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A class of Newton's methods with third-order convergence

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Abstract

In this work, a class of iterative Newton's methods, known as power mean Newton's methods, is proposed. Some known results can be regarded as particular cases. It is shown that the order of convergence of the proposed methods is 3. Numerical results are given to verify the theory and demonstrate the performance.

Keywords: Variants of Newton's method; Simple roots; Power mean; Cubic convergence; Generalization

1. Introduction

Perhaps the most celebrated of all one-dimensional root-finding routines is the classical Newton's (CN) method, also called the Newton–Raphson method. This method requires the evaluation of both the function $f(x)$ and the derivative $f'(x)$, at arbitrary points x. The classical Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{1}$$

It is known that the classical Newton's method converges quadratically to simple zeros and linearly to multiple zeros.

Some annotations were more helpful for the understanding of the construction of Newton's method, for instance, the Newton–Leibniz formula and Taylor's expansion formula. To get a method with a higher order of convergence, some new variants of Newton's method have been proposed [1–3]. The methods given by Fernando et al. [1] and Özban [2] suggest a class of methods developed in this work.

The work is organized as follows. In Section 2, some fundamental concepts are given. In Section 3, first, we introduce some existing variants of Newton's method. Then, a class of Newton's methods based on the power mean is given, called the power mean Newton's (PN) methods. The cubic convergence of PN methods is shown in Section 4. In Section 5, some numerical results are discussed to demonstrate the convergence of this class of Newton's methods.

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2. Fundamental concepts

Definition 1 (*See [1,3]*). Let $\beta \in \mathcal{R}$, $x_n \in \mathcal{R}$, $n = 0, 1, 2, \ldots$. Then, the sequence x_n is said to converge to β if

$$\lim_{n\to\infty} |x_n - \beta| = 0.$$

If, in addition, there exists a constant $c \geq 0$, an integer $n_0 \geq 0$ and $p \geq 0$ such that for all $n > n_0$,

$$|x_{n+1} - \beta| \leq c|x_n - \beta|^p$$

then $\{x_n\}$ is said to converge to β with q-order at least p. If $p = 1, 2$, or 3, the convergence is said to be linear, quadratic or cubic, respectively.

Let $e_n = x_n - \beta$ be the error in the nth iterate. The relation

$$e_{n+1} = ce_n^p + O(e_n^{p+1})$$

is called the error equation for the method, p being the order of convergence.

Definition 2. Let a and b be positive scalars. For a finite real number α, the α-power mean of a and b is defined as (see [4])

$$m_\alpha = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}}.$$

Several particular power means are well known. For example, setting $\alpha = 2, \alpha = 1, \alpha = -1$, we have

$$m_2(a,b) = \left(\frac{a^2 + b^2}{2}\right)^{\frac{1}{2}}, \qquad m_1(a,b) = \frac{a+b}{2} \qquad m_{-1}(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

which are called the *square mean, arithmetic mean* and *harmonic mean* of a and b, respectively. For $\alpha = 0$, $m_0(a, b)$ can be defined by the limit of $m_\alpha(a, b)$ with $\alpha \to 0$, that is

$$m_0(a,b) = \lim_{\alpha\to 0} m_\alpha(a,b) = \sqrt{ab}$$

which is the so-called *geometric mean* of a and b.

3. Description of the methods

In the iterative formula of the classical Newton's method, if $f'(x_n)$ is approximated by the arithmetic mean of $f'(x_n)$ and $f'(x_{n+1})$, that is by $\frac{f'(x_n)+f'(x_{n+1})}{2}$, we have

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})}.$$

To overcome the implicit problem in the right hand side of the above equation, the $(n + 1)$st value of Newton's method z_{n+1} is used instead of x_{n+1}. That is

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_{n+1})}, \qquad z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Since (2) is obtained by using the arithmetic mean of $f'(x_n)$ and $f'(z_{n+1})$, it is called the *arithmetic mean Newton's* (AN) method (see [2]).

Also replacing $f'(x_n)$ with the harmonic mean of $f'(x_n)$ and $f'(z_{n+1})$, we get

$$x_{n+1} = x_n - \frac{f(x_n)\left(f'(x_n) + f'(z_{n+1})\right)}{2f'(x_n)f'(z_{n+1})} \quad n = 0, 1, 2, \ldots$$

which is called the *harmonic mean Newton's* (HN) method (see [2]).

