JOURNAL OF FUNCTIONAL ANALYSIS **145**, 175–204 (1997)
ARTICLE NO. FU963002

Weak Amenability of Banach Algebras on Locally Compact Groups

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Received December 12, 1995; accepted March 20, 1996

Recent work by various authors has considered the implications of Banach

quences of weak amenability. Here the homomorphism property fails in general, however it remains true for suitable direct summands. It is this technique that we make much use of here.

0. INTRODUCTION

Let A be a Banach algebra, X a Banach A-bimodule. Then X^* is a Banach A-bimodule under the actions

$$\langle a \cdot m, \xi \rangle = \langle m, \xi \cdot a \rangle, \quad \langle m \cdot a, \xi \rangle = \langle m, a \cdot \xi \rangle \qquad (a \in A, \xi \in X, m \in X^*).$$

A *derivation* $D: A \to X$ is a (bounded) linear map such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

The derivation D is *inner* if it is of the form $a \mapsto a \cdot \xi - \xi \cdot a$ for some $\xi \in X$. The cohomology space $H^1(A, X)$ is the quotient of the space of derivations by the inner derivations, and in many situations triviality of this space is of considerable importance. In particular, A is *contractible* if, for every Banach A-bimodule X, $H^1(A, X) = \{0\}$, *amenable* if, for every Banach A-bimodule X, $H^1(A, X^*) = \{0\}$, and *weakly amenable* if $H^1(A, A^*) = \{0\}$. In the case that A is commutative, then weak amenability is equivalent to every derivation into a commutative bimodule being zero, [2, Theorem 1.5].

0022-1236/97 $25.00

As examples of these notions, for G a locally compact group and $A = L^1(G)$, A is contractible if and only if G is finite, A is amenable if and only if G is amenable as a topological group, and A is always weakly amenable. If A is a C*-algebra, then A is contractible if and only if it is finite dimensional, A is amenable if and only if it is nuclear, and A is always weakly amenable.

Recent papers, [19, 43, 44, 15], have considered the implications of amenability for various algebras defined over locally compact groups. In the present paper we continue this investigation, with emphasis on weak amenability. There are two results along these lines given towards the end of [19], and in fact one of the basic tools used here is an abstract form of the technique used there, cf. Proposition 4.14 below. Other results are given in [14]. Weak amenability for other specific classes of Banach algebras are given in [2, 4].

It is a well known and very useful fact that a continuous homomorphic image of an amenable algebra is again amenable. This is also easily verified for weak amenability in the commutative case, but is false in general. We are grateful to Dr. N. Grønbæk for pointing out that a counterexample is provided by the algebra of nuclear operators on a Banach space without the approximation property, see [28]. A sufficient condition for weak amenability of a homomorphic image is given in [27, Propositions 1.3 and 2.4], but as noted below it fails to apply in the situations of interest here.

An outline of the present paper is as follows. Basic properties of introverted subspaces of Banach algebras are given in Section 1, many of the algebras we consider are of this form. Section 2 contains some simple yet crucial facts about derivations and homomorphic images. The results proper start in Section 3 with a study of weak amenability of the measure algebra $M(G)$ of a locally compact group G. We prove, amongst other things, that if G is connected and amenable as a discrete group, and $M(G)$ is weakly amenable, then G must be trivial. In Section 4 we consider the following general problem: given closed left introverted subspaces $Y \subseteq X$ of $L^\infty(G)$ such that X^* is weakly amenable, when does it follow that Y^* is weakly amenable? The same question is addressed in Sections 6 and 7 for subspaces of the von Neumann algebra $VN(G)$ generated by the left regular representation of a locally compact group G, and the spaces $PM_p(G)$, $1 < p < \infty$ in $\mathscr{B}(L^p(G))$ generated by left translations. In particular, we show that if X is a left introverted subspace of $L^\infty(G)$ containing the almost periodic functions, and X^* is weakly amenable, then G cannot have an infinite abelian quotient. We also show that if A is a commutative W*-algebra such that the predual A_* is a two-sided G-module satisfying $x \cdot \phi \geqslant 0$ and $\phi \cdot x \geqslant 0$ for $x \in G$ and each $\phi \in A_*^+$, then any derivation D: $L^1(G) \to A$ is inner.

1. PRELIMINARIES

For a linear space E, and a functional f on E, we will variously denote the value of f on $x \in E$ by $f(x)$ and $\langle f, x \rangle$.

Given a Banach algebra A, for each $f \in A^*$ and $\phi \in A$, define elements $\phi \cdot f$ and $f \cdot \phi$ of A^* by

$$\langle f \cdot \phi, \psi \rangle = \langle f, \phi\psi \rangle, \quad \langle \phi \cdot f, \psi \rangle = \langle f, \psi\phi \rangle \qquad (\psi \in A).$$

A subspace X of A^* is *A-left invariant* (resp. *A-right invariant*) if $X \cdot \phi \subseteq X$ (resp. $\phi \cdot X \subseteq X$) for each $\phi \in A$. To simplify notation, A- will be omitted whenever the ambient algebra is clear. X is *invariant* if it is both left and right invariant. Note that a closed invariant subspace of A^* is thus a Banach A-bimodule.

Let $X \subseteq A^*$ be a left invariant subspace. For $m \in X^*$ and $f \in X$ define $m \cdot f \in A^*$ by $\langle m \cdot f, \phi \rangle = \langle m, f \cdot \phi \rangle$, $(\phi \in A)$. The subspace X is *A-left introverted* if $X^* \cdot X \subseteq X$. Again, A- will be omitted whenever the ambient algebra is clear. Clearly every left introverted subspace of A^* is invariant. Following [1], define a product on the left introverted subspace X by

$$\langle mn, f \rangle = \langle m, n \cdot f \rangle \qquad (m, n \in X^*, f \in X).$$

This product makes X^* into a Banach algebra.

We shall repeatedly require the following facts about introverted subspaces of the dual of a Banach algebra.

LEMMA 1.1. *Let A be a Banach algebra, $Y \subset X$ two left introverted subspaces of A^*. Then*

$$Y^{\perp} = \{m \in X^* : \langle m, f \rangle = 0 \textit{ for } f \in Y\}$$

is a weak closed ideal of X^*.*

Proof. Certainly $Y^{\perp}$ is weak* closed. Take $m \in Y^{\perp}$, $f \in Y$ and $\phi \in A$. Then $\langle m \cdot f, \phi \rangle = \langle m, f \cdot \phi \rangle = 0$ since $f \cdot \phi \in Y$ by A-left invariance of Y. Thus for $n \in X^*$, $\langle nm, f \rangle = \langle n, m \cdot f \rangle = 0$, so that $nm = 0 \in Y^{\perp}$. Further, if $n \in X^*$, then $n \cdot f \in Y$ by left introversion, whence $\langle mn, f \rangle = \langle m, n \cdot f \rangle = 0$. Thus $mn \in Y^{\perp}$. ∎

Remark. The ideals $Y^{\perp}$ are typical of the kernels of the homomorphisms we consider below. The proof of Lemma 1.1 shows, in particular, that they have zero products, so certainly have no bounded approximate identities. Thus [27, Proposition 1.3] does not apply.

For a Banach algebra A define

$$W(A^*) = \{ f \in A^* : A \to A^* : \phi \mapsto \phi \cdot f \text{ is weakly compact} \},$$

and, for each $f \in A^*$,

$$\mathscr{K}(f) = \{ \phi \cdot f : \phi \in A, \|\phi\| \leqslant 1 \}.$$

LEMMA 1.2. *Let A be a Banach algebra, $X \subseteq A^*$ a left invariant subspace. Then X is left introverted if, and only if, for each $f \in X$ the weak* closure of $\mathscr{K}(f)$ is contained in X. In particular*

(a) *Any weak* closed invariant subspace of A^* is left introverted.*

(b) *Any norm closed invariant subspace of $W(A^*)$ is left introverted.*

Proof. Suppose that X is left introverted, and let $f \in X$. Given g in the weak* closure of $\mathscr{K}(f)$, there is a net $(\phi_\alpha) \subset A$ with $\|\phi_\alpha\| \leqslant 1$ and $\phi_\alpha \cdot f \to g$ weak*. Take a weak* cluster point m of (ϕ_α) in A^{**}. Then for each $\psi \in A$,

$$\langle g, \psi \rangle = \lim_\alpha \langle \phi_\alpha \cdot f, \psi \rangle = \lim_\alpha \langle \phi_\alpha, f \cdot \psi \rangle = \langle m, f \cdot \psi \rangle = \langle m \cdot f, \psi \rangle,$$

so that $g = m \cdot f \in X$.

Conversely, suppose that the weak* closure of $\mathscr{K}(f)$ lies in X for each $f \in X$. For $m \in X^*$, let $n \in A^{**}$ be a norm preserving extension of m. By Goldstine's theorem there is a net $(\phi_\alpha) \subset A$ with $\|\phi_\alpha\| \leqslant \|m\|$ and $\phi_\alpha \to n$ weak* in A^{**}. Then for $\psi \in A$,

$$\langle m \cdot f, \psi \rangle = \langle m, f \cdot \psi \rangle = \lim_\alpha \langle \phi_\alpha, f \cdot \psi \rangle = \lim_\alpha \langle \phi_\alpha \cdot f, \psi \rangle,$$

whence $m \cdot f$ lies in the weak* closure of $\mathscr{K}(f)$, and hence in X. Thus X is left introverted.

(a) This is immediate from above.

(b) Let X be a norm-closed invariant subspace of $W(A^*)$. For $f \in X$ the norm closure of $\mathscr{K}(f)$ is weakly compact in A^*, hence weak* compact, so that the weak* closure of $\mathscr{K}(f)$ lies in X. ∎

LEMMA 1.3. *Let A be a Banach algebra, X be a left introverted subspace of A^*. Suppose $P: A^* \to X$ is a continuous projection which commutes with the operators $T_\phi : f \mapsto f \cdot \phi$, $\phi \in A$. Then P^* is an algebra homomorphism of X^* into A^{**}.*

Proof. For $m, n \in X^*$ and $f \in A^*$,

$$\langle P^*(mn), f \rangle = \langle mn, Pf \rangle = \langle m, n \cdot (Pf) \rangle$$

where, for $\phi \in A$,

$$\begin{aligned}\langle n\cdot(Pf),\phi\rangle &= \langle n,(Pf)\cdot\phi\rangle = \langle n, P(f\cdot\phi)\rangle\\ &= \langle P^*n, f\cdot\phi\rangle = \langle (P^*n)\cdot f,\phi\rangle.\end{aligned}$$

It follows that $(P^*n)\cdot f = n\cdot(Pf) = P(n\cdot(Pf))$, and so

$$\begin{aligned}\langle P^*(mn), f\rangle &= \langle m, P(n\cdot(Pf))\rangle = \langle P^*m, n\cdot(Pf)\rangle\\ &= \langle P^*m,(P^*n)\cdot f\rangle = \langle (P^*m)(P^*n), f\rangle. \quad ▮\end{aligned}$$

LEMMA 1.4. *Let A be a commutative Banach algebra, X a closed left introverted subspace of A^*. Then the following are equivalent*:

(a) $X \subseteq W(A^*)$;

(b) *the product in X^* is separately weak* continuous on bounded spheres*;

(c) *X^* is a commutative Banach algebra.*

Proof. (a) ⇒ (b). For $m \in X^*$, the map $X^* \to X^*: n \mapsto n\cdot m$ is weak*-weak* continuous.

