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Certificates of convexity for basic semi-algebraic sets[☆]

J.B. Lasserre [*]

LAAS-CNRS and Institute of Mathematics, University of Toulouse, France
LAAS, 7 Avenue du Colonel Roche, 31077 Toulouse Cédex 4, France

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ABSTRACT

We provide two certificates of convexity for arbitrary basic closed semi-algebraic sets of $\mathbb{R}^n$. The first one is based on a necessary and sufficient condition whereas the second one is based on a sufficient (but simpler) condition only. Both certificates are obtained from any feasible solution of a related semidefinite program and so, in principle, can be obtained numerically (however, up to machine precision).

1. Introduction

With $\mathbb{R}[x]$ being the ring of real polynomials in the variables $x_1, \ldots, x_n$, consider the basic closed semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$ defined by:

$$\mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0,\ j = 1, \ldots, m\} \tag{1.1}$$

for some given polynomials $g_j \in \mathbb{R}[x], j = 1, \ldots, m$.

By definition, $\mathbf{K} \subset \mathbb{R}^n$ is convex if and only if

$$x,\ y \in \mathbf{K} \Rightarrow \lambda\, x + (1 - \lambda)\, y \in \mathbf{K} \quad \forall \lambda \in [0, 1]. \tag{1.2}$$

The above geometric condition does *not* depend on the representation of $\mathbf{K}$ but requires uncountably many tests and so cannot be checked in general.

Of course concavity of g_j for every $j = 1, \ldots, m$, provides a certificate of convexity for $\mathbf{K}$ but not every convex set $\mathbf{K}$ in (1.1) is defined by concave polynomials. Hence an important issue is to analyze whether there exists a necessary and sufficient condition of convexity in terms of the representation (1.1) of $\mathbf{K}$ because after all, very often (1.1) is the only information available about $\mathbf{K}$. Moreover, a highly desirable feature would be that such a condition can be checked, at least numerically.

In a recent work [1], the author has provided an algorithm to obtain a numerical *certificate* of convexity for $\mathbf{K}$ in (1.1) by using the condition:

$$\langle \nabla g_j(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathbf{K} \text{ with } g_j(y) = 0, \tag{1.3}$$

which is equivalent to (1.2) provided that the Slater[1] condition holds and the nondegeneracy condition $\nabla g_j(y) \neq 0$ holds whenever $y \in \mathbf{K}$ and $g_j(y) = 0$. This certificate consists of an integer p_j and two polynomials $\theta_1^j, \theta_2^j \in \mathbb{R}[x, y]$ for each

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* Tel.: +33 561336415; fax: +33 561336936.
E-mail address: lasserre@laas.fr.

[1] The Slater condition holds if there exists $x_0 \in \mathbf{K}$ such that $g_k(x_0) > 0$ for every $k = 1, \ldots, m$.

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$j = 1, \ldots, m$, and their characterization obviously implies that (1.3) holds true and so $\mathbf{K}$ is convex (whence the name certificate); see Lasserre [1, Corollary 4.4]. More precisely, for every $j = 1, \ldots, m$, define the $2m+1$ polynomials $h_\ell \in \mathbb{R}[x, y]$ by $h_\ell(x, y) = g_\ell(x)$, $h_{m+\ell}(x, y) = g_\ell(y)$ for every $l = 1, \ldots, m$, and $h_{2m+1}(x, y) = -g_j(y)$. The *preordering* $P_j \subset \mathbb{R}[x, y]$ generated by the polynomials $(h_\ell) \subset \mathbb{R}[x, y]$ is defined by:

$$P_j = \left\{ \sum_{J\subseteq\{1,\ldots,2m+1\}} \sigma_J(x, y) \left(\prod_{\ell\in J} h_\ell(x, y) \right) : \quad \sigma_J \in \Sigma[x, y] \right\}, \tag{1.4}$$

