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Extensions of barrier sets to nonzero roots of the matching polynomial

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ABSTRACT

In matching theory, barrier sets (also known as Tutte sets) have been studied extensively due to their connection to maximum matchings in a graph. For a root θ of the matching polynomial, we define θ-barrier and θ-extreme sets. We prove a generalized Berge–Tutte formula and give a characterization for the set of all θ-special vertices in a graph.

1. Introduction

All the graphs in this paper are simple and finite.

Definition 1.1. An *r-matching* in a graph G is a set of r edges, no two of which have a vertex in common. The number of r-matchings in G will be denoted by $p(G,r)$. Set $p(G,0)=1$. The *matching polynomial* of G is defined by

$$\mu(G,x)=\sum_{r=0}^{\lfloor n/2\rfloor}(-1)^r p(G,r)x^{n-2r}.$$

In [1], Chen and Ku developed a Gallai–Edmonds decomposition associated to a root θ of the matching polynomial, generalizing the usual one which is the special case where $\theta = 0$. Note that 0 is a root of the matching polynomial if and only if the graph has no perfect matching. In this paper, we extend the notions of barrier and extreme sets to θ-barrier and θ-extreme sets and show connections with the Gallai–Edmonds decomposition for general θ.

We shall denote the multiplicity of θ as a root of $\mu(G,x)$ by $\mathrm{mult}(\theta,G)$. In particular, $\mathrm{mult}(\theta,G)=0$ if and only if θ is not a root of $\mu(G,x)$.

The following are properties of $\mu(G,x)$.

Theorem 1.2 (*Theorem 1.1 on p. 2 of [2]*)**.**

(a) $\mu(G\cup H,x)=\mu(G,x)\mu(H,x)$ *when G and H are disjoint graphs,*
(b) $\mu(G,x)=\mu(G-e,x)-\mu(G\setminus uv,x)$ *if* $e=\{u,v\}$ *is an edge of* G,
(c) $\mu(G,x)=x\mu(G\setminus u,x)-\sum_{i\sim u}\mu(G\setminus ui,x)$, *where* $i\sim u$ *means i is adjacent to u,*
(d) $\frac{\mathrm{d}}{\mathrm{d}x}\mu(G,x)=\sum_{i\in V(G)}\mu(G\setminus i,x)$, *where* $V(G)$ *is the vertex set of* G.

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It is well known that all roots of $\mu(G, x)$ are real (see [5] and in particular [2, Corollary 1.2]). By Theorem 5.3 on p. 29 and Theorem 1.1 on p. 96 of [2], one can easily deduce the following lemma (see also [4]).

Lemma 1.3. *If G is a graph and $u \in V(G)$, then*

$$\text{mult}(\theta, G) - 1 \leq \text{mult}(\theta, G \setminus u) \leq \text{mult}(\theta, G) + 1.$$

As a consequence of Lemma 1.3, we can classify the vertices in a graph with respect to θ as follows.

Definition 1.4 (*See [3, Section 3]*)**.** For any $u \in V(G)$,

(a) u is θ-*essential* if $\text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G) - 1$,
(b) u is θ-*neutral* if $\text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G)$,
(c) u is θ-*positive* if $\text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G) + 1$.

Furthermore, when u is not θ-essential but is adjacent to some θ-essential vertex, we say that u is θ-special.

It turns out that θ-special vertices play an important role in the Gallai–Edmonds decomposition of a graph (see [1]). One of the results in this paper is a characterization of the set of these vertices in terms of θ-barriers.

Note that, if $\text{mult}(\theta, G) = 0$, then, for any $u \in V(G)$, u is either θ-neutral or θ-positive, and no vertices in G can be θ-special. By Corollary 4.3 of [3], a θ-special vertex is θ-positive. Let $D_\theta(G)$, $A_\theta(G)$, and $N_\theta(G)$, respectively, denote the sets of θ-essential, θ-special, and θ-neutral vertices, and let $P_\theta(G)$ denote the set of vertices that are θ-positive but not θ-special. These four sets partition $V(G)$.

Note that there are no 0-neutral vertices. If there were, then there would be a vertex, say u, with $\text{mult}(0, G) = \text{mult}(0, G \setminus u)$. There is then a maximum matching that does not cover u, and so $u \in D_0(G)$, a contradiction, for $D_0(G)$ is the set of all points in G which are not covered by at least one maximum matching of G (see [11, Section 3.2 on p. 93] for the details). Thus $N_0(G) = \varnothing$ and $V(G) = D_0(G) \cup A_0(G) \cup P_0(G)$.

Definition 1.5 (*See [3, Section 3]*)**.** A graph G is said to be θ-*critical* if all vertices in G are θ-essential and $\text{mult}(\theta, G) = 1$.

The Gallai–Edmonds structure theorem describes a certain canonical decomposition of $V(G)$ with respect to the zero root of $\mu(G, x)$.

