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Harmonic Liouville Theorem for Exterior Domains[1]

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We give a very simple function theoretic proof to a Liouville type theorem for harmonic functions defined on exterior domains obtained and proved in a convexity theoretic method by F. Cammaroto and A. Chinnì. The theorem itself is also slightly generalized.

Key Words: harmonic functions; Liouville type theorem; exterior domain; Picard principle.

We denote by $\mathbb{C}$ the complex plane so that points z in $\mathbb{C}$ are complex numbers $z = x + iy$, where x and y are real numbers. Cammaroto and Chinnì [2] obtained in essence the following result.

THEOREM 1. *Let K be a compact set in $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected and u be a harmonic function on $\mathbb{C} \setminus K$ bounded from below. Then the function u is constant on $\mathbb{C} \setminus K$ if and only if the inequality*

$$|\nabla u(z)|\,|\nabla u_{xx}(z)| \le |\nabla u_x(z)|^2 \tag{1}$$

holds for every $z = x + iy$ in $\mathbb{C} \setminus K$.

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Actually K was supposed to be a nonempty compact *convex* set of $\mathbb{C}$ and the additional hypothesis

$$\lim_{z\to\infty} \frac{u(z)}{|z|} = 0 \tag{2}$$

was postulated in [2], where ∞ is the point at infinity of $\mathbb{C}$ so that $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane (i.e., Riemann sphere). Especially in [2] it is explicitly asked whether (2) can be removed or not. Thus we claim here that Theorem 1 above is slightly more general than the original Cammaroto–Chinnì main theorem in [2] in the respect that the convexity of K and the hypothesis (2) are not postulated.

The Cammaroto–Chinnì proof, which is very interesting in its own right, of their main theorem which is more restricted than the above Theorem 1, relies essentially upon a functional analytic technique, especially on a result in convexity theory. The *purpose* of this paper is to give an elementary, simple, and direct proof of Theorem 1 by using only an introductory function theory of the undergraduate level.

Proof of Theorem 1. The constancy of u trivially implies (1) and thus we only have to prove assuming (1) that u is constant on $\mathbb{C} \setminus \overline{\Delta}(0, R)$ in view of the uniqueness theorem for harmonic functions, where $\Delta(0, R)$ is the open disc of radius $R > 0$ centered at the origin 0 containing K and $\overline{\Delta}(0, R)$ is the closure of $\Delta(0, R)$. On replacing u by $u - \inf_{\mathbb{C}\setminus K} u$ if necessary, we may assume without loss of generality that $u \geq 0$ on $\mathbb{C} \setminus \Delta(0, R)$. Hence u is harmonic and nonnegative on $\mathbb{C} \setminus \Delta(0, R)$. Then $u(1/z)$ is harmonic and nonnegative on $\overline{\Delta}(0, 1/R) \setminus \{0\}$ and hence the Picard principle or the principle of positive singularity or the Bôcher theorem (cf., e.g., [1, p. 50; 3]) assures that there are a nonnegative real constant c and a harmonic function w on $\overline{\Delta}(0, 1/R)$ such that

$$u\left(\frac{1}{z}\right) = c \log \frac{1}{|z|} + w(z) \tag{3}$$

for $z \in \overline{\Delta}(0, 1/R) \setminus \{0\}$. Observe that v given by $v(z) := w(1/z)$ is harmonic and automatically bounded on $\hat{\mathbb{C}} \setminus \Delta(0, R)$ and

$$u(z) = c \log|z| + v(z) \tag{4}$$

for z in $\mathbb{C} \setminus \Delta(0, R)$. Then we have

$$\begin{cases} u_x(z) = v_x(z) + c\dfrac{x}{x^2 + y^2}, \\ u_y(z) = v_y(z) + c\dfrac{y}{x^2 + y^2}. \end{cases} \tag{5}$$

The function f given by

$$f(z) = u_x(z) - iu_y(z) \tag{6}$$

is holomorphic on $\mathbb{C} \setminus \Delta(0, R)$. We also consider the function g given by

$$g(z) = v_x(z) - iv_y(z). \tag{7}$$

Then g is holomorphic on $\mathbb{C} \setminus \Delta(0, R)$. From (5), (6), and (7), it follows that

$$f(z) = g(z) + \frac{c}{z} \tag{8}$$

for z in $\mathbb{C} \setminus \Delta(0, R)$. In order to conclude that u is constant on $\mathbb{C} \setminus \Delta(0, R)$ it suffices to prove that $c = 0$ and $g(z) \equiv 0$ on $\mathbb{C} \setminus \Delta(0, R)$.

We denote by f' the complex derivative df/dz of f which is identical with $\partial f/\partial x$. Observe that

$$\begin{cases} |\nabla u|^2 = u_x^2 + u_y^2 = |u_x - iu_y|^2 = |f|^2, \\ |\nabla u_x|^2 = u_{xx}^2 + u_{xy}^2 = |(u_x - iu_y)_x|^2 = |f'|^2, \\ |\nabla u_{xx}|^2 = u_{xxx}^2 + u_{xxy}^2 = |(u_x - iu_y)_{xx}|^2 = |f''|^2. \end{cases}$$

These with (1) imply that

$$|f(z)|\,|f''(z)| \le |f'(z)|^2 \tag{9}$$

for every z in $\mathbb{C} \setminus \Delta(0, R)$.