Now, for generalization, approximating $f'(x_n)$ with the power mean of $f'(x_n)$ and $f'(z_{n+1})$, we have

$$x_{n+1} = x_n - \frac{2^{\frac{1}{\alpha}} f(x_n)}{\text{sign }(f'(x_n))\,(f'(x_n)^\alpha + f'(z_{n+1})^\alpha)^{\frac{1}{\alpha}}}, \quad n = 0, 1, 2, \ldots \tag{3}$$

which can be called the *α-power mean Newton's* (PN) method. The foregoing analysis in Section 1 shows the former two variants of Newton's methods are all particular cases of PN.

4. Convergence analysis

Theorem 3. *Let β be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathcal{R} \to \mathcal{R}$ for an open interval I. If x_0 is sufficiently close to β, then for all $\alpha \in \mathcal{R}$ the methods defined by (3) converge cubically with the following error equation:*

$$e_{n+1} = \frac{1}{2}(c_2^2 + \alpha c_2^2 + c_3)e_n^3 + O(e_n^4)$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\beta)}{f'(\beta)}$, $k = 2, 3, \ldots$ and $e_n = |x_n - \beta|$.

Proof. Let β be a simple zero of f. Since f is sufficiently differentiable, expanding $f(x_n)$ and $f'(x_n)$ at β, we have

$$f(x_n) = f'(\beta)(e_n + c_2e_n^2 + c_3e_n^3 + \cdots) \tag{4}$$

and

$$f'(x_n) = f'(\beta)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 \cdots). \tag{5}$$

Then

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4),$$

thus, for z_{n+1} given in (2) we have

$$z_{n+1} = \beta + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4).$$

By Taylor's expansion,

$$f'(z_{n+1}) = f'(\beta)(1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + O(e_n^4)). \tag{6}$$

For $\alpha = 0$, from (5) and (6), we get

$$\text{sign}\,(f'(x_n))\sqrt{f'(x_n)f'(z_{n+1})} = f'(\beta)\sqrt{1 + 2c_2e_n + (2c_2^2 + 3c_3)e_n^2 + 4(c_2c_3 + c_4)e_n^3 + O(e_n^4)}.$$

Hence,

$$\begin{aligned}\frac{f(x_n)}{\text{sign}\,(f'(x_n))\sqrt{f'(x_n)f'(z_{n+1})}} &= (e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)) \\ &\quad \times \left(1 - c_2e_n + \frac{1}{2}(c_2^2 - 3c_3)e_n^2 + \frac{1}{2}(5c_2c_3 - 4c_4 + c_2^3)e_n^3 + O(e_n^4)\right) \\ &= e_n - \frac{1}{2}(c_2^2 + c_3)e_n^3 + O(e_n^4),\end{aligned}$$

that is

$$e_{n+1} = \frac{1}{2}(c_2^2 + c_3)e_n^3 + O(e_n^4). \tag{7}$$

For $\alpha \in \mathcal{R} \setminus \{0\}$,

$$f'(x_n)^\alpha = f'(\beta)^\alpha \Bigg(1 + 2\alpha c_2 e_n + (2\alpha(\alpha-1)c_2^2 + 3\alpha c_3)e_n^2 + \left(4\alpha c_4 + 6\alpha(\alpha-1)c_2c_3 + \frac{4}{3}\alpha(\alpha-1)(\alpha-2)c_2^3\right)e_n^3 + O(e_n^4)\Bigg) \tag{8}$$

and

$$f'(z_{n+1})^\alpha = f'(\beta)^\alpha(1 + 2\alpha c_2^2 e_n^2 + 4\alpha c_2(c_3 - c_2^2)e_n^3 + O(e_n^4)). \tag{9}$$

From (8) and (9) we get

$$\operatorname{sign}\left(f'(x_n)\right)\left(\frac{f'(x_n)^\alpha + f'(z_{n+1})^\alpha}{2}\right)^{\frac{1}{\alpha}} = f'(\beta)\Bigg(1 + c_2 e_n + \frac{1}{2}(c_2^2 + \alpha c_2^2 + 3c_3)e_n^2 + \frac{1}{2}(-c_2^3 - 3\alpha c_2^3 + c_2c_3 + 3\alpha c_2 c_3 + 4c_4)e_n^3 + O(e_n^4)\Bigg). \tag{10}$$