Now let $f \in X \subseteq W(A^*)$. The norm closure Z of $\mathscr{K}(f)$ is weakly compact in A^*, hence weak* compact, and moreover these two topologies agree on Z. In fact $Z = \{m\cdot f : m \in X^*, \|x\| \leqslant 1\}$. For Z is certainly weak* closed, convex, and contains $\mathscr{K}(f)$. Conversely, for $m \in X^*$, $\|m\| \leqslant 1$, Goldstine's theorem gives $(\phi_\alpha) \subset A$, $\|\phi_\alpha\| \leqslant 1$ with $\phi_\alpha \to m$ weak* in A^*, whence $\phi_\alpha\cdot f \to m\cdot f$ weak* in A^*, so that $m\cdot f \in Z$.

Suppose now that $(n_\alpha) \subset X^*$ is weak* convergent to $n \in X^*$, with $\|n_\alpha\| \leqslant 1$, $\|n\| \leqslant 1$. Then $n_\alpha\cdot f \to n\cdot f$ weak* in A^*, and hence weakly as this occurs in Z. So for $m \in X^*$,

$$\langle m\cdot n_\alpha, f\rangle = \langle m, n_\alpha\cdot f\rangle \to \langle m, n\cdot f\rangle = \langle m\cdot n, f\rangle,$$

that is, $m\cdot n_\alpha \to m\cdot n$ weak* in X^*.

(b) ⇒ (c). Note that $\phi\cdot m = m\cdot\phi$ for $\phi \in A$, $m \in X^*$ by commutativity of A. Thus $m\cdot n = n\cdot m$ for $m, n \in X^*$.

(c) ⇒ (a). Take $f \in X$. Then the map $X^* \to X: m \mapsto m\cdot f$ is weak*-weak continuous. Thus $\mathscr{K}(f)$, as a subset of the weakly compact set Z, is weakly precompact, which is to say $f \in W(A^*)$. ▮

Throughout this paper, G will always denote a locally compact group with fixed left Haar measure; G_d will denote G with the discrete topology. Given a function $f: G \to \mathbb{C}$, the left (right) translation of f by $x \in G$ is defined by $(\ell_x f)(y) = f(xy)$ $((r_x f)(y) = f(yx))$. The standard Lebesgue spaces with respect to left Haar measure will be denoted $L^p(G)$, $1 \leqslant 1 \leqslant \infty$;

$CB(G)$ will denote the space of all bounded continuous complex valued functions on G with the supremum norm, $CB_{\mathbb{R}}(G)$ its (real) subspace of real valued functions; $C_0(G)$ the closed subspace of $CB(G)$ of functions vanishing at infinity, $C_{00}(G)$ the dense subspace of $C_0(G)$ consisting of the functions with compact support. Note that $CB(G)$ is isometrically isomorphic to a closed, translation invariant subspace of $L^\infty(G)$.

Given a subspace X of $L^\infty(G)$, for each $f \in X$ and $\phi \in L^1(G)$, $f \cdot \phi = (1/\Delta)\,\tilde{\phi} * f$ and $\phi \cdot f = f * \tilde{\phi}$. Here Δ is the modular function of G, and $\tilde{\phi}(x) = \phi(x^{-1})$ for $x \in G$.

Of major interest to us here is the left introverted space $C_0(G)$. It is well known that $C_0(G)^*$ is the space $M(G)$ of regular Borel measures on G, and that the above product agrees with the usual convolution of measures, [30, Theorem 19.10].

Let $LUC(G)$ denote the space of those $f \in CB(G)$ such that the map $x \mapsto \ell_x f \colon G \to (CB(G), \|\cdot\|)$ is continuous. This in fact coincides with the space of those $f \in CB(G)$ such that $x \mapsto \ell_x f \colon G \to (CB(G), \text{weak})$ is continuous. Thus $LUC(G)$ is $L^1(G)$-invariant; it is in fact the maximal $\ell^1(G)$-left introverted subspace of $\ell^\infty(G)$ contained in $CB(G)$, [3, Theorem 5.7]. Note that $LUC(G)$ is exactly the space of *right* uniformly continuous functions of $CB(G)$ as defined in [30]. If $f \in LUC(G)$, $n \in LUC(G)^*$, then for $x \in G$, $(n \cdot f)(x) = \langle n, \ell_x f \rangle$, [37, Lemma 3].

The right orbit of a function $f \in CB(G)$ is given by $\mathscr{RO}(f) = \{r_x f \colon x \in G\}$. Recall that a function $f \in CB(G)$ is almost periodic (weakly almost periodic) if $\mathscr{RO}(f)$ is precompact in the norm topology (weak topology) of $CB(G)$. We shall denote the spaces of such functions by $AP(G)$ and $W(G)$ respectively; they are also left introverted subspaces of $L^\infty(G)$, [52, Lemma 6.4].

If X is a weak* closed subspace of $L^\infty(G)$, then X is $L^1(G)$-invariant if and only if X is left and right translation invariant, [52, Lemma 6.3]. If X is a norm-closed subspace of $W(G)$, then X is $L^1(G)$-invariant if and only if X is left and right translation invariant.

LEMMA 1.5. *Let G and H be locally compact groups, $x \mapsto \bar{x}$ a surjective homomorphism. Let X (resp. Y) be a left introverted subspaces of $L^\infty(G)$ (resp. $L^\infty(H)$) contained in $LUC(G)$ (resp. $LUC(H)$). Suppose that $J \colon Y \to X$ is a linear mapping such that for $x \in G$, $f \in X$,*

$$(Jf)(\bar{x}) = f(x), \qquad J(\ell_x f) = \ell_{\bar{x}}(Jf).$$

Then $J^* \colon X^* \to Y^*$ is an algebraic homomorphism.

Proof. Take $\mu, \nu \in X^*$ and $f \in Y$, so that

$$\langle J^*(\mu\nu), f \rangle = \langle \mu\nu, Jf \rangle = \langle \mu, \nu \cdot (Jf) \rangle.$$

Now, for $x \in G$,

$$(\nu \cdot (Jf))(\bar{x}) = \langle \nu, \ell_{\bar{x}}(Jf) \rangle = \langle \nu, J(\ell_x f) \rangle = \langle J^*\nu, \ell_x f \rangle$$
$$= ((J^*\nu) \cdot f)(x) = J((J^*\nu) \cdot f))(\bar{x}).$$

Thus

$$\langle J^*(\mu\nu), f \rangle = \langle J^*\mu, J^*\nu \cdot f \rangle = \langle J^*\mu J^*\nu, f \rangle. \quad \blacksquare$$

Finally, note that *subspace* will always refer to norm closed subspace, and all linear mappings are taken to be norm continuous.

2. HOMOMORPHIC IMAGES

We need some elementary results concerning derivations on Banach algebras. These are surely well known, but some care must be taken in the noncommutative case, so proofs are included. Recall that for a Banach algebra A, $d \in A^*$ is a point derivation at a character ψ of A if $d(xy) = d(x)\,\psi(y) + d(y)\,\psi(x)$ for $x, y \in A$. Of course, this means exactly that d is a derivation into the one dimensional module $\mathbb{C}$, with commutative module action given by $\alpha \cdot x = x \cdot \alpha = \psi(x)\,\alpha$, $(\alpha \in \mathbb{C})$.

LEMMA 2.1. *Let A and B be Banach algebras, $\Phi\colon A \to B$ a homomorphism with dense range. Let $\phi\colon B \to \mathbb{C}$ be a (non-zero) character, and let $d\colon B \to \mathbb{C}$ be a non-zero point derivation at ϕ. Then $\psi = \phi\Phi$ is a (non-zero) character on A, and $\Delta = d\Phi$ is a non-zero point derivation at ψ. In particular A is not weakly amenable.*

Proof. It is clear that ψ is a character, non-zero because of the density of $\Phi(A)$. For the same reason, $\Delta \neq 0$. Further, for $x, y \in A$,

$$\begin{aligned}\Delta(xy) &= (d\Phi)(xy) = d(\Phi(x)\,\Phi(y)) \\ &= (d\Phi)(x)(\phi\Phi)(y) + (\phi\Phi)(x)(d\Phi)(y) \\ &= \Delta(x)\,\psi(y) + \psi(x)\,\Delta(y).\end{aligned}$$

Now set $D\colon A \to A^*\colon x \mapsto \Delta(x)\,\psi$. Then for $x, y, z \in A$,

$$\begin{aligned}\langle D(xy), z \rangle &= \langle \Delta(xy)\,\psi, z \rangle \\ &= \Delta(x)\,\psi(z)\,\psi(y) + \psi(x)\,\Delta(y)\,\psi(z) \\ &= \Delta(x)\langle \psi, yz \rangle + \Delta(y)\langle \psi, zx \rangle \\ &= \Delta(x)\langle \psi \cdot y, z \rangle + \Delta(y)\langle x \cdot \psi, z \rangle \\ &= \langle D(x) \cdot y, z \rangle + \langle x \cdot D(y), z \rangle,\end{aligned}$$

so that $D\colon A \to A^*$ is a derivation. If D were inner, say $D(x) = x \cdot \xi - \xi \cdot x$ for $x \in A$, for some fixed $\xi \in A^*$, then for $x, y \in A$ we have

$$\begin{aligned} \Delta(x)\,\psi(y) &= \langle \Delta(x)\,\psi, y \rangle = \langle x \cdot \xi - \xi \cdot x, y \rangle \\ &= \langle \xi, yx - xy \rangle = -\langle \Delta(y)\,\psi, x \rangle = -\Delta(y)\,\psi(x). \end{aligned}$$

This means that $\psi = c\Delta$ for some $c \neq 0$. But then for $x \in A$,

$$\psi(x)^2 = \psi(x^2) = 2c\Delta(x)\,\psi(x),$$

whence $\psi = 2c\Delta$, which is absurd. ∎

For a commutative Banach algebra A and finite-dimensional bimodule X, existence of non-zero point derivations is necessary and sufficient for $H^1(A, X) \neq \{0\}$. However, this simple characterisation fails even in the general commutative situation, see [2, Section 1] for details. In the absence of non-zero point derivations more care is needed to pass to homomorphic images.

