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Local cohomology properties of direct summands

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ABSTRACT

In this article, we prove that if $R \to S$ is a homomorphism of Noetherian rings that splits, then for every $i \geq 0$ and ideal $I \subset R$, $\mathrm{Ass}_R H^i_I(R)$ is finite when $\mathrm{Ass}_S H^i_{IS}(S)$ is finite. In addition, if S is a Cohen–Macaulay ring that is finitely generated as an R-module, such that all the Bass numbers of $H^i_{IS}(S)$, as an S-module, are finite, then all the Bass numbers of $H^i_I(R)$, as an R-module, are finite. Moreover, we show these results for a larger class a functors introduced by Lyubeznik [5]. As a consequence, we exhibit a Gorenstein F-regular UFD of positive characteristic that is not a direct summand, not even a pure subring, of any regular ring.

1. Introduction

Throughout this article all rings are commutative Noetherian with unity. Let R denote a ring. If M is an R-module and $I \subset R$ is an ideal, we denote the i-th local cohomology of M with support in I by $H^i_I(M)$. If I is generated by the elements $f_1, \dots, f_\ell \in R$, these cohomology groups can be computed by the Čech complex,

$$0 \to M \to \oplus_j M_{f_j} \to \dots \to M_{f_1 \cdots f_\ell} \to 0.$$

The structure of these modules has been widely studied by several authors. Among the results obtained, one encounters the following finiteness properties for certain regular rings:

(1) the set of associated primes of $H^i_I(R)$ is finite;
(2) the Bass numbers of $H^i_I(R)$ are finite;
(3) $\mathrm{inj.dim} H^i_I(R) \leq \dim \mathrm{Supp} H^i_I(R)$.

Huneke and Sharp proved these properties for characteristic $p > 0$ [3]. Lyubeznik showed them for local regular rings of equal characteristic zero and finitely generated regular algebras over a field of characteristic zero [5].

These properties have been proved for a larger family of functors introduced by Lyubeznik [5]. If $Z \subset \mathrm{Spec}(R)$ is a closed subset and M is an R-module, we denote by $H^i_Z(M)$ the i-th local cohomology module of M with support in Z. We notice that $H^i_Z(M) = H^i_I(M)$, where $Z = \mathcal{V}(I) = \{P \in \mathrm{Spec}(R) : I \subset P\}$. For two closed subsets of $\mathrm{Spec}(R)$, $Z_1 \subset Z_2$, there is a long exact sequence of functors

$$\dots \to H^i_{Z_1} \to H^i_{Z_2} \to H^i_{Z_1/Z_2} \to \dots . \qquad (1)$$

We denote by $\mathcal{T}$ any functor of the form $\mathcal{T} = \mathcal{T}_1 \circ \dots \circ \mathcal{T}_t$, where every functor $\mathcal{T}_j$ is either H^i_Z or $H^i_{Z_1/Z_2}$ for some closed subsets Z, Z_1, Z_2 of $\mathrm{Spec}(R)$, or the kernel, image or cokernel of some morphism in the previous long exact sequence for some closed subsets Z_1, Z_2 of $\mathrm{Spec}(R)$.

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Our aim in this manuscript is to prove the finiteness properties (1) and (2) for direct summands. We need to make some observations before we are able to state our theorems precisely. Let $R \to S$ be a homomorphism of Noetherian rings. For an ideal $I \subset R$, we have two functors associated with it, $H^i_I(-) : R\text{-mod} \to R\text{-mod}$ and $H^i_{IS}(-) : S\text{-mod} \to S\text{-mod}$, which are naturally isomorphic when we restrict them to S-modules. Moreover, for two ideals of R, $I_2 \subset I_1$, the natural morphism $H^i_{I_1}(-) \to H^i_{I_2}(-)$ is the same as the natural morphism $H^i_{I_1S}(-) \to H^i_{I_2S}(-)$ when we restrict the functors to S-modules. Thus, their kernel, cokernel and image are naturally isomorphic as S-modules. Hence, every functor $\mathcal{T}$ for R is a functor of the same type for S when we restrict it to S-modules.