where $\Sigma[x, y] \subset \mathbb{R}[x, y]$ is the set of polynomials that are *sums of squares* (in short s.o.s.), and where by convention, $\prod_{\ell\in J} h_\ell(x, y) = 1$ when $J = \emptyset$. Then by a direct application of Stengle's Positivstellensatz [2, Theor. 4.4.2, p. 92] (more precisely, a Nichtnegativstellensatz version) (1.3) holds if and only if

$$\theta_1^j(x, y)\langle\nabla g_j(y), x - y\rangle = (\langle\nabla g_j(y), x - y\rangle)^{2p_j} + \theta_2^j(x, y), \tag{1.5}$$

for some integer p_j and some polynomials $\theta_1^j, \theta_2^j \in P_j$. In addition, bounds (p, d) are available for the integer p_j and the degrees of the s.o.s. polynomials σ_J appearing in the definition (1.4) of polynomials $\theta_1^j, \theta_2^j \in P_j$, respectively. Observe that in (1.5) one may replace p_j with the fixed bound p (multiply each side with $(\langle\nabla g_j(y), x - y\rangle)^{2(p-p_j)}$) and take $d := d + p$. Next, recall that s.o.s. polynomials of bounded degree can be obtained from feasible solutions of an appropriate *semidefinite program*[2] (see e.g. [4]). Hence, in principle, checking whether (1.5) has a feasible solution (θ_1^j, θ_2^j) reduces to checking whether a *single* semidefinite program has a feasible solution.

And so, when both Slater and the nondegeneracy condition hold, checking whether $\mathbf{K}$ is convex reduces to checking if each of the semidefinite programs associated with (1.5), $j = 1, \ldots, m$, has a feasible solution. When $\mathbf{K}$ is convex, the $2m$ polynomials $\theta_1^j, \theta_2^j \in \mathbb{R}[x, y]$, $j = 1, \ldots, m$, provide the desired certificate of convexity through (1.3); see [1, Corollary 4.4]. However, it is only a *numerical* certificate because it comes from the output of a numerical algorithm, and so subject to unavoidable numerical inaccuracies. Moreover, the size of each semidefinite program equivalent to (1.5) is out of reach for practical computation, and in practice, one will solve a semidefinite program associated with (1.5) but for reasonable bounds $(p', d') \ll (p, d)$, hoping to obtain a solution when $\mathbf{K}$ is convex. An alternative and more tractable certificate of convexity using *quadratic modules* rather than preorderings is also provided in [1, Assumption 4.6], but it only provides a sufficient condition of convexity (almost necessary when $\mathbf{K}$ is compact and satisfies some technical condition).

The present contribution is to provide a certificate of convexity for *arbitrary* basic closed semi-algebraic sets (1.1), i.e., with *no* assumption on $\mathbf{K}$. This time, by certificate we mean an obvious guarantee that the geometric condition (1.2) holds true (instead of (1.3) in [1]). To the best of our knowledge, and despite the result being almost straightforward, it is the first of this type for arbitrary basic closed semi-algebraic sets. As in [1] our certificate also consists of two polynomials of $\mathbb{R}[x, y]$ and is also based on the powerful Stengle's Positivstellensatz in real algebraic geometry. In addition, a numerical certificate can also be obtained as the output of a semidefinite program (hence valid only up to machine precision). We also provide another certificate based on a simpler characterization which now uses only a sufficient condition for a polynomial to be nonnegative on $\mathbf{K}$; so in this case, even if $\mathbf{K}$ is convex, there is no guarantee to obtain the required certificate. Finally, we also provide a sufficient condition that permits us to obtain a numerical certificate of non-convexity of $\mathbf{K}$ in the form of points $x, y \in \mathbf{K}$ which violate (1.2).