Lemma 2.2. *Let A, B be Banach algebras, $\Phi\colon A \to B$ a homomorphism with dense range. Let $D\colon B \to B^*$ be a non-zero derivation. Then $\Delta = \Phi^* D \Phi\colon A \to A^*$ is a non-zero derivation.*

Proof. This is just a calculation. For $x, y, z \in A$,

$$\begin{aligned} \langle \Delta(xy), z \rangle &= \langle (\Phi^* D \Phi)(xy), z \rangle \\ &= \langle D(\Phi(x)\,\Phi(y)), \Phi(z) \rangle \\ &= \langle (D\Phi)(x) \cdot \Phi(y) + \Phi(x) \cdot (D\Phi)(y), \Phi(z) \rangle. \end{aligned}$$

Now

$$\begin{aligned} \langle (D\Phi)(x) \cdot \Phi(y), \Phi(z) \rangle &= \langle (D\Phi)(x), \Phi(y)\,\Phi(z) \rangle \\ &= \langle (D\Phi)(x), \Phi(yz) \rangle \\ &= \langle \Delta(x), yz \rangle \\ &= \langle \Delta(x) \cdot y, z \rangle, \end{aligned}$$

and, similarly

$$\langle \Phi(x) \cdot (D\Phi)(y), \Phi(z) \rangle = \langle x \cdot \Delta(y), z \rangle.$$

It follows that Δ is a derivation.

If $\Delta = 0$, then for $x, y \in A$, $\langle \Delta(x), y \rangle = 0$, that is, $\langle (\Phi^* D \Phi)(x), y \rangle = 0$. Thus $\langle (D\Phi)(x), \Phi(y) \rangle = 0$, so by density $(D\Phi)(A) = 0$, and so $D = 0$. ∎

Lemma 2.3. *Let A be a Banach algebra such that $A = B \oplus I$ for a closed subalgebra B and closed (two-sided) ideal I. Suppose that A is weakly amenable. Then B is weakly amenable.*

Proof. Let $\pi: A \to B$ be the natural projection of A onto B, so that π is a homomorphism with kernel I. Take $D: B \to B^*$ a derivation. From Lemma 2.2, $\Delta = \pi^* D\pi: A \to A^*$ is a derivation, so by hypothesis there is $\xi \in A^*$ such that $\Delta(x) = x \cdot \xi - \xi \cdot x$ for $x \in A$. Set $\eta = \xi|_B$. Then for $x, y \in B$,

$$\begin{aligned}\langle Dx, y\rangle &= \langle (D\pi)(x), \pi(y)\rangle = \langle \Delta(x), y\rangle \\ &= \langle x \cdot \xi - \xi \cdot x, y\rangle = \langle \xi, yx\rangle - \langle \xi, xy\rangle \\ &= \langle \eta, yx\rangle - \langle \eta, xy\rangle = \langle x \cdot \eta - \eta \cdot x, y\rangle,\end{aligned}$$

where the penultimate step uses the fact that B is a subalgebra. Thus D is inner, as required. ∎

3. WEAK AMENABILITY OF $M(G)$

Let N be a compact normal subgroup of a locally compact group G, and set

$$\begin{aligned}X_N &= \{f \in C_0(G) : f \text{ is constant on cosets of } N\} \\ &= \{f \in C_0(G) : \ell_h f = f \text{ for } h \in N\}.\end{aligned}$$

By Lemma 1.2, X_N is a left introverted subspace of $L^\infty(G)$. Denoting by $\bar{x}$ the coset xN, it is standard that the map $J: X_N \to C_0(G/N)$ given by $(Jf)(\bar{x}) = f(x)$ is an isometric isomorphism. Now, for $x, y \in G$, $f \in X_N$,

$$\ell_{\bar{x}}(Jf)(\bar{y}) = (Jf)(\bar{x}\bar{y}) = f(xy) = (\ell_x f)(y) = J(\ell_x f)(\bar{y}),$$

so that $J(\ell_x f) = \ell_{\bar{x}}(Jf)$. Thus $J^*: C_0(G/N)^* \to (X_N)^*$ is an algebraic epimorphism by Lemma 1.5.

Define a linear map $P: C_0(G) \to X_N$ by the formula

$$(Pf)(x) = \int_N f(xh)\, dh.$$

It is well known that P is a projection of $C_0(G)$ onto X_N, see for example, [16, Proposition V.3.5]. It is clear that P commutes with left translations, and, taking, normalized Haar measure on N, $\|P\| = 1$. In fact much more is true.

LEMMA 3.1. *Let N be a compact normal subgroup of G. Then $P^*\colon (X_N)^* \to M(G)$ is an isometric isomorphism into, and there is the direct sum decomposition*

$$M(G) = P^*(X_N)^* \oplus X_N^{\perp},$$

where $P^*(X_N)^*$ is a closed subalgebra, and $X_N^{\perp}$ is a weak closed ideal of $M(G)$.*

Proof. Note firstly that for $\mu \in (X_N)^*$,

$$\begin{aligned}\|\mu\| \geqslant \|P^*\mu\| &= \sup\{|\langle P^*\mu, f\rangle| : f \in C_0(G), \|f\| \leqslant 1\}\\ &= \sup\{|\langle \mu, Pf\rangle| : f \in C_0(G), \|f\| \leqslant 1\}\\ &\geqslant \sup\{|\langle \mu, f\rangle| : f \in C_0(G/N), \|f\| \leqslant 1\}\\ &= \|\mu\|,\end{aligned}$$

since $Pf = f$ for $f \in X_N$. Thus P^* is an isometry. Also, for $x \in G$, $f \in C_0(G)$, $P(\ell_x f) = \ell_x(Pf)$. So by Lemma 1.5 P^* is a homomorphism.

Clearly $X_N^{\perp}$ has trivial intersection with $P^*(X_N)^*$. For $\mu \in M(G)$, set μ' to be the restriction of μ to X_N. Then $\mu - P^*\mu' \in X_N^{\perp}$, so we do, indeed, have a direct sum decomposition.

That $X_N^{\perp}$ is a weak* closed ideal follows from Lemma 1.1. ∎

The following should be compared with the corresponding results for amenability proved in [43].

LEMMA 3.2. *Let N be a compact normal subgroup of G. Then $M(G)$ weakly amenable implies $M(G/N)$ is weakly amenable.*

Proof. This is an immediate consequence of Lemmas 2.3 and 3.1. ∎

THEOREM 3.3. *Let G be a connected locally compact group such that G_d is amenable. Then $M(G)$ is weakly amenable if and only if $G = \{e\}$.*

Proof. Suppose firstly that G is a Lie Group. Then G is solvable, [49, Theorem 3.9]. Let

$$G = G_1 \rhd G_2 \rhd \cdots \rhd G_k = \{e\}$$

be a normal series for G with each G_i/G_{i+1} abelian. Then certainly G_1/G_2 is abelian. So, unless G_1/G_2 is discrete, $M(G_1/G_2)$ admits a non-zero point derivation, [5], whence $M(G)$ is not weakly amenable by Lemma 3.2, which contradicts the hypothesis. Thus G_2 is open, so that $G_1 = G_2$ by connectivity. That $G = \{e\}$ now follows by induction.

In the general case, there is a directed set (N_i) of compact normal subgroups of G such that each G/N_i is a (connected) Lie group, and $N_i \downarrow \{e\}$. By Lemma 3.2, $M(G/N_i)$ is weakly amenable for each i, hence trivial by above, since $(G/N_i)_d$ is certainly amenable. Thus G is trivial. ∎

For a general connected group, factoring out the largest compact normal subgroup reduces the question to the Lie group case.

THEOREM 3.4. *Let G be a non-compact, connected $[IN]$-group. Then $M(G)$ is not weakly amenable.*

Proof. If G was in fact a [SIN]-group, then G would have the form $\mathbb{R}^n \times K$ for some compact subgroup K. Thus by Lemma 3.2, $M(G)$ weakly amenable would imply $M(\mathbb{R}^n)$ weakly amenable, which is not the case.

More generally, if G is an [IN]-group, then G has a compact normal subgroup N such that G/N is a [SIN]-group, so the result follows from Lemma 3.2 and the previous case. ∎

4. LEFT INTROVERTED SUBSPACES OF $L^\infty(G)$

The general problem concerning us in this section is the following. Given left introverted spaces $Y \subset X$ of $L^\infty(G)$, such that X^* is weakly amenable, when does it follow that Y^* is weakly amenable? Note that in this situation, the restriction map $X^* \to Y^*$ provides a contractive algebraic homomorphism of X^* onto Y^*. Lemma 3.2 provides an affirmative answer to this question in the case $X = C_0(G)$, and $Y = X_N$ for some compact normal subgroup N of G.

Let X be a left introverted subspace of $L^\infty(G)$ which is contained in $CB(G)$. Given $\mu \in M(G)$, define $\tau(\mu) \in X^*$ by

$$\langle \tau(\mu), f \rangle = \int f(x)\, d\mu(x) \qquad (f \in X).$$

Provided $C_0(G) \subseteq X$, $\tau(\mu)$ is the unique norm preserving extension of $\mu \in C_0(G)^*$ to X, see [45, Lemma 1], so that τ is a linear isometry of $M(G)$ into X^*.

LEMMA 4.1. *Let X be a left introverted subspace of $L^\infty(G)$ such that $C_0(G) \subseteq X \subseteq CB(G)$. Then there is an isometric algebraic isomorphism τ: $M(G) \to X^*$ and an isometric direct sum decomposition*

$$X^* = \tau(M(G)) \oplus C_0(G)^\perp,$$

where $C_0(G)^\perp$ is a weak closed ideal of X^*.*

Proof. For $\mu, \nu \in M(G)$, both $\tau(\mu\nu)$ and $\tau(\mu) \cdot \tau(\nu)$ are norm preserving extensions of $\mu\nu$ from $C_0(G)$ to X, so by uniqueness $\tau(\mu\nu) = \tau(\mu) \cdot \tau(\nu)$. Thus τ is a homomorphism.