As per the previous discussion, for an S-module, M, we will make no distinction in the notation or meaning of $\mathcal{T}(M)$ whether it is induced by ideals of R or their extensions to S and, therefore, by the corresponding closed subsets of their respective spectra. Now, we are ready to state our main results.

Theorem 1.1. *Let $R \to S$ be a homomorphism of Noetherian rings that splits. Suppose that $\operatorname{Ass}_S \mathcal{T}(S)$ is finite for a functor $\mathcal{T}$ induced by extension of ideals of R. Then, $\operatorname{Ass}_R \mathcal{T}(R)$ is finite. In particular, $\operatorname{Ass}_R H^i_I(R)$ is finite for every ideal $I \subset R$, if $\operatorname{Ass}_S H^i_{IS}(S)$ is finite.*

Theorem 1.2. *Let $R \to S$ be a homomorphism of Noetherian rings that splits. Suppose that S is a Cohen–Macaulay ring such that all the Bass numbers of $\mathcal{T}(S)$, as an S-module, are finite for a functor $\mathcal{T}$ induced by extension of ideals of R. If S is a finitely generated R-module, then all the Bass numbers of $\mathcal{T}(R)$, as an R-module, are finite. In particular, for every ideal $I \subset R$ the Bass numbers of $H^i_I(R)$ are finite, if the Bass numbers of $H^i_{IS}(S)$ are finite.*

The first theorem holds when S is a polynomial ring over a field and R is the invariant ring of an action of a linearly reductive group over S. It also holds when $R \subset K[x_1, \dots, x_n]$ is an integrally closed ring that is finitely generated as a K-algebra by monomials. This is because such a ring is a direct summand of a possibly different polynomial ring (cf. Proposition 1 and Lemma 1 in [1]).

We would like to mention another case in which an inclusion splits. This is when $R \to S$ is a module finite extension of rings containing a field of characteristic zero such that S has finite projective dimension as an R-module. Moreover, such a splitting exists when Koh's conjecture holds (cf. [4,8,9]). Therefore, if Koh's conjecture applies to $R \to S$ and $\mathcal{T}(S)$ has finite associated primes or finite Bass numbers, so does $\mathcal{T}(R)$.

We point out that property (3) does not hold for direct summands of regular rings, even in the finite extension case. A counterexample is $R = K[x^3, x^2y, xy^2, y^3] \subset S = K[x, y]$, where S is the polynomial ring in two variables with coefficients in a field K. The splitting of the inclusion is the map $\theta : S \to R$ defined in the monomials by $\theta(x^\alpha y^\beta) = x^\alpha y^\beta$ if $\alpha + \beta \in 3\mathbb{Z}$ and as zero otherwise. We have that the dimension of $\operatorname{Supp}(H_{(x^3,x^2y,xy^2,y^3)}(R))$ is zero, but it is not an injective module, because R is not a Gorenstein ring, since $R/(x^3, y^3)R$ has a two dimensional socle.

The manuscript is organized as follows. In Section 2, we prove Theorem 1.1, and we show some consequences. In particular, in Corollary 2.1, we exhibit a Gorenstein F-regular UFD of positive characteristic that is not a direct summand, not even a pure subring, of any regular ring. In Section 3, we give a proof for Theorem 1.2.