2. Main result

Observe that in fact, (1.2) is equivalent to the simpler condition

$$x,\ y \in \mathbf{K} \Rightarrow (x + y)/2 \in \mathbf{K}. \tag{2.1}$$

Indeed if $\mathbf{K}$ is convex then of course (2.1) holds. Conversely, if $\mathbf{K}$ is not convex then there exists $x, y \in \mathbf{K}$ and $0 < \lambda < 1$ such that $z := x + \lambda(y - x)$ is not in $\mathbf{K}$. As $\mathbf{K}$ is closed, moving on the line segment $[x, y]$ from x to y, there necessarily exist $\tilde{x} \in \mathbf{K}$ (the first exit point of $\mathbf{K}$) and $\tilde{y} \in \mathbf{K}$ (the first re-entry point in $\mathbf{K}$), with $\tilde{x} \neq \tilde{y}$. Thus, as $\tilde{x}$ and $\tilde{y}$ are the only points of $[\tilde{x}, \tilde{y}]$ contained in $\mathbf{K}$, the mid-point $\tilde{z} := (\tilde{x} + \tilde{y})/2$ is not contained in $\mathbf{K}$.

Given the basic closed semi-algebraic set $\mathbf{K}$ defined in (1.1), let $\widehat{\mathbf{K}} := \mathbf{K} \times \mathbf{K} \subset \mathbb{R}^n \times \mathbb{R}^n$ be the associated basic closed semi-algebraic set defined by:

$$\widehat{\mathbf{K}} := \{(x, y) : \hat{g}_j(x, y) \geq 0, j = 1, \ldots, 2m\}, \tag{2.2}$$

where:

$$(x, y) \mapsto \hat{g}_j(x, y) := g_j(x), \quad j = 1, \ldots, m \tag{2.3}$$

$$(x, y) \mapsto \hat{g}_j(x, y) := g_{m-j}(y), \quad j = m + 1, \ldots, 2m, \tag{2.4}$$

[2] A semidefinite program is a convex optimization problem with the nice property that it can be solved efficiently. More precisely, up to arbitrary fixed precision, it can be solved in time polynomial in its input size. For more details on semidefinite programming and its applications, the interested reader is referred to e.g. [3].

and let $P(\hat{g}) \subset \mathbb{R}[x, y]$ be the preordering associated with the polynomials $(\hat{g}_j)$ that define $\widehat{\mathbf{K}}$ in (2.2), i.e.,

$$P(\hat{g}) := \left\{ \sum_{J\subseteq\{1,\dots,2m\}} \phi_J \left(\prod_{k\in J} \hat{g}_k \right) : \phi_J \in \Sigma[x, y] \right\}, \tag{2.5}$$

where $\Sigma[x, y] \subset \mathbb{R}[x, y]$ is the set of s.o.s. polynomials. Our necessary and sufficient condition of convexity is as follows.

Theorem 2.1. *Let $\mathbf{K} \subset \mathbb{R}^n$ be the basic closed semi-algebraic set defined in (1.1). Then $\mathbf{K}$ is convex if and only if for every $j = 1, \dots, m$, there exist polynomials $\sigma_j, h_j \in P(\hat{g})$ and an integer $p_j \in \mathbb{N}$ such that:*

$$\sigma_j(x, y)\, g_j((x + y)/2) = g_j((x + y)/2)^{2p_j} + h_j(x, y), \quad \forall\, x, y \in \mathbb{R}^n. \tag{2.6}$$

Proof. The set $\mathbf{K}$ is convex if and only if (2.1) holds, that is, if and only if for every $j = 1, \dots, m$,

$$g_j((x + y)/2) \geq 0 \quad \forall (x, y) \in \widehat{\mathbf{K}}. \tag{2.7}$$

But then (2.6) follows from a direct application of Stengle's Positivstellensatz [2, Theor. 4.4.2, p. 92] to (2.7) (in fact, a Nichtnegativstellensatz version). □