That $C_0(G)^\perp$ is a weak* closed ideal follows from Lemma 1.1.

The argument of [18, Lemma 1.1] suffices to show that if $m = \mu + \nu$ where $\mu \in \tau(M(G))$ and $\nu \in C_0(G)^\perp$, then $\|m\| = \|\mu\| + \|\nu\|$. ▮

THEOREM 4.2. *Let X be a left introverted subspace of $L^\infty(G)$, $C_0(G) \subseteq X \subseteq CB(G)$. Then X^* weakly amenable implies that $M(G)$ is weakly amenable. In particular, Theorems* 3.3 *and* 3.4 *apply.*

Proof. Use Lemmas 2.3 and 4.1. ▮

Let G^{ap} be the almost periodic compactification of G. Thus G^{ap} is the spectrum of the Banach algebra $AP(G)$, taken with the usual weak* topology. Under the product

$$\langle \xi \cdot \eta, f \rangle = \langle \xi, \eta \cdot f \rangle \qquad (\xi, \eta \in G^{ap}, f \in AP(G)),$$

where $(\eta \cdot f)(x) = \langle \eta, \ell_x f \rangle$, G^{ap} is a compact topological group containing as a dense subgroup the image of G under $\pi: x \mapsto \delta_x$.

The map $\check{\pi}: C(G^{ap}) \to AP(G)$ given by $(\check{\pi} f)(x) = f(x)$, $(x \in G)$ is an isometric surjective $*$-isomorphism, so that $\check{\pi}^*$ is certainly a linear isometry between the Banach spaces $AP(G)^*$ and $M(G^{ap})$. To see that it is a homomorphism, note firstly that for $x, y \in G$, $\check{\pi}^*(\delta_x * \delta_y) = \check{\pi}^*(\delta_x) * \check{\pi}^*(\delta_y)$ is immediate from the definition of π. It then follows that $\check{\pi}^*(mn) = \check{\pi}^*(m) * \check{\pi}^*(n)$ for $m, n \in \mathrm{sp}\{\delta_x : x \in G\}$. Now weak* density of $\mathrm{sp}\{\delta_x : x \in G\}$ in $AP(G)^*$, together with weak* continuity of multiplication, shows that $\check{\pi}^*$ is a homomorphism.

The following is an analogue of [14, Theorem 3.2].

THEOREM 4.3. *Let X be a left introverted subspace of $L^\infty(G)$ containing $AP(G)$. Suppose X^* is weakly amenable. Then any closed normal subgroup N of G with G/N abelian has finite index. In particular, if G is connected then G has no proper closed normal subgroup with abelian quotient.*

Proof. Take such a subgroup N of G and set $T = G/N$. Let $q: G \to T$ be the quotient homomorphism and $\pi: T \to T^{ap}$ the natural embedding $\pi(t) = \delta_t$. Then $\rho = \pi \circ q: G \to T^{ap}$ extends continuously to $\bar{\rho}: G^{ap} \to T^{ap}$, [3]. Now ρ has dense range, and $\bar{\rho}$ has closed range by compactness of G^{ap}, so in fact $\bar{\rho}$ is surjective. Define $\check{\rho}: C(T^{ap}) \to C(G^{ap})$ by $\check{\rho}(f)(t) = f(\bar{\rho}(t))$. Then $\check{\rho}$ is a monomorphism, and its adjoint $\check{\rho}^*: M(G^{ap}) \to M(T^{ap})$ is a homomorphism.

The map $\check{\rho}^*$ is weak*-weak* continuous, and has weak* dense range, in that it contains all the point masses. Given $\nu \in M(T^{ap})$ with $\nu \geqslant 0$, $\|\nu\| = 1$, there is a net $(\nu_\alpha) \subset M(T^{ap})$ of convex combination of point masses converging weak* to ν. Take a net $(\mu_\alpha) \subset M(G^{ap})$ with $\check{\rho}^*(\mu_\alpha) = \nu_\alpha$, $\mu_\alpha \geqslant 0$, $\|\mu_\alpha\| \leqslant 1$. By weak* compactness (μ_α) has a cluster point $\mu \in M(G^{ap})$, and by weak*-weak* continuity $\check{\rho}^*(\mu) = \nu$. It follows that $\check{\rho}^*$ is surjective.

Now T^{ap} is an abelian compact group, so if is it not finite, $M(T^{ap})$ admits a non-zero point derivation at some character. By Lemma 2.1, the same holds for $M(G^{ap})$.

Now $AP(G)^\perp$, the kernel of the restriction map $X^* \to AP(G)^*$, is a weak* closed ideal in X^* by Lemma 1.1. Since $M(G^{ap}) \simeq AP(G)^* \simeq X^*/AP(G)^\perp$, by Lemma 2.1 again, X^* is not weakly amenable, contrary to hypothesis. ∎

COROLLARY 4.4 ([14, Corollary 3.3]). *Suppose G is abelian, and let X be an introverted subspace of $L^\infty(G)$ containing $AP(G)$. Then X^* weakly amenable implies that G is finite. In particular, if $L^1(G)^{**}$ is weakly amenable then G is finite.*

COROLLARY 4.5. *For G the three dimensional Heisenberg group, the "$ax+b$" group or the motion group, X^* cannot be weakly amenable for any left introverted subspace X of $L^\infty(G)$ containing $AP(G)$.*

Set $W_0(G)$ to be the space of all $f \in W(G)$ such that 0 lies in the weak closure of $\{r_x f : x \in G\}$. By [10, or 6], $W_0(G)$ is a closed translation invariant subspace of $W(G)$, and in terms of the (unique) invariant mean m_G on $W(G)$, $W_0(G) = \{f \in W(G) : m_G(|f|) = 0\}$. In fact $W(G) = AP(G) \oplus W_0(G)$, but once again we need more. Let Q: $W(G) \to AP(G)$ be the natural projection with kernel $W_0(G)$. It is well known that $W(G) \simeq C(G^w)$, where the spectrum G^w is the weakly almost periodic compactification of G. This latter is a semitopological semigroup with unique minimal ideal which is a compact group, [6]. If e is the identity of this group then for $f \in W(G)$, $Q(f) = r_e f$. In particular, $\|Q\| = 1$. As in Lemma 3.1 this implies that Q^*: $AP(G)^* \to W(G)^*$ is an isometry.

LEMMA 4.6. *There is the direct sum decomposition*

$$W(G)^* = Q^*(AP(G)^*) \oplus AP(G)^\perp,$$

where $Q^*(AP(G)^*)$ is a closed subalgebra, and $AP(G)^\perp$ is a weak closed ideal of $W^*(G)$.*

Proof. Since Q commutes with right translations, and $(n \cdot f)(x) = \langle n, \ell_x f \rangle$ for $n \in W(G)^*$, $f \in W(G)$ and $x \in G$, the argument of Lemma 1.3 shows that Q^* is a homomorphism.

For $m \in W(G)^*$ let m' be the restriction of m to $AP(G)$. Then for $f \in AP(G)$,

$$\langle m - Q^*(m'), f \rangle = m(f) - m'(Qf) = m(f) - m(f) = 0.$$

It follows that $W(G)^* = Q^*(AP(G)^*) \oplus AP(G)^\perp$.

That $AP(G)^\perp$ is a weak* closed ideal follows from Lemma 1.1. ▮

THEOREM 4.7. *Suppose that $W(G)^*$ is weakly amenable. Then $M(G)$ and $AP(G)^*$ are weakly amenable.*

Proof. Weak amenability of $M(G)$ is a special case of Theorem 4.2. The result for $AP(G)^*$ follows immediately from Lemmas 2.3 and 4.6. ▮

Now let N be a closed normal subgroup of G, and set

$$\begin{aligned} W_N &= \{ f \in W(G) : f \text{ is constant on cosets of } N \} \\ &= \{ f \in W(G) : \ell_h f = f \text{ for } h \in N \}. \end{aligned}$$

Then W_N is a left introverted subspace contained in $W(G)$. Let m_N be the unique invariant mean on $W(N)$. Given $f \in W(G)$, the function $f^N: x \mapsto m_N(\ell_x f)$ is constant on the cosets of N, so lies in W_N. Set $P: W(G) \to W_N: f \mapsto f^N$.

LEMMA 4.8. *There is the direct sum decomposition*

$$W(G)^* = P^*((W_N)^*) \oplus W_N^\perp,$$

where $W_N^\perp$ is a weak closed ideal of $W(G)^*$.*

Proof. For $x, y \in G$, $f \in W(G)$,

$$\begin{aligned} (P(\ell_x f))(y) &= m_N(\ell_y(\ell_x f)) = m_N(\ell_{yx} f) \\ &= f^N(yx) = (\ell_x(f^N))(y) = (\ell_x(Pf))(y), \end{aligned}$$

so that P commutes with left translations. As in Lemma 3.1. P^* is an isometry, and Lemma 1.5 shows that it is a homomorphism. That $W_N^\perp$ is an ideal follows from Lemma 1.1. ▮

THEOREM 4.9. *Suppose that $W(G)^*$ is weakly amenable. Then $W(G/N)^*$ is weakly amenable for any closed, normal subgroup N of G.*

Proof. By [7, Lemma 2.3], the map $J: W_N \to W(G/N)$ given by $(Jf)(\bar{x}) = f(x)$ is an isometric surjection. By Lemma 1.5, $J^*: W(G/N)^* \to (W_N)^*$ is an isometric algebra epimorphism. Now use Lemma 4.8. ▮

COROLLARY 4.10. *Suppose that $W(G)^*$ is weakly amenable.*

(i) *If G is connected and G_d is amenable, then $G = \{e\}$.*

(ii) *If G is a connected $[IN]$-group, then G is compact.*

LEMMA 4.11. *Suppose that G is amenable, and let $A \neq \{0\}$ be a weak* closed self-adjoint translation invariant subalgebra of $L^\infty(G)$. Then there exists a norm one projection P: $L^\infty(G) \to A$ which is a left $L^1(G)$-module homomorphism, and P^* is an isometric algebraic isomorphism from A^* into $L^\infty(G)$. Thus there is the direct sum decomposition*

$$L^\infty(G)^* = P^*(A^*) \oplus A^\perp,$$

where $A^\perp$ is a weak closed ideal of $L^\infty(G)^*$.*

Proof. By Lemma 1.2, A is is left introverted. Since G is amenable, [39, Theorem 3.3] shows there exists a norm one projection P from $L^\infty(G)$ to A which commutes with all weak*-weak* continuous operators on $L^\infty(G)$ which commute with right translation. The map P^*: $A^* \to L^\infty(G)^*$ is certainly a linear isometry; we need to show it is a homomorphism.