Recall that v is harmonic on $\hat{\mathbb{C}} \setminus \Delta(0, R)$. Consider the function $G := v + iv^*$ with the harmonic conjugate v^* to v on $\hat{\mathbb{C}} \setminus \Delta(0, R)$, which is holomorphic and automatically bounded on $\hat{\mathbb{C}} \setminus \Delta(0, R)$ since $\hat{\mathbb{C}} \setminus \Delta(0, R)$ is contained in a simply connected domain $\hat{\mathbb{C}} \setminus \overline{\Delta}(0, R')$ with $0 < R' < R$ sufficiently close to R so that $\Delta(0, R')$ contains K and v is harmonic on $\hat{\mathbb{C}} \setminus \overline{\Delta}(0, R')$. Then $G' = v_x + iv_x^* = v_x - iv_y = g$ on $\mathbb{C} \setminus \Delta(0, R)$. Let the Taylor expansion of $G(z)$ about ∞ be

$$G(z) = \sum_{j \ge 0} \frac{a_j}{z^j}$$

for z in $\mathbb{C} \setminus \Delta(0, R)$. Then from $g = G'$ on $\mathbb{C} \setminus \Delta(0, R)$ it holds that

$$g(z) = \sum_{j \ge 2} \frac{c_j}{z^j} \tag{10}$$

for z in $\mathbb{C} \setminus \Delta(0, R)$ by setting $c_j := -(j-1)a_{j-1}$ $(j \ge 2)$.

We now show that $c = 0$ and $g(z) \equiv 0$. To begin with we prove that $c = 0$. For this purpose we use (8) and (10) to deduce the relations

$$\begin{cases} f(z) = \dfrac{c}{z} + \displaystyle\sum_{j \geq 2} \dfrac{c_j}{z^j}, \\ f'(z) = -\dfrac{c}{z^2} - \displaystyle\sum_{j \geq 2} \dfrac{jc_j}{z^{j+1}}, \\ f''(z) = \dfrac{2c}{z^3} + \displaystyle\sum_{j \geq 2} \dfrac{j(j+1)c_j}{z^{j+2}}, \end{cases}$$

which with (9) yield that

$$\left| \frac{c}{z} + \sum_{j \geq 2} \frac{c_j}{z^j} \right| \left| \frac{2c}{z^3} + \sum_{j \geq 2} \frac{j(j+1)c_j}{z^{j+2}} \right| \leq \left| \frac{c}{z^2} + \sum_{j \geq 2} \frac{jc_j}{z^{j+1}} \right|^2 \tag{11}$$

or, by multiplying $|z|\,|z^3| = |z^2|^2$ to both sides of (11), we have

$$\left| \sum_{j \geq 2} \frac{c_j}{z^{j-1}} + c \right| \left| \sum_{j \geq 2} \frac{j(j+1)c_j}{z^{j-1}} + 2c \right| \leq \left| \sum_{j \geq 2} \frac{jc_j}{z^{j-1}} + c \right|^2 \tag{12}$$

for z in $\mathbb{C} \setminus \overline{\Delta}(0, R)$. On letting $z \to \infty$ in (12) we deduce $2|c|^2 \leq |c|^2$, which concludes that $c = 0$.

Finally we wish to show that $g(z) \equiv 0$. On the contrary assume that $g(z) \not\equiv 0$. Then there is the smallest $k \geq 2$ among numbers $j \geq 2$ with $c_j \neq 0$. Then

$$g(z) = \sum_{j \geq k} \frac{c_j}{z^j} \qquad (c_k \neq 0,\ k \geq 2)$$

for $z \in \mathbb{C} \setminus \overline{\Delta}(0, R)$. In view of $c = 0$, (11) takes the form

$$\left| \sum_{j \geq k} \frac{c_j}{z^j} \right| \left| \sum_{j \geq k} \frac{j(j+1)c_j}{z^{j+2}} \right| \leq \left| \sum_{j \geq k} \frac{jc_j}{z^{j+1}} \right|^2.$$

Multiplying $|z^k|\,|z^{k+2}| = |z^{k+1}|^2$ to the both sides of the above displayed inequality, we obtain

$$\left| c_k + \sum_{j > k} \frac{c_j}{z^{j-k}} \right| \left| k(k+1)c_k + \sum_{j > k} \frac{j(j+1)c_j}{z^{j-k}} \right| \leq \left| kc_k + \sum_{j > k} \frac{jc_j}{z^{j-k}} \right|^2$$

for z in $\mathbb{C} \setminus \overline{\Delta}(0, R)$. On making $z \to \infty$ in the above displayed inequality, we see that $|c_k|\,|k(k+1)c_k| \leq |kc_k|^2$ so that $k + 1 \leq k$ since $|c_k| \neq 0$, a contradiction. ▮

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