The polynomials $\sigma_j, h_j \in P(\hat{g}), j = 1, \dots, m$, obtained in (2.6) indeed provide an obvious certificate of convexity for $\mathbf{K}$. This is because if (2.6) holds then for every $x, y \in \mathbf{K}$ one has $\sigma_j(x, y) \geq 0$ and $h_j(x, y) \geq 0$ because $\sigma_j, h_j \in P(\hat{g})$; and so $\sigma_j(x, y)g_j((x + y)/2) \geq 0$. Therefore if $\sigma_j(x, y) > 0$ then $g_j((x + y)/2) \geq 0$ whereas if $\sigma_j(x, y) = 0$ then $g_j((x + y)/2)^{2p_j} = 0$ which in turn implies $g_j((x + y)/2) = 0$. Hence for every $j = 1, \dots, m$, $g_j((x + y)/2) \geq 0$ for every $x, y \in \mathbf{K}$, that is, (2.1) holds and so $\mathbf{K}$ is convex.

A numerical certificate of convexity

Again, as (2.6) is coming from Stengle's Positivstellensatz, bounds (p, d) are available for the integer p_j and the degrees of the s.o.s. polynomials ϕ_J appearing in the definition (2.5) of polynomials $\sigma_j, h_j \in P(\hat{g})$, respectively. Hence, with the same arguments as in the discussion just after (1.5), checking whether (2.6) holds reduces to checking whether some (single) appropriately defined semidefinite program has a feasible solution.

Hence checking convexity of the basic closed semi-algebraic set $\mathbf{K}$ reduces to checking whether each semidefinite program associated with (2.6), $j = 1, \dots, m$, has a feasible solution, and any feasible solution $\sigma_j, h_j \in P(\hat{g})$ of (2.6), $j = 1, \dots, m$, provides a certificate of convexity for $\mathbf{K}$. However the certificate is only "numerical" as the coefficients of the polynomials σ_j, h_j are obtained numerically and are subject to unavoidable numerical inaccuracies. Moreover, the bounds (p, d) being out of reach, in practice one will solve a semidefinite program associated with (2.6) but for reasonable bounds $(p', d') \ll (p, d)$, hoping to obtain a solution when $\mathbf{K}$ is convex.

2.1. An easier sufficient condition for convexity

While Theorem 2.1 provides a necessary and sufficient condition for convexity, it is very expensive to check because for each $j = 1, \dots, m$, the certificate of convexity $\sigma_j, h_j \in P(\hat{g})$ in (2.6) involves computing $2 \times 2^{2m} = 2^{2m+1}$ s.o.s. polynomials ϕ_J in the definition (2.5) of σ_j and h_j. However, one also has the following sufficient condition:

Theorem 2.2. *Let $\mathbf{K} \subset \mathbb{R}^n$ be the basic semi-algebraic set defined in (1.1). Then $\mathbf{K}$ is convex if for every $j = 1, \dots, m$:*

$$g_j((x + y)/2) = \sigma_0(x, y) + \sum_{k=1}^{m} \sigma_k^j(x, y)g_k(x) + \psi_k^j(x, y)\, g_k(y), \quad \forall x, y \in \mathbb{R}^n, \tag{2.8}$$

for some s.o.s. polynomials $\sigma_k^j, \psi_k^j \in \Sigma[x, y]$.

Proof. Observe that if (2.8) holds then $g_j((x + y)/2) \geq 0$ for every $j = 1, \dots, m$ and all $(x, y) \in \widehat{\mathbf{K}}$; and so $\mathbf{K}$ is convex because (2.1) holds. □

Again, checking whether (2.8) holds with an *a priori* bound $2d$ on the degrees of the s.o.s. polynomials σ_k^j, ψ_k^j, reduces to solving a semidefinite program. But it now only involves $2m + 1$ unknown s.o.s. polynomials (to be compared with 2^{2m+1} previously). On the other hand, Theorem 2.2 only provides a sufficient condition, that is, even if $\mathbf{K}$ is convex it may happen that (2.8) does not hold.