Note firstly that for $\phi \in L^1(G)$, the map T_ϕ on $L^\infty(G)$ defined by $T_\phi(f) = \phi * f$ is weak*-weak* continuous and trivially commutes with right translations. Thus P commutes with such T_ϕ, and so P^* is a homomorphism by Lemma 1.3.

That $A^\perp$ is a weak* closed ideal follows from Lemma 1.1. ∎

LEMMA 4.12. *Let G be amenable, and suppose that $L^1(G)^{**}$ is weakly amenable. Then for any weak* closed, self-adjoint, translation invariant subalgebra A of $L^\infty(G)$, the algebra A^* is weakly amenable.*

Proof. Immediate from Lemmas 4.11 and 2.3. ∎

THEOREM 4.13. *Let G be amenable, N a closed normal subgroup of G. Then $L^1(G)^{**}$ weakly amenable implies the same for $L^1(G/N)^{**}$.*

Proof. Define a map T: $C_{00}(G) \to C_{00}(G/N)$ by

$$(Tf)(\bar{x}) = \int_N f(xh)\, dh. \tag{4.1}$$

This map is in fact surjective, and extends to a continuous algebraic epimorphism of $L^1(G)$ onto $L^1(G/N)$, [50]. By scaling of Haar measures, we may assume that the Weil formula holds:

$$\int_{G/N} \left(\int_N f(xh)\, dh \right) d\bar{x} = \int_G f(x)\, dx \qquad (f \in L^1(G)).$$

Define

$$A = \{ f \in L^\infty(G) : \ell_x f = f \text{ for all } x \in N \}.$$

Then A is a weak* closed, self-adjoint, translation invariant subalgebra A of $L^\infty(G)$. By Lemma 4.12 it suffices to show that $L^1(G/N)^{**}$ and A^* are isomorphic as Banach algebras; we will show that T^{**} is the desired isomorphism.

For any $f' \in CB(G/N)$, the function $f(x) = f'(\bar{x})$ lies in A. Taking $\phi \in C_{00}(G)$, we have

$$\begin{aligned} \langle T^* f', \phi \rangle = \langle f', T\phi \rangle &= \int_{G/N} f'(\bar{x}) \int_N \phi(xh)\, dh\, d\bar{x} \\ &= \int_{G/N} \int_N f(xh)\, \phi(xh)\, dh\, d\bar{x} \\ &= \int_G f(x)\, \phi(x)\, dx = \langle f, \phi \rangle, \end{aligned}$$

showing that $T^*(f') = f$.

It is clear that $\|T^*\| \leqslant 1$. For $f' \in L^\infty(G/N)$ and $\varepsilon > 0$, take $\phi' \in C_{00}(G/N)$ such that $\|\phi'\|_1 \leqslant 1$, and $\langle f', \phi' \rangle \geqslant \|f'\| - \varepsilon$. The proof ([50]) of the surjectivity of T: $C_{00}(G) \to C_{00}(G/N)$ shows that there is $\phi \in C_{00}(G)$ such that $\|\phi\|_1 = \|\phi'\|_1$ and $T(\phi) = \phi'$. Thus

$$\|f'\| \geqslant |\langle T^* f', \phi \rangle| = |\langle f', T\phi \rangle = |\langle f', \phi' \rangle| \geqslant \|f'\| - \varepsilon.$$

It follows that T^* is an isometry on $CB(G/N)$.

Now the range of T^*, being norm closed, is also weak* closed. Since the range contains $CB(G) \cap A$ which is weak* dense by [39, Lemma 2.1], it must thus be all of A. So we have that T^* is a linear isometry of $L^\infty(G/N)$ onto A. Thus T^{**} is a linear isometry of $L^\infty(G/N)^*$ onto A^*.

Finally, T^{**} is a homomorphism since T is a homomorphism [8, Theorem 6.1]. ∎

We remark that any weak* closed, self-adjoint, translation invariant subalgebra A of $L^\infty(G)$ is isomorphic to $L^\infty(G/N)$ for some closed normal subgroup N of G, [46]. However, to ensure the desired algebraic isomorphism of the duals we need to be more explicit.

As indicated earlier, the following related results are implicit in [19].

PROPOSITION 4.14. (i) *Suppose that* $L^1(G)^{**}$ *is weakly amenable. Then* $LUC(G)^*$ *is weakly amenable.*

(ii) *Suppose that $LUC(G)^*$ is weakly amenable. Then $M(G)$ is weakly amenable.*

Proof. (i) follows from Lemma 2.3 and the decomposition

$$L^1(G)^{**} = EL^1(G)^{**} \oplus (I-E)L^1(G)^{**},$$

where E is a right identity of $L^1(G)^{**}$ with $E \geqslant 0$, $\|E\| = 1$. The isomorphism $EL^1(G)^{**} \simeq LUC(G)^*$ is shown in [17].

(ii) This time we use the decomposition, [18],

$$LUC(G)^* = M(G) \oplus C_0(G)^{\perp}. \quad \blacksquare$$

The conclusions of the following are the same as in Corollary 4.10.

COROLLARY 4.15. *Suppose that $L^1(G)^{**}$ is weakly amenable.*

(i) *If G is connected and G_d is amenable, then $G = \{e\}$.*

(ii) *If G is an $[IN]$-group, then G is compact.*

Proof. Theorems 3.3 and 3.4. ∎

5. WEAK AMENABILITY OF $L^1(G)$

A Banach space X is a left Banach G-module if it is a left G-module such that

(i) there is $k \geqslant 0$ such that $\|x \cdot \phi\| \leqslant k \, \|\phi\|$ for all $x \in G$, $\phi \in X$;

(ii) for $\phi \in X$, the map $G \to X$: $x \mapsto x \cdot \phi$ is continuous.

Similarly for right Banach G-modules, and two-sided Banach G-modules, where in the latter case we require $G \times G \to X$: $(x, y) \mapsto x \cdot \phi \cdot y$ to be continuous. Any left G-module can, and will, be taken to be two-sided with the trivial action $\phi \cdot x = \phi$ on the right.

For X a left Banach G-module, define an action of G on X^* by

$$\langle f \cdot x, \phi \rangle = \langle f, x \cdot \phi \rangle \qquad (f \in X^*, \phi \in X, x \in G).$$

Thus X^* becomes a right G-module, however the continuity condition (ii) need not be satisfied.

Further, for $\phi \in X$, $\mu \in M(G)$, define

$$\phi \cdot \mu = \int x \cdot \phi \, d\mu(x).$$

This makes X into a right Banach $M(G)$-module, and by restriction, a right Banach $L^1(G)$-module. For the latter we have $X \cdot L^1(G) = X$ by the Cohen factorization theorem. Similarly for X a right or two-sided Banach G-module. For more details see [31, Section 2].

For any Banach space Y, we will say that a net $(m_\alpha) \subset Y^*$ converges weak$^\sim$ to $m \in Y^*$ if $m_\alpha \to m$ weak* *and* $\|m_\alpha\| \to \|m\|$. Often this will be used on a sphere, where of course it coincides with weak* convergence. In particular, if $\mu \in M(G)$, let $\nu \in L^\infty(G)^*$ be a norm preserving extension of μ. By Goldstine's theorem, there is a net $(\phi_\alpha) \subset L^1(G)$ with $\|\phi_\alpha\| \leqslant \|\mu\|$, and $\phi_\alpha \to \nu$ weak*. By passing to a suitable subnet we may assume that $\|\phi_\alpha\| \to \|\mu\|$, so that, in fact, $\phi_\alpha \to \mu$ weak$^\sim$.

The proof of the next result is similar to [11] and has its roots in [32], [26], [34]. We avoid explicit use of a bounded approximate identity in $L^1(G)$ because of a later analogue (see Lemma 6.8) where this avenue is not available.

LEMMA 5.1. *Suppose that X is a two-sided G-module.*

(a) *Any derivation $\bar{D}: M(G) \to X^*$ is weak$^\sim$-weak* continuous, and hence is an extension of a derivation $D: L^1(G) \to X$.*

(b) *Any derivation $D: L^1(G) \to X^*$ extends to a unique derivation $\bar{D}: M(G) \to X$.*

Proof. (a) If $\bar{D}: M(G) \to X^*$ is a derivation, then for any $\theta_1, \theta_2 \in L^1(G)$, $\mu \in M(G)$, and $x \in X$,

$$\langle \bar{D}\mu, \theta_1 \cdot x \cdot \theta_2 \rangle = \langle \bar{D}(\mu \cdot \theta_1), x \cdot \theta_2 \rangle - \langle D\theta_1, x \cdot \theta_2 \cdot \mu \rangle. \tag{5.1}$$

Now suppose that $\mu_\alpha \to \mu$ weak$^\sim$ in $M(G)$. Then by [22], $\|\mu_\alpha * \phi - \mu * \phi\| \to 0$ for $\phi \in L^1(G)$. Further, if $\mu \mapsto \mu^\sim$ denotes the natural isometric involution on $M(G)$, [30, Section 20], $\mu_\alpha^\sim \to \mu^\sim$ weak$^\sim$, so that

$$\|\phi * \mu_\alpha - \phi * \mu\| = \|(\mu_\alpha^\sim * \phi^\sim - \mu^\sim * \phi^\sim)^\sim\| \to 0.$$

It follows from (5.1) that

$$\langle \bar{D}\mu_\alpha, \theta_1 \cdot x \cdot \theta_2 \rangle \to \langle \bar{D}(\mu \cdot \theta_1), x \cdot \theta_2 \rangle - \langle D\theta_1, x \cdot \theta_2 \cdot \mu \rangle = \langle \bar{D}\mu, \theta_1 \cdot x \cdot \theta_2 \rangle.$$

Since $L^1(G) \cdot X \cdot L^1(G) = X$ by the Cohen factorization theorem, we thus have that $\bar{D}\mu_\alpha \to \bar{D}\mu$ weak*.

(b) For $\mu \in M(G)$ take a net $(\phi_\alpha) \subset L^1(G)$ with $\phi_\alpha \to \mu$ weak$^\sim$. As above, the net $(D\phi_\alpha)$ converges in the weak* topology.