2. Associated primes

Lemma 2.1. *Let $R \to S$ be an injective homomorphism of Noetherian rings, and let M be an S-module. Then, $\operatorname{Ass}_R M \subset \{Q \cap R : Q \in \operatorname{Ass}_S M\}$.*

Proof. Let $P \in \operatorname{Ass}_R M$ and $u \in M$ be such that $\operatorname{Ann}_R u = P$. We have that $(\operatorname{Ann}_S u) \cap R = P$. Let $Q_1, \dots, Q_t$ denote the minimal primes of $\operatorname{Ann}_S u$. We obtain that

$$P = \sqrt{P} = \sqrt{\operatorname{Ann}_S u} \cap R = (\cap_j Q_j) \cap R = \cap_j (Q_j \cap R),$$

so there exists a Q_j such that $P = Q_j \cap R$. Since Q_j is a minimal prime for $\operatorname{Ann}_S u$, we have that $Q_j \in \operatorname{Ass}_S M$ and the result follows. □

Definition 2.1. We say that a homomorphism of Noetherian rings $R \to S$ is pure if $M = M \otimes_R R \to M \otimes_R S$ is injective for every R-module M. We also say that R is a pure subring of S.

Proposition 2.1 (*Cor. 6.6 in [2]*). *Suppose that $R \to S$ is a pure homeomorphism of Noetherian rings and that $\mathcal{G}$ is a complex of R-modules. Then, the induced map $j : H^i(\mathcal{G}) \to H^i(\mathcal{G} \otimes_R S)$ is injective.*

Proposition 2.2. *Let $R \to S$ be a pure homomorphism of Noetherian rings. Suppose that $\operatorname{Ass}_S H^i_I(R)$ is finite for some ideal $I \subset R$ and $i \geq 0$. Then, $\operatorname{Ass}_S H^i_{IS}(S)$ is finite.*

Proof. Since $H^i_I(R) \to H^i_{IS}(S)$ is injective by Proposition 2.1, $\operatorname{Ass}_R H^i_I(R) \subset \operatorname{Ass}_R H^i_{IS}(S)$ and the result follows by Lemma 2.1. □

Proof of Theorem 1.1. The splitting between R and S makes $\mathcal{T}(R)$ into a direct summand of $\mathcal{T}(S)$; in particular, $\mathcal{T}(R) \subset \mathcal{T}(S)$. Therefore, $\operatorname{Ass}_R \mathcal{T}(R) \subset \operatorname{Ass}_R \mathcal{T}(S)$ and the result follows by Lemma 2.1. □

If R is a ring containing a field of characteristic $p > 0$, Theorem 1.1 gives a method for showing that R is not a direct summand of a regular ring. In Corollary 2.1, we used this method to prove that there exists a Gorenstein strongly F-regular UFD of characteristic $p > 0$ that is not a direct summand of any regular ring. This is, to the best of the author's knowledge, the first such example.

Theorem 2.1 (*Theorem 5.4 in [7]*). *Let K be a field of prime characteristic, and consider the hypersurface*

$$R = \frac{K[r, s, t, u, v, w, x, y, z]}{(su^2x^2 + sv^2y^2 + tuxvy + rw^2z^2)}.$$

Then, R is a unique factorization domain for which the local cohomology module $H^3_{(x,y,z)}(R)$ has infinitely many associated prime ideals. This is preserved if R is replaced by the localization at its homogeneous maximal ideal. The hypersurface R has rational singularities if K has characteristic zero, and it is F-regular if K has positive characteristic.

Corollary 2.1. *Let R be as in the previous theorem taking K of positive characteristic. Then, R is a Gorenstein F-regular UFD that is not a pure subring of any regular ring. In particular, R is not a direct summand of any regular ring.*

Proof. Since $H^3_{(x,y,z)}(R)$ has infinitely many associated prime ideals, it cannot be a direct summand or pure subring of a regular ring by Theorem 1.1, Proposition 2.1 and finiteness properties of regular rings of positive characteristic (cf. [6]). $\square$

Theorem 2.2 (*Theorem 1 in [10]*). *Assume that $S = K[x_1, \dots, x_n]$ is a polynomial ring in n variables over a field K of characteristic $p > 0$. Suppose that $I = (f_1, \dots, f_s)$ is an ideal of S such that $\sum_i \deg f_i < n$. Then $\dim S/Q \geq n - \sum_i \deg f_i$ for all $Q \in \mathrm{Ass}_S H^i_I(S)$.*