Dividing (4) by (10), we have

$$\frac{2^{\frac{1}{\alpha}} f(x_n)}{\operatorname{sign}\left(f'(x_n)\right)\left(f'(x_n)^\alpha + f'(z_{n+1})^\alpha\right)^{\frac{1}{\alpha}}} = e_n - \frac{1}{2}(c_2^2 + \alpha c_2^2 + c_3)e_n^3 + O(e_n^4).$$

Thus

$$e_{n+1} = \frac{1}{2}(c_2^2 + \alpha c_2^2 + c_3)e_n^3 + O(e_n^4). \tag{11}$$

From (7) and (11), it can be concluded that for all $\alpha \in \mathcal{R}$, the α-power mean Newton's method converges cubically. □

5. Numerical results

In this section, we will give the results of some numerical tests to demonstrate the convergence efficiencies of PN methods. Moreover, numerical results for CN, HN and AN methods for the same test problems are also given to compare their efficiencies. All numerical computations have been carried out in a Matlab 6.5 environment with a P4-1.5 MHz based PC. The stopping criterion has been taken as $|x_{n+1} - \beta| + |f(x_{n+1})| < 10^{-14}$.

The following test functions have been used (see [1,2]):

(a) $x^3 + 4x^2 - 10$, $\beta = 1.365230013414097$;
(b) $\sin^2 x - x^2 + 1$, $\beta = 1.404491648215341$;
(c) $x^2 + e^x - 3x + 2$, $\beta = 0.2575302854398608$;
(d) $\cos x - x$, $\beta = 0.7390851332151607$;
(e) $(x-1)^3 - 1$, $\beta = 2$;
(f) $(x-1)^6 - 1$, $\beta = 2$;
(g) $(x-1)^8 - 1$, $\beta = 2$;
(h) $xe^{x^2} - \sin^2 x + 3\cos x + 5$, $\beta = -1.207647827130919$;
(i) $e^{x^2+7x-30} - 1$, $\beta = 3$;
(j) $\prod_{m=0}^{4}(x - (1 + 0.1m))$, $\beta = 1$;
(k) $\prod_{m=0}^{5}(x - (m+1))$, $\beta = 1$;
(l) $(x-2)^3(x+2)^4$, $\beta_1 = 2$ and $\beta_2 = -2$.

In Table 1, "×" means that the method does not converge in 1000 iterations. It is easy to see that PN methods, especially the ones with $\alpha < 0$, can converge to the zeros of functions more quickly than the classical Newton's method on the whole.

Table 1
Numerical results for test functions

$f(x)$	x_0	Number of iterations							
		CN	HN	AN	PN				
					$\alpha = 0$	$\alpha = 2$	$\alpha = -2$	$\alpha = 3$	$\alpha = -3$
a	−0.5	131	43	6	4	4	4	4	4
	1	5	3	3	3	3	3	4	3
	2	5	3	3	3	4	3	4	3
b	1	6	3	4	4	4	4	4	4
	3	6	3	3	4	4	4	4	4
c	2	5	4	4	4	4	3	4	3
	3	6	4	4	4	5	4	5	4
d	1	4	3	2	3	3	3	3	3
	1.7	4	3	3	3	3	3	3	3
	−0.3	5	4	3	3	4	4	4	4
e	1.5	7	4	5	4	5	4	5	4
	2.5	6	3	4	4	4	3	4	3
	3	6	4	4	4	4	4	5	4
	3.5	7	4	5	4	5	4	5	4
f	1.5	15	7	467	12	658	8	737	8
	2.5	7	4	5	5	5	4	5	4
	3.5	10	6	7	6	7	5	7	5
g	1.5	27	13	×	351	×	13	×	13
	2.5	8	5	5	5	6	4	6	4
	3.5	12	7	8	7	9	6	9	6
h	−2	8	5	6	5	6	5	6	5
	−3	14	8	9	9	10	7	11	7
i	3.5	12	7	8	7	9	6	9	6
	3.25	8	5	6	5	6	5	6	4
j	−0.5	16	9	11	10	11	8	12	8
k	−2	11	6	7	7	8	6	8	6
l	1.4	78	41	51	46	56	37	60	33
	−1	111	59	74	66	81	53	87	49

6. Conclusion

This work proposes a class of variants of Newton's method, which can be viewed as the generalization of some existing improvements of Newton's method. Theoretical analysis shows that the order of convergence of such Newton's methods is three for simple roots. It will be interesting to consider other variants of the method obtained by replacing x_n by the power mean of x_n and z_{n+1} in (1) and for multiple roots.

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