of PDIE and BSDE with jumps we do have a similar estimate, call it a weak form of Krylov estimate related to the BSDE, which may be useful for the stochastic analysis and its applications.

Theorem 12. (A weak form of Krylov estimate). Under the assumption of Theorem 7 for the solution \((x_s, q_s, p_s)\) of BSDE (10) one has
\[
E_t, x \int_t^\tau |f(s, y_s, x_s, q_s, p_s(\cdot))| ds 
\leq k_{T, p, d} \| f(r, y, u(r, y), u'_s(r, y) \cdot \sigma(r, y), u(r, y + c(r, y, \cdot)) - u(r, y) \|_{L^p},
\]
where \( y_s \) is the solution of (9), \( L_p = L_p([0, T] \times D) \),
\[
\tau = \inf(s > t: y_s \notin D); \quad \tau = T, \text{ for } \inf(\phi).
\]
and \( 0 \leq k_{T, p, d} \) is a universal constant, \( u(t, x) \) is the solution of PDIE (8) related to \( f \), and \( D \) is a bounded domain in \( \mathbb{R}^d \).

Proof. By Theorem 1 of Marhno (1976) and Theorem 7 here one has
\[
E_t, x \int_t^\tau |f(s, y_s, x_s, q_s, p_s(\cdot))| ds 
= E_t, u \int_t^\tau |f(s, y_s, u(s, y_s), u'_s(s, y_s) \cdot \sigma(s, y_s), u(s, x_s + c(s, y_s, \cdot)) - u(s, y_s))| ds,
\leq k_{T, p, d} \| f(r, y, u(r, y), u'_s(r, y) \cdot \sigma(r, y), u(r, y + c(r, y, \cdot)) - u(r, y)) \|_{L^p},
\]
\( \square \)
Acknowledgements and additional notes

The results of this paper were announced at the 23rd Conference on Stochastic Processes and their Applications, 19–23 June 1995 held at the National University of Singapore. The author would like to thank the conference organizers for inviting him to participate.

After having completed this paper, the author visited the University of New South Wales in Australia and found that a new paper by Mao (1995) had also discussed the existence of adapted solutions for BSDE with non-Lipschitzian coefficients. But there are obvious differences between the two papers. Mao discussed the BSDE in the continuous case and with fixed deterministic terminal time, while in our paper we treated the problem for the BSDE with jumps and with bounded random stopping time as a terminal time.

Suppose that we keep the assumption on g in Mao’s paper unchanged but replace the hypothesis on f in that paper with our weaker assumption, it is easily seen that the unique solution of BSDE with jumps can still be derived by means of the technique developed in our paper but cannot be solved by Mao’s method. Moreover, since our technique can also be applied to the continuous BSDE and since our assumption of f is obviously weaker, it shows that our result implies that obtained by Mao, in case we make the same assumption on g. Finally, the author would like to thank professor Layton, the Dean of the Faculty of Commerce, UNSW for his kind invitation and hospitality, and the author would also like to thank the referee for his careful reading of the manuscript, pointing out the misprints, and making nice suggestions.

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