Define

$$\bar{D}\mu = \text{w*-lim}\, D\phi_\alpha \qquad (\mu \in M(G), (\phi_\alpha) \subset L^1(G), \phi_\alpha \to \mu \text{ weak}^\sim).$$

Then $\bar{D}: M(G) \to X^*$ is a well-defined bounded linear operator extending D. Furthermore, for $\theta_1, \theta_2 \in L^1(G)$, $\mu \in M(G)$, (5.1) holds.

To see that $\bar{D}$ is a derivation, take $\mu \in M(G)$. As above, take a net $(\phi_\alpha) \subset L^1(G)$ with $\phi_\alpha \to \mu$ weak$^\sim$. Then for $\eta \in M(G)$, $\phi \in L^1(G)$, and $x \in X$,

$$\begin{aligned}\langle \bar{D}(\eta\mu), \phi \cdot x \rangle &= \lim_\alpha \langle \bar{D}(\eta \cdot \phi_\alpha), \phi \cdot x \rangle \\ &= \lim_\alpha \langle \bar{D}\eta, (\phi_\alpha \phi) \cdot x \rangle + \lim_\alpha \langle D\phi_\alpha, \phi \cdot x \cdot \eta \rangle \\ &= \langle \bar{D}\eta \cdot \mu, \phi \cdot x \rangle + \langle \bar{D}\mu \cdot \eta, \phi \cdot x \rangle,\end{aligned}$$

using (5.1) and the fact that $\|\phi_\alpha * \phi - \mu * \phi\| \to 0$. Since $L^1(G) \cdot X = X$ the derivation identity follows.

Suppose Δ is any derivation extending D to $M(G)$, let $\mu \in M(G)$, and take $(\phi_\alpha) \subset L^1(G)$ with $\phi_\alpha \to \mu$ weak$^\sim$. Then $\phi_\alpha \to \mu$ weak$^\sim$, whence $\Delta\mu = \lim_\alpha D\phi_\alpha = \bar{D}\mu$ by (a). ∎

For a W*-algebra A, we write A^s for the set of self-adjoint elements of A. For $x \in A$ we write $x_1 = \frac{1}{2}(x + x^*)$, $x_2 = (1/2i)(x - x^*)$, so that $x_i \in A^s$ and $x = x_1 + ix_2$. The set of positive normal functionals on A will be written A_*^+.

LEMMA 5.2. *Let G be a locally compact group, A a W*-algebra such that the predual A_* is a two-sided G-module such that $x \cdot \phi \geqslant 0$ and $\phi \cdot x \geqslant 0$ for each $x \in G$ and each $\phi \in A_*^+$. Let $D: L^1(G) \to A$ be a derivation, $\bar{D}: M(G) \to A$ its extension. Define*

$$h_i^\phi(t) = \langle (\delta_{t^{-1}} \bar{D}\delta_t)_i, \phi \rangle \qquad (t \in G, \phi \in A_*^+, i = 1, 2).$$

Suppose that m is a right invariant function on $CB_{\mathbb{R}}(G)$ such that

$$m(f + c\mathbf{1}) = m(f) + c \qquad (f \in CB_{\mathbb{R}}(G), c \in \mathbb{R}),$$

and that there exist $f_1, f_2 \in A^s$ such that

$$\langle m, h_i^\phi \rangle = \langle f_i, \phi \rangle \qquad (\phi \in A_*^+, i = 1, 2).$$

Then $\bar{D}$ is inner, indeed, with $f = f_1 + if_2$, $\bar{D}\mu = \mu \cdot f - f \cdot \mu$, $(\mu \in M(G))$.

Proof. For $i = 1, 2$ and $\phi \in A_*^+$, $\|h_i^\phi\| \leqslant \|\bar{D}\| \, \|\phi\|$. Further, if $(t_\alpha) \subset G$, with $t_\alpha \to t$, then $\delta_{t_\alpha} \to \delta_t$ weak$^\sim$, so that

$$\begin{aligned}|h_i^\phi(t_\alpha) - h_i^\phi(t)| &= |\langle \bar{D}\delta_{t_\alpha}, \phi \cdot \delta_{t_\alpha^{-1}} \rangle - \langle \bar{D}\delta_t, \phi \cdot \delta_{t^{-1}} \rangle| \\ &\leqslant \|\bar{D}\| \, \|\phi \cdot \delta_{t_\alpha^{-1}} - \phi \cdot \delta_{t^{-1}}\| + |\langle \bar{D}\delta_{t_\alpha} - \bar{D}\delta_t, \phi \cdot \delta_{t^{-1}} \rangle| \\ &\to 0,\end{aligned}$$

by Lemma 5.1 and the continuity of the action of G on A_*. This ensures that $h_i^\phi \in CB_{\mathbb{R}}(G)$. Now for $t, x \in G$,

$$\delta_{t^{-1}} \cdot \bar{D}\delta_t = \delta_{t^{-1}} \cdot \bar{D}(\delta_{tx^{-1}} * \delta_x)$$
$$= \delta_{x^{-1}} \cdot [(\delta_{(tx^{-1})^{-1}} \bar{D}(\delta_{tx^{-1}})] \cdot \delta_x + \delta_{x^{-1}} \cdot \bar{D}(\delta_x).$$

Thus for $\phi \in A_*^+$, and $i = 1, 2$,

$$\langle (\delta_{t^{-1}} \cdot \bar{D}(\delta_t)_i, \phi \rangle$$
$$= \langle \delta_{(tx^{-1})^{-1}} \bar{D}(\delta_{tx^{-1}})_i, \delta_x \cdot \phi \cdot \delta_{x^{-1}} \rangle + \langle \delta_{x^{-1}} \cdot \bar{D}(\delta_x)_i, \phi \rangle.$$

Applying m to both sides, as functions of the variable t, we deduce that

$$\langle m \circ h_i, \phi \rangle = \langle \delta_x \cdot (m \circ h_i) \cdot \delta_x, \phi \rangle + \langle \delta_{x^{-1}} \cdot \bar{D}(\delta_x)_i, \phi \rangle,$$

where $\langle m \circ h_i, \phi \rangle = \langle m, h_i^\phi \rangle$.

It follows that $\bar{D}\delta_x = \delta_x \cdot f - f \cdot \delta_x$ for $x \in G$. But for $\mu \in M(G)$ with $\mu \geqslant 0$, $\|\mu\| = 1$, there is a convex combination of point masses converging weak*, and hence weak$^\sim$, to μ. Thus by Lemma 5.1 we have $\bar{D}\mu = \mu \cdot f - f \cdot \mu$. But then $\bar{D}$ is inner, whence so is its restriction to $L^1(G)$. ▮

We remark that any right invariant positive functional on $CB_{\mathbb{R}}(G)$ with norm one satisfies the condition of Lemma 5.2. Also, if the sets $\{\delta_{t^{-1}} \cdot \bar{D}(\delta_t)_i: t \in G\}$ have a least upper bound in A^s, then the above conditions are satisfied by taking $m(h) = \sup\{h(t): t \in G\}$ (which is right invariant, but not linear). This is always the case when A is commutative, since then $A \simeq L^\infty(Z, \nu)$ for some measure space (Z, ν) that is a direct sum of finite measure spaces, and so A^s is a complete vector lattice. In particular, we have the following result.

THEOREM 5.3. *Let A a commutative W^*-algebra such that the predual A_* is a two-sided G-module with $x \cdot \phi \geqslant 0$ and $\phi \cdot x \geqslant 0$ for each $x \in G$ and each $\phi \in A_*^+$. Then any derivation D: $L^1(G) \to A$ is inner.*

Let Z be a locally compact Hausdorff space, and suppose there is given a jointly continuous action $G \times Z \to Z$ for which Z has a quasi-invariant measure ν. For $s \in G$, set $\nu_s(E) = \nu(s^{-1}E)$ for each Borel set $E \subset Z$. Then each $\nu_s \ll \nu$; let Θ_s be the Radon–Nikodym derivative: $\nu_s = \Theta_s \nu$. The space $L^1(Z, \nu)$ is a non-degenerate left G-module under the actions

$$\phi \cdot s = \phi, \quad (s \cdot \phi)(t) = \Theta_s(t)\phi(s^{-1}t) \qquad (s, t \in G, \phi \in L^1(Z, \nu)),$$

see [24].

COROLLARY 5.4. *Any derivation* $D: L^1(G) \to L^\infty(Z, \nu)$ *is inner.*

Remark. It follows from [40, Theorem 4.1] that if G is *not* amenable, then there exists a Banach G-module such that $x \cdot s = x$ for $x \in X$, $s \in G$, and a bounded derivation $D: L^1(G) \to X^*$ which is not inner.

For a closed subgroup H of G, there always exists an essentially unique quasi-invariant measure ν on the coset space G/H so we have the following.

COROLLARY 5.5. *For any closed subgroup* H *of* G, *any derivation* $D: L^1(G) \to L^\infty(G/H)$ *is inner, regarding* $L^1(G/H)$ *as a left* G*-module.*

This result should be compared with [33, Corollary 2.2].

6. INTROVERTED SUBSPACES OF $VN(G)$

Let $P(G) \subset CB(G)$ be the set of continuous positive definite functions on G, $B(G)$ its linear span. The space $B(G)$ can be identified with the dual of the group C*-algebra $C^*(G)$, this latter being the completion of $L^1(G)$ under its largest C*-norm. Indeed, we have the duality

$$\langle \phi, f \rangle = \int_G \phi f \qquad (\phi \in B(G), f \in L^1(G)).$$

With pointwise multiplication and dual norm, $B(G)$ is a commutative Banach algebra, the *Fourier–Stieltjes algebra* of G, [12].

Denote by $P_\rho(G)$ the closure of $P(G) \cap C_{00}(G)$ in the compact-open topology, and $B_\rho(G)$ its linear span. Then $B_\rho(G)$ is a closed ideal in $B(G)$, and is the dual of the reduced C*-algebra $C^*_\rho(G)$, this latter being the norm closure in $\mathscr{B}(L^2(G))$ of the convolution operators $\{\rho(f): f \in L^1(G)\}$, where $\rho(f)(h) = f * h$, $h \in L^2(G)$. It is well known that $B_\rho(G) = B(G)$ if and only if G is amenable.