However, when $\mathbf{K}$ is compact, convex, and if for some $M > 0$ the quadratic polynomial $x \mapsto M - \|x\|^2$ can be written

$$M - \|x\|^2 = \sigma_0(x) + \sum_{k=1}^{m} \sigma_k(x)\, g_j(x), \tag{2.9}$$

for some s.o.s. polynomials $(\sigma_k) \subset \Sigma[x]$, then (2.8) is *almost* necessary because for every $\epsilon > 0$:

$$g_j((x+y)/2) + \epsilon = \sigma^j_{0\epsilon}(x,y) + \sum_{k=1}^m \sigma^j_{k\epsilon}(x,y) g_k(x) + \psi^j_{k\epsilon}(x,y)\, g_k(y), \tag{2.10}$$

for some s.o.s. polynomials $\sigma^j_{k\epsilon}, \psi^j_{k\epsilon} \in \Sigma[x, y]$. Indeed, consider the quadratic polynomial

$$(x, y) \mapsto \Delta(x, y) := 2M - \|x\|^2 - \|y\|^2.$$

From (2.9), Δ belongs to the quadratic module $Q(\hat{g}) \subset \mathbb{R}[x, y]$ generated by the polynomials $\hat{g}_k$ that define $\widehat{\mathbf{K}}$, that is, the set

$$Q(\hat{g}) := \left\{ \sigma_0(x,y) + \sum_{k=1}^m \sigma_k(x,y)\, g_k(x) + \psi_k(x,y)\, g_k(y) : \sigma_k, \psi_k \in \Sigma[x,y] \right\}.$$

In addition, its level set $\{(x, y) : \Delta(x, y) \geq 0\}$ is compact, which implies that $Q(\hat{g})$ is Archimedean (see e.g. [5]). Therefore, as $g_j((x+y)/2) + \epsilon > 0$ on $\widehat{\mathbf{K}}$, (2.10) follows from Putinar's Positivstellensatz [6].

2.2. A certificate on non-convexity

In this final section we provide a numerical certificate of *non-convexity* of $\mathbf{K}$ when the optimal value of a certain semidefinite program is strictly negative and some moment matrix associated with an optimal solution satisfies a certain rank condition.

Given a sequence $\mathbf{z} = (z_{\alpha\beta})$ indexed in the canonical basis $(x^\alpha y^\beta)$ of $\mathbb{R}[x, y]$, let $L_\mathbf{z} : \mathbb{R}[x, y] \to \mathbb{R}$ be the linear functional:

$$f \left(= \sum_{\alpha,\beta} f_{\alpha\beta}\, x^\alpha y^\beta \right) \mapsto L_\mathbf{z}(f) = \sum_{\alpha,\beta} f_{\alpha\beta}\, z_{\alpha\beta},$$

and as in [4], the *moment* matrix $M_s(\mathbf{z})$ associated with $\mathbf{z}$ is the real symmetric matrix with rows and columns indexed in the canonical basis $(x^\alpha y^\beta)$ and with entries

$$M_s(\mathbf{z})((\alpha, \beta), (\alpha', \beta')) = z_{(\alpha+\alpha')(\beta+\beta')}$$

for every $(\alpha, \beta), (\alpha', \beta') \in \mathbb{N}^{2n}_s$, where $\mathbb{N}^n_s := \{\alpha \in \mathbb{N}^n : \sum_i \alpha_i \leq s\}$.

Similarly, with a polynomial $(x, y) \mapsto \theta(x, y) = \sum_{\alpha,\beta} \theta_{\alpha\beta}\, x^\alpha y^\beta$, the *localizing* matrix $M_s(\theta\, \mathbf{z})$ associated with θ and $\mathbf{z}$, is the real symmetric matrix with rows and columns indexed in the canonical basis $(x^\alpha y^\beta)$ and with entries

$$M_s(\theta\, \mathbf{z})((\alpha, \beta), (\alpha', \beta')) = \sum_{\alpha'',\beta''} \theta_{\alpha''\beta''}\, z_{(\alpha+\alpha'+\alpha'')(\beta+\beta'+\beta'')},$$

for every $(\alpha, \beta), (\alpha', \beta') \in \mathbb{N}^{2n}_s$.