Corollary 2.2. *Let $S = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K of characteristic $p > 0$. Let $R \to S$ be a splitting homomorphism of Noetherian rings such that S is a finitely generated R-module. Suppose that $I = (f_1, \dots, f_s)$ is an ideal of R such that $\sum_i \deg(f_i) < \dim R$. If S is a finitely generated R-module, then $\dim R/P \geq \dim R - \sum_i \deg f_i$ for all $P \in \mathrm{Ass}_R H^i_I(R)$.*

Proof. Since $H^i_I(-)$ commutes with a direct sum of R-modules, we have that a splitting of $R \hookrightarrow S$ over R induces a splitting of $H^i_I(R) \hookrightarrow H^i_I(S)$ over R. Then, by Lemma 2.1, for any $P \in \mathrm{Ass}_R H^i_I(R) \subset \mathrm{Ass}_R H^i_I(S)$ there exists $Q \in \mathrm{Ass}_R H^i_I(S)$ such that $P = Q \cap R$ and then $\dim R/P = \dim S/Q > n - \sum_i \deg f_i$, and the result follows. $\square$

3. Bass numbers

Lemma 3.1. *Let (R, m, K) be a local ring and M be an R-module. Then, the following are equivalent:*

(a) $\dim_K(\mathrm{Ext}^j_R(K, M))$ *is finite for all $j \geq 0$;*
(b) $\mathrm{length}(\mathrm{Ext}^j_R(N, M))$ *is finite for every finite length module N for all $j \geq 0$;*
(c) *there exists one module N of finite length such that* $\mathrm{length}(\mathrm{Ext}^j_R(N, M))$ *is finite for all $j \geq 0$.*

Proof. (a) $\Rightarrow$ (b): Our proof will be by induction on $h = \mathrm{length}(N)$. If $h = 1$, then $N = K$, and the proof follows from our assumption. We will assume that the statement is true for h and prove it when $\mathrm{length}(N) = h + 1$. In this case, there is a short exact sequence $0 \to K \to N \to N' \to 0$, where N' has length h. From the induced long exact sequence

$$\dots \to \mathrm{Ext}^{j-1}_R(N', M) \to \mathrm{Ext}^j_R(K, M) \to \mathrm{Ext}^j_R(N, M) \to \dots,$$

we see that $\mathrm{length}(\mathrm{Ext}^i_R(N, M))$ is finite for all $i \geq 0$.

(b) $\Rightarrow$ (c): Clear.

(c) $\Rightarrow$ (a): We will prove the contrapositive. Let j be the minimum non-negative integer such that $\dim_K(\mathrm{Ext}^j_R(K, M))$ is infinite. We claim that $\mathrm{length}(\mathrm{Ext}^i_R(N, M)) < \infty$ for $i < j$ and $\mathrm{length}(\mathrm{Ext}^j_R(N, M)) = \infty$ for any module N of finite length. Our proof will be by induction on $h = \mathrm{length}(N)$. If $h = 1$, then $N = K$ and it follows from our choice of j. We will assume that this is true for h and prove it when $\mathrm{length}(N) = h + 1$. We have a short exact sequence $0 \to K \to N \to N' \to 0$, where N' has length h. From the induced long exact sequence

$$\dots \to \mathrm{Ext}^{j-1}_R(N', M) \to \mathrm{Ext}^j_R(K, M) \to \mathrm{Ext}^j_R(N, M) \to \dots,$$

we have that $\mathrm{length}(\mathrm{Ext}^i_R(N, M)) < \infty$ for $i < j$ and that the map $\mathrm{Ext}^j_R(K, M)/\mathrm{Im}(\mathrm{Ext}^{j-1}_R(N', M)) \to \mathrm{Ext}^j_R(N, M)$ is injective. Therefore, $\mathrm{length}(\mathrm{Ext}^j_R(N, M)) = \infty$. $\square$

Lemma 3.2. *Let $R \to S$ be a pure homomorphism of Noetherian rings. Assume that S is a Cohen–Macaulay ring. If S is finitely generated as an R-module, then R is a Cohen–Macaulay ring.*