Finally, the *Fourier algebra* $A(G)$ of G is the closed linear span of $P(G) \cap C_{00}(G)$ in $B(G)$. It is a closed ideal of $B(G)$, and is contained in $B_\rho(G)$. Each $\phi \in A(G)$ has the form $\phi(x) = \langle \lambda(x) h, k \rangle$ where $h, k \in L^2(G)$ and λ is the left regular representation of G on $L^2(G)$. Consequently, any $\phi \in A(G)$ may be regarded as a σ-weak continuous linear functional on $VN(G)$, the von Neumann algebra generated by the representation λ. Indeed, $A(G)$ is the predual of $VN(G)$.

In the case when G is abelian, with dual group $\hat{G}$, we have isometric isomorphisms $B(G) \simeq M(\hat{G})$, $A(G) \simeq L^1(\hat{G})$ and $VN(G) \simeq L^\infty(\hat{G})$.

Since $A(G)$ is an ideal in $B(G)$, $VN(G)$ is a commutative $B(G)$-bimodule under the action

$$\langle \phi \cdot T, \psi \rangle = \langle T, \psi\phi \rangle \qquad (\phi \in B(G), \psi \in A(G), T \in VN(G)).$$

Left introverted subspaces of $VN(G)$ include

$$UC(\hat{G}) = \text{norm closed linear span of } A(G) \cdot VN(G),$$

$$W(\hat{G}) = \{T \in VN(G) : A(G) \to VN(G) : \phi \mapsto \phi \cdot T \text{ is weakly compact}\},$$

$$AP(\hat{G}) = \{T \in VN(G) : A(G) \to VN(G) : \phi \mapsto \phi \cdot T \text{ is compact}\},$$

$$C^*_\delta(G) = \text{norm closed linear span of } \{\lambda(x) \colon x \in G\},$$

and, of course $C^*_\rho(G)$. In general $W(\hat{G}) \supseteq AP(\hat{G}) \supseteq C^*_\delta(G)$, and $W(\hat{G}) \supseteq C^*_\rho(G)$. When G is abelian, $UC(\hat{G})$ is precisely the algebra of bounded uniformly continuous functions on the dual group $\hat{G}$, $W(\hat{G})$ is the space of weakly almost periodic functions on $\hat{G}$, and $AP(\hat{G}) = C^*_\delta(G)$ is the algebra of almost periodic functions on $\hat{G}$ [23, 36, 38].

Proposition 6.1. *Suppose that $W(\hat{G})^*$ is weakly amenable. Then for any $AP(G)$-invariant, closed subspace X of $W(\hat{G})^*$, X^* is weakly amenable. In particular, $AP(\hat{G})$ is weakly amenable.*

Proof. Note firstly that X^* is left introverted by Lemma 1.2. By Lemma 1.4 or [36, Theorem 5.6], $W(\hat{G})^*$ is commutative. The restriction map $\Psi\colon W(\hat{G})^* \to X^*$ is a continuous algebra homomorphism of the *commutative* Banach algebra $W(\hat{G})^*$ onto X^*, so that X^* is weakly amenable. ▮

Corollary 6.2. *Suppose that $W(\hat{G})^*$ is weakly amenable. Then for any closed subspace X of $W(\hat{G})^*$, X^* is weakly amenable.*

Proof. By Lemma 1.4 or [36, Theorem 5.6], $X \subseteq W(\hat{G})$. ▮

Proposition 6.3. (a) *Suppose that G is amenable and $VN(G)^*$ is weakly amenable. Then $UC(\hat{G})^*$ is weakly amenable.*

(b) *Suppose that $UC(\hat{G})^*$ is weakly amenable. Then $B_\rho(G)$ and $A(G)$ are weakly amenable.*

Proof. (a) When G is amenable, $A(G)$ has a bounded approximate identity (e_α) such that $\|e_\alpha\| = 1$ and $e_\alpha \geqslant 0$, [36]. Take a weak* cluster point E of (e_α). Then E is a right identity for $VN(G)^*$, $E \geqslant 0$ and $\|E\| = 1$.

Furthermore, $UC(\hat{G})^*$ is isometrically isomorphic to $E \cdot VN(G)^*$, see [42]. But then

$$VN(G)^* = E \cdot VN(G)^* \oplus (I - E) \cdot VN(G)^*,$$

where $(I - E) \cdot VN(G)^*$ is a closed ideal. Lemma 2.3 shows that $E \cdot VN(G)^*$, and hence $UC(\hat{G})^*$, is weakly amenable.

(b) By [44, Lemma 5.2],

$$UC(\hat{G})^* = B_\rho(G) \oplus C_\rho(G)^\perp.$$

Thus Lemma 2.3 shows $B_\rho(G)$ is weakly amenable. Consequently, $A(G)$ is weakly amenable by [14, Proposition 3.6]. ∎

Remark. It follows from Lemma 6.5 below that if $VN(G)^*$ is weakly amenable, then $UC(\hat{G})^* = B_\rho(G)$ and $A(G)$ is weakly amenable, [13, Theorem 3].

For any $X \subseteq VN(G)$, define

$$\Sigma(X) = \{x \in G : \lambda(x) \in X\},$$

here $\lambda = \lambda_G$: $G \to VN(G)$ is the left regular representation of G. For a non-empty closed subgroup H of G, define $\Phi_H(G) = \mathrm{sp}\{\rho_G(x): x \in H\}$, and set $VN_H(G)$ to be its ultraweak closure. If X is an invariant weak* closed subalgebra of $VN(G)$, then $H = \Sigma(X)$ is a non-empty closed subgroup of G, and $X = VN_H(G)$, [51, Theorems 6 and 8].

LEMMA 6.4. *Let X be an invariant W^*-algebra subalgebra of $VN(G)$ such that $\Sigma(X)$ is a normal subgroup of G. Suppose that $VN(G)^*$ is weakly amenable. Then X^* is weakly amenable.*

Proof. Note that X is left introverted by [36, Lemma 5.1]. Using [41, Theorem 2] there exists a continuous projection P: $VN(G) \to X$ such that $P(\phi \cdot T) = \phi \cdot P(T)$, $(T \in VN(G), \phi \in A(G))$. In fact $P(T)$ lies in the ultraweak closure of $\mathscr{RO}(T)$. Thus $\|P(T)\| \leqslant \|T\|$. It follows that P^*: $X^* \to VN(G)^*$ is a linear isometry. An argument analogous to that of Lemma 1.3 shows that P^* is an algebraic homomorphism. Thus

$$VN(G)^* = P^*(X^*) \oplus X^\perp,$$

where, by Lemma 1.1, $X^\perp$ is a weak* closed ideal in $VN(G)^*$. So $P^*(X^*)$, and consequently X^*, is weakly amenable by Lemma 2.3. ∎

LEMMA 6.5. *Let X be a left introverted subspace of $VN(G)$ with $C^*_\delta(G) \subseteq X$. If X^* weakly amenable, then G is discrete.*

Proof. It has been shown by Forrest, [14, Theorem 3.2], that every abelian subgroup of G is finite, and, moreover, G must be totally disconnected. Let H be a compact open subgroup H of G; such exists in any compact open neighbourhood of the identity in G, [30, Theorem 7.5]. If H is infinite, then by [53, Theorem 2] H would contain an infinite abelian subgroup, which is impossible. Thus H is a finite open subgroup of G, so that G must be discrete. ▮

The following is an analogue of Theorem 4.2. Note that if G is abelian, and H is a closed subgroup of G, then $VN(H) \simeq L^\infty(\hat{H}) \simeq L^\infty(\hat{G}/H^\perp)$, where $H^\perp = \{\chi \in \hat{G} : \chi(h) = 1 \text{ for all } h \in H\}$.

THEOREM 6.6. *Suppose that $VN(G)^*$ is weakly amenable. Then G is discrete, and furthermore $VN(H)^*$ is weakly amenable for any normal subgroup H of G.*

Proof. Discreteness is immediate from Lemma 6.5. The restriction map $Q: A(G) \to A(H)$ is a surjective, norm decreasing homomorphism, [12, Theorem 2.20, 48]. Thus $Q^*: VN(H) \to VN(G)$ is one-to-one.

Then for $x \in H$, $Q^*(\lambda_H(x)) = \lambda_G(x)$, so that Q^* maps $\Phi_H(H) = \text{sp}\{\rho_H(x): x \in H\}$ onto $\Phi_H(G)$. Since the range of Q is closed, the range of Q^* is weak* closed, and so we deduce that $Q^*(VN(H)) \supseteq VN_H(G)$.

Now for $T \in VN(H)$ with $\|T\| \leqslant 1$, the Kaplansky density theorem gives a net $(T_\alpha) \subset \Phi_H(H)$ such that $\|T_\alpha\| \leqslant 1$, and $T_\alpha \to T$ ultraweakly in $VN(H)$. Thus $Q^*(T_\alpha) \to Q^*(T)$ ultraweakly in $VN(G)$. But $(Q^*(T_\alpha))$ is a bounded net in $\Phi_H(G)$, and so converges to an element of $VN_H(G)$. Thus Q^* is a one-to-one map of $VN(H)$ onto $VN_H(G)$.

It is clear that $Q^*: \Phi_H(H) \to \Phi_H(G)$ is a $*$-algebraic isomorphism. By the ultraweak continuity of the involution, and separate continuity of multiplication, it follows that Q^* is a $*$-algebraic isomorphism of $VN(H)$ onto $VN_H(G)$. As a conquence Q^* is an isometry.

By [8, Theorem 6.1], Q^{**} is a homomorphism, and so we have that Q^{**} is an isometric algebraic isomorphism of $VN_H(G)^*$ onto $VN(H)^*$. But the former algebra is weakly amenable by Lemma 6.3, so that $VN(H)^*$ is weakly amenable. ▮

COROLLARY 6.7 (Granirer [23]). *Suppose that $VN(G)^*$ is amenable, then G is finite.*

Proof. When $VN(G)^*$ is amenable, it has a bounded approximate identity, so that G is compact by [38, Proposition 3.2(b)], and hence finite since discrete. ▮

Remark. This is an analogue of a recent result of [19]. The compactness result was in fact noted in [44, Section 4], it also appears in [14, Theorem 4.4].

The proof of the next result is analogous to that of Lemma 5.1, so only a brief outline will be given, indicating the necessary changes.