Let $v_k := \lceil (\deg \hat{g}_k)/2 \rceil$, $k = 1, \ldots, 2m$, and for every $j = 1, \ldots, m$, and $s \geq v := \max_k v_k$, consider the semidefinite program:

$$\begin{cases} \rho_{js} = \min\limits_{\mathbf{z}} & L_\mathbf{z}(g_j((x+y)/2)) \\ \quad\quad\ \text{s.t.} & M_s(\mathbf{z}) \succeq 0 \\ & M_{s-v_k}(\hat{g}_k\, \mathbf{z}) \succeq 0, \quad k = 1, \ldots, 2m \\ & \mathbf{z}_0 = 1, \end{cases} \tag{2.11}$$

where for a real symmetric matrix A, the notation $A \succeq 0$ stands for A is positive semidefinite. The semidefinite program (2.11) is a convex relaxation of the global optimization problem

$$g_j^* := \min_{x,y} \{ g_j((x+y)/2) : (x, y) \in \widehat{\mathbf{K}} \}$$

and so $\rho_{js} \leq g_j^*$ for every $s \geq v$. Moreover, $\rho_{js} \uparrow g_j^*$ as $s \to \infty$; for more details see e.g. [4].

Theorem 2.3. *Let $\mathbf{K} \subset \mathbb{R}^n$ be as in* (1.1) *and let $\mathbf{z}$ be an optimal solution of the semidefinite program* (2.11) *with optimal value ρ_{js}. If $\rho_{js} < 0$ and*

$$\operatorname{rank} M_s(\mathbf{z}) = \operatorname{rank} M_{s-v}(\mathbf{z}) \quad (=: t) \tag{2.12}$$

then the set $\mathbf{K}$ is not convex and one may extract t points $(x(i), y(i)) \in \widehat{\mathbf{K}}$, $i = 1, \ldots, t$, such that

$$g_j((x(i) + y(i))/2) < 0, \quad \forall\, i = 1, \ldots, t.$$

Hence each mid-point $(x(i) + y(i))/2 \notin \widehat{\mathbf{K}}$ is a certificate that $\mathbf{K}$ is not convex.

Proof. By the flat extension theorem of Curto and Fialkow [7] (see also [8]), the rank condition (2.12) ensures that $\mathbf{z}$ is the moment sequence of a t-atomic probability measure μ supported on $\widehat{\mathbf{K}}$. That is:

$$z_{\alpha\beta} = \int_{\widehat{\mathbf{K}}} x^\alpha y^\beta \mathrm{d}\mu, \quad \forall\, (\alpha, \beta) \in \mathbb{N}^{2n}_{2s}.$$

Let $(x(i), y(i))_{i=1}^{t} \subset \widehat{\mathbf{K}}$ be the support of μ which is a positive linear combination of Dirac measures $\delta_{(x(i),y(i))}$ with positive weights (γ_i) such that $\sum_i \gamma_i = 1$. Then

$$\begin{aligned} g_j^* \geq \rho_{js} = L_{\mathbf{z}}(g_j((x+y)/2)) &= \int_{\widehat{\mathbf{K}}} g_j((x+y)/2)\mathrm{d}\mu \\ &= \sum_{i=1}^{t} \gamma_i \, g_j((x(i)+y(i))/2) \\ &\geq \sum_{i=1}^{t} \gamma_i \, g_j^* = g_j^*, \end{aligned}$$

which shows that $\rho_{js} = g_j^*$ and so, $g_j((x(i)+y(i))/2) = g_j^*$ for every $i = 1, \ldots, t$. But then the result follows from $\rho_{js} < 0$. □

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