Proof. Let $P \subset R$ be a prime ideal. Let $\underline{x} = x_1, \ldots, x_d$ denote a system of parameters of R_P, where $d = \dim(R_P)$. It suffices to show that $H_i(\mathcal{K}(\underline{x}; R_P)) = 0$ for $i \neq 0$, where $\mathcal{K}$ is the Koszul complex with respect to $\underline{x}$. We notice that the natural inclusion $R_P \to S_P$ is a pure homeomorphism of rings. This induces an injective morphism of R-modules $H_i(\mathcal{K}(\underline{x}; R_P)) \to H_i(\mathcal{K}(\underline{x}; S_P))$ by Proposition 2.1. Thus, it is enough to show that $H_i(\mathcal{K}(\underline{x}; S_P)) = 0$ for $i \neq 0$. Since S_P is a module finite extension of R_P, we have that every maximal ideal $Q \subset S_P$ contracts to PR_P and $\underline{x}$ is a system of parameters for S_Q. Then, $H_i(\mathcal{K}(\underline{x}; S_Q)) = 0$ for $i \neq 0$ and every maximal ideal $Q \subset S_P$. Hence, $H_i(\mathcal{K}(\underline{x}; S_P)) = 0$ for $i \neq 0$ and the result follows. □

Proposition 3.1. *Let $R \to S$ be a homomorphism of Noetherian rings that splits. Assume that S is a Cohen–Macaulay ring and S is finitely generated as an R-module. Let N be an R-module and M be an S-module. Let $N \to M$ be a morphism of R-modules that splits. If all the Bass numbers of M, as an S-module, are finite, then all the Bass numbers of N, as an R-module, are finite.*

Proof. Since $N \hookrightarrow M$ splits, we have that $\mathrm{Ext}^i_{R_P}(R_P/PR_P, N_P)$ is a direct summand of $\mathrm{Ext}^i_{R_P}(R_P/PR_P, M_P)$, so, we may assume that $N = M$.

Let P be a fixed prime ideal of R and let K_P denote R_P/PR_P. Since we want to show that $\dim_{K_P}(\mathrm{Ext}^i_{R_P}(K_P, M_P))$ is finite, we may assume without loss of generality that R is local and P is its maximal ideal. Let $\underline{x} = x_1, \ldots, x_n$ be a system of parameters for R. Since R is Cohen–Macaulay by Lemma 3.2, we have that the Koszul complex, $\mathcal{K}_R(\underline{x})$, is a free resolution for R/I, where $I = (x_1, \ldots, x_n)$. We also have that for every maximal ideal $Q \subset S$ lying over P, $\underline{x}$ is a system of parameters of S_Q because $\dim R = \dim S_Q$ and S_Q/IS_Q is a zero dimensional ring. From the Cohen–Macaulayness of S and the previous fact, we have that the Koszul complex $\mathcal{K}_S(\underline{x})$ is a free resolution for S/IS. Therefore, $\mathrm{Ext}^i_R(R/I, M) = H^i(\mathrm{Hom}_R(\mathcal{K}_R(\underline{x}), M)) = H^i(\mathrm{Hom}_S(\mathcal{K}_S(\underline{x}), M)) = \mathrm{Ext}^i_S(S/IS, M)$. Since $\mathrm{Ext}^i_S(S/IS, M) = \oplus_Q \mathrm{Ext}^i_{S_Q}(S_Q/IS_Q, M_Q)$ has finite length as an S-module by Lemma 3.1, we have that $\mathrm{Ext}^i_R(R/I, M)$ has finite length as an R-module because S is finitely generated. Then, we have that $\dim_{K_P}(\mathrm{Ext}^i_R(K_P, M))$ is finite by Lemma 3.1. □

Proof of Theorem 1.2. The splitting between R and S induces a splitting between $\mathcal{T}(R) \hookrightarrow \mathcal{T}(S)$. The rest follows from Proposition 3.1. □

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