LEMMA 6.8. *Let X be a weak*-closed invariant subspace of $VN(G)$.*

(a) *Any derivation $\bar{D}: B_\rho(G) \to X$ is weak$^\sim$-weak* continuous, and hence an extension of a derivation $D: A(G) \to X$.*

(b) *Any derivation $D: A(G) \to X$ extends to a unique derivation $\bar{D}: B_\rho(G) \to X$.*

Proof. (a) Let $\bar{D}: B_\rho(G) \to X$ is a derivation, and suppose $\psi \in B_\rho(G)$ and $\phi_\alpha \to \psi$ weak$^\sim$ in $B_\rho(G)$. Then as in Lemma 5.1, for $\theta_1, \theta_2 \in A(G)$,

$$\langle \bar{D}\psi_\alpha, \theta_1\theta_2 \rangle \to \langle D(\psi \cdot \theta_1), \theta_2 \rangle - \langle D\theta_1, \psi \cdot \theta_2 \rangle = \langle \bar{D}\psi, \theta_1\theta_2 \rangle.$$

Since the empty set is a set of synthesis, [12], $A(G)^2$ is norm dense in $A(G)$. The net $(\bar{D}(\psi_\alpha))$ is bounded and so it follows that $\bar{D}\psi_\alpha \to \bar{D}\psi$ weak*.

(b) For $\psi \in B_\rho(G)$ there is a net $(\phi_\alpha) \subset A(G)$ and $\phi_\alpha \to \psi$ weak$^\sim$. Thus by [22] we have $\|\phi_\alpha \cdot \phi - \psi \cdot \phi\| \to 0$ for each $\phi \in A(G)$.

Now let $\psi \in B_\rho(G)$, and take $(\phi_\alpha) \subset A(G)$ with $\phi_\alpha \to \psi$ weak$^\sim$. The same argument as for Lemma 5.1 shows that that $(D\phi_\alpha)$ converges in the $\sigma(VN(G), A(G)^2)$-topology, and hence, as above, $(D\phi_\alpha)$ converges weak* in $VN(G)$. The rest of the proof follows Lemma 5.1. ∎

For H a closed subgroup of G, $B(G)$ acts on $A(H)$ by $\psi \cdot \phi = \psi|_H \phi$, $(\psi \in B(G),\ \phi \in A(G))$, and $\|\psi \cdot \phi\| \leqslant \|\psi|_H\| \, \|\phi\| \leqslant \|\psi\| \, \|\phi\|$ by [12, Theorem 2.20]. Hence we may define an action of $B(G)$ on $VN(H)$ by

$$\langle T \cdot \psi, \phi \rangle = \langle T, \psi \cdot \phi \rangle = \langle T, \psi|_H \phi \rangle,$$

where $\psi \in B(G)$, $\phi \in A(G)$, $T \in VN(H)$.

THEOREM 6.9. *Let H be a closed subgroup of G.*

(a) *Any derivation $\bar{D}: B_\rho(G) \to VN(H)$ is weak$^\sim$-weak* continuous, and hence is an extension of a derivation $D: A(G) \to VN(H)$.*

(b) *Any derivation $D: A(G) \to VN(H)$ extends uniquely to a derivation $\bar{D}: B_\rho(G) \to VN(H)$.*

(c) *Suppose $A(G)$ is weakly amenable. Then any derivation $\bar{D}: B_\rho(G) \to VN(H)$ is necessarily zero.*

Proof. Let $Q: A(G) \to A(H)$ denote the restriction map. As in the proof of Theorem 6.4, Q^* is a $*$-isomorphism of the von Neumann algebras $VN(H)$ and $VN_H(G)$. Since $VN_H(G)$ is an invariant weak* closed subalgebra of $VN(G)$ and $Q^*(\psi \cdot T) = \psi \cdot Q^*(T)$ for any $\psi \in B_\rho(G)$, $T \in VN(H)$, (a) and (b) follow from Lemma 6.4. Finally, (c) follows from (a). ∎

7. INTROVERTED SUBSPACES OF $A_p(G)^*$

In this section we will indicate how some of the results of Section 6 can be extended to left introverted subspaces of $PM_p(G) = A_p(G)^*$, $1 < p < \infty$. We first need a little additional notation.

Take $1 < p < \infty$, and $x \in G$. The operator ℓ_x: $L^p(G) \to L^p(G)$ is defined in the usual way. For $f \in L^1(G)$, the operator $\lambda(f)$: $L^p(G) \to L^p(G)$ is defined by $\lambda(f)h = f * h$, $h \in L^p(G)$. The Figa–Talamanca–Herz algebra $A_p(G)$ is the space of functions f: $G \to \mathbb{C}$ which can be represented (non-uniquely)

$$f = \sum_{n=1}^{\infty} v_n * \tilde{u}_n,$$

where $(u_n) \subset L^p(G)$, $(v_n) \subset L^q(G)$, $1/p + 1/q = 1$, and $\sum \|u_n\|_p \|v_n\|_q < \infty$. The norm of f is defined as

$$\|f\| = \inf \sum_n \|u_n\|_p \|v_n\|_q,$$

the infimum taken over all possible representations of f. It is well known that $A_p(G)$ is a subspace of $C_0(G)$, and equipped with the above norm and pointwise multiplication is a regular tauberian commutative Banach algebra with spectrum G. The Fourier algebra $A(G)$ is just $A_2(G)$, [29].

The weak operator closure of $L^1(G)$ in $\mathscr{B}(L^p(G))$ will be denoted by $PM_p(G)$. Then $PM_p(G) = A_p(G)^*$, in particular, $PM_2(G) = VN(G)$.

Let $B_p(G)$ be the multiplier algebra of $A_p(G)$, that is, the continuous functions v on G such that $v\phi \in A_p(G)$ for all $\phi \in A_p(G)$. Define a norm on $B_p(G)$ by

$$\|v\| = \sup\{ \|v\phi\|_{A_p(G)} : \|\phi\|_{A_p(G)} \leqslant 1\}.$$

The multipliers act on $PM_p(G)$ in the obvious way:

$$\langle \phi \cdot T, \psi \rangle = \langle T, \phi\psi \rangle \qquad (\phi \in B_p(G), \psi \in A_p(G)).$$

Set

$$UC_p(\hat{G}) = \text{norm closed linear span of } A_p(G) \cdot PM_p(G),$$

$$W_p(\hat{G}) = \{ T \in PM_p(G) : \phi \mapsto \phi \cdot T \text{ is weakly compact}\},$$

$$AP_p(\hat{G}) = \{ T \in PM_p(G) : \phi \mapsto \phi \cdot T \text{ is compact}\}.$$

See [21].

LEMMA 7.1. *The spaces, $AP_p(\hat{G})$, $W_p(\hat{G})$, and $UC_p(\hat{G})$ are left introverted subspaces of $PM_p(G)$.*

Proof. Lemma 1.2 shows that $AP_p(\hat{G})$ and $W_p(\hat{G})$ are introverted. For $UC_p(\hat{G})$, let $m \in UC_p(\hat{G})^*$, and take $\phi \in A_p(G)$, $S \in PM_p(G)$. Then for $T = \phi \cdot S$, $m \cdot T = \phi \cdot (m \cdot S) \in UC_p(\hat{G})$. More generally, if $T \in UC_p(\hat{G})$ take $(T_n) \subset \operatorname{span} A_p(G) \cdot UC_p(\hat{G})$ with $\|T_n - T\| \to 0$. Then $\|m \cdot T_n - m \cdot T\| \leqslant \|m\| \, \|T_n - T\| \to 0$, whence $m \cdot T \in UC_p(\hat{G})$. ∎

Remark. It follows from Lemmas 1.2 and 1.4 that Proposition 6.1 and Corollary 6.2 remain valid when $W(G)^*$, $A(G)$ and $AP(\hat{G})$ are replaced by $W_p(G)$, $A_p(G)$ and $AP_p(\hat{G})$, $1 < p < \infty$, respectively.

PROPOSITION 7.2. *Let $S \subset B_p(G)$ $(1 < p < \infty)$ be a norm bounded semigroup, and set*

$$F_S = \{ f \in PM_p(G) : \phi \cdot f = f \text{ for all } \phi \in S \}.$$

Then F_S is left introverted. Furthermore, if $PM_p(G)^*$ is weakly amenable, then F_S^* is weakly amenable.

Proof. Note that F_S is clearly $A_p(G)$-invariant and weak* closed in $PM_p(G)^*$, so by Lemma 1.2(a) F_S is left introverted. By [20, Theorem 6] there is a linear projection P: $PM_p(G) \to F_S$ such that $P(\phi \cdot f) = \phi \cdot P(f)$ for $\phi \in A_p(G)$. Thus Lemma 1.3 shows that P^*: $F_S^* \to PM_p(G)^*$ is an algebra homomorphism. But P^* is one-to-one, and so is an algebraic isomorphism of F_S^* onto $P^*(F_S^*)$. This latter is closed since $F_S = P(PM_p(G))$ is closed, and we see that P^* is a topological isomorphism. Thus

$$PM_p(G)^* = P^*(F_S^*) \oplus F_S^{\perp},$$

where $F_S^{\perp}$ is a weak* closed ideal by Lemma 1.1. Thus $P^*(F_S^*)$, and hence F_S^* is weakly amenable by Lemma 2.3. ∎

In the case $p = 2$, let H be a closed subgroup of G, and set

$$P_H = \{ \phi \in P(G) : \phi(x) = 1 \text{ for all } x \in H \}.$$

Then $\|\phi\| = \phi(e) = 1$ for $\phi \in P_H$, so that P_H is certainly a bounded semigroup in $B_p(G)$. Define

$$F(H) = \{ T \in VN(G) : \phi \cdot T = T \text{ for all } \phi \in P_H \}.$$

COROLLARY 7.3. *Suppose that $VN(G)^*$ is weakly amenable. Then for any subgroup H of G, $F(H)$ is weakly amenable.* ∎

Corollary 7.3 should be compared with Lemma 6.4. Note that if X is an invariant W*-algebra of $VN(G)$ such that $\Sigma(X)=H$ is a normal subgroup of G, then $X=F(H)$, see [41, Lemma 6].

8. CONCLUDING REMARKS

It should be noted that our inheritance results in Sections 3, 4, 6 and 7 which make use of Lemma 2.3 all remain valid when weak amenability is replaced by amenability.

We close with three open problems; the latter two can also be asked for amenability.

1. If $VN(G)^*$ is weakly amenable, must G be finite?

2. If H is an *open* subgroup of G, does $M(G)$ not weakly amenable imply $M(H)$ not weakly amenable? (This may fail for closed subgroups: $\mathbb{Z} \subset \mathbb{R}$.)

3. Does G compact, $M(G)$ weakly amenable, imply G finite?

ACKNOWLEDGMENTS

This paper was written when the second author was visiting the Department of Mathematical Sciences, University of Alberta in October 1995. The hospitality of the department is gratefully acknowledged. The work is supported by an NSERC grant.

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