Theorem 1.6 (*Theorem 1.5 of [1]*)**.** *Let G be a graph with θ a root of $\mu(G, x)$. If $u \in A_\theta(G)$, then*

(i) $D_\theta(G \setminus u) = D_\theta(G)$,
(ii) $P_\theta(G \setminus u) = P_\theta(G)$,
(iii) $N_\theta(G \setminus u) = N_\theta(G)$,
(iv) $A_\theta(G \setminus u) = A_\theta(G) \setminus \{u\}$.

Theorem 1.7 (*Theorem 1.7 of [1]*)**.** *If G is connected and every vertex of G is θ-essential, then* $\text{mult}(\theta, G) = 1$.

By Theorems 1.6 and 1.7, it is not hard to deduce the following, whose proof is omitted.

Corollary 1.8.

(i) $A_\theta(G \setminus A_\theta(G)) = \varnothing$, $D_\theta(G \setminus A_\theta(G)) = D_\theta(G)$, $P_\theta(G \setminus A_\theta(G)) = P_\theta(G)$, *and* $N_\theta(G \setminus A_\theta(G)) = N_\theta(G)$.
(ii) *$G \setminus A_\theta(G)$ has exactly $|A_\theta(G)| + \text{mult}(\theta, G)$ θ-critical components.*
(iii) *If H is a component of $G \setminus A_\theta(G)$, then either H is θ-critical or* $\text{mult}(\theta, H) = 0$.
(iv) *The subgraph induced by $D_\theta(G)$ consists of all the θ-critical components in $G \setminus A_\theta(G)$.*

Consider the Gallai–Edmonds decomposition of the graph G in Fig. 1 for $\theta = 0$ and $\theta = 1$. For $\theta = 0$, it is the usual Gallai–Edmonds decomposition (see [11, Section 3.2 on p. 93]). First note that $\text{mult}(1, G) = 1 = \text{mult}(0, G)$.

For $\theta = 1$, we have $A_1(G) = \{u_1\}$, $D_1(G) = \{u_2, u_3, u_4, u_5\}$, $P_1(G) = \{u_7, u_{10}\}$, and $N_1(G) = \{u_6, u_8, u_9, u_{11}, u_{12}, u_{13}\}$. Now C_1, C_2, C_3, C_4 are the only components in $G \setminus A_1(G)$. Note that C_1 and C_2 are 1-critical, and $\text{mult}(1, C_3) = 0 = \text{mult}(1, C_4)$.

For $\theta = 0$, we have $A_0(G) = \{u_2, u_4, u_7, u_8, u_{10}\}$, $D_0(G) = \{u_1, u_3, u_5, u_6, u_9, u_{12}\}$, and $P_0(G) = \{u_{11}, u_{13}\}$. Now all components in $G \setminus A_0(G)$ consist of a single vertex except H (see Fig. 2). The single vertex is 0-critical, and $\text{mult}(0, H) = 0$.

Let G be a graph. The *deficiency* of G, denoted by $\text{def}(G)$, is defined to be the number of points left uncovered by any maximum matching. Let the number of odd components in G be denoted by $o(G)$. Then $\text{def}(G) = \max_{X \subseteq V(G)} o(G \setminus X) - |X|$ (see [11, Theorem 3.1.14 on p. 90]), and this is called the Berge–Tutte formula. Note that the multiplicity of 0 as a root of $\mu(G, x)$ is $|V(G)|$ minus the largest r for which there is a matching of size r. Therefore $\text{mult}(0, G) = \text{def}(G)$, and the following theorem follows.

Theorem 1.9. $\text{mult}(0, G) = \max_{X \subseteq V(G)} o(G \setminus X) - |X|$.

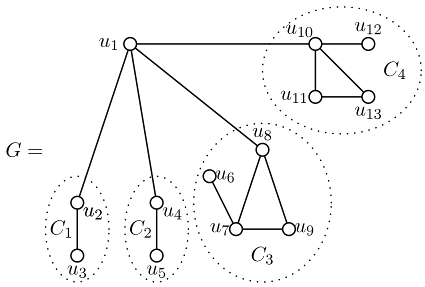

Fig. 1.

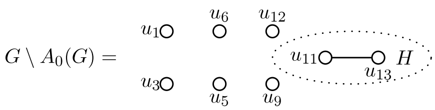

Fig. 2.

Definition 1.10. Motivated by the Berge–Tutte formula, a *barrier set* is defined to be a set $X \subseteq V(G)$ for which $\text{mult}(0, G) = o(G \setminus X) - |X|$. An *extreme* set is defined to be a set for which $\text{mult}(0, G \setminus X) = \text{mult}(0, G) + |X|$.

It should be noted that the standard terminology for a barrier set is a Tutte set in the classical matching theory.

Properties of extreme and barrier sets can be found in [11, Section 3.3]. In fact a barrier set is an extreme set. An extreme set is not necessarily a barrier set, but it can be shown that an extreme set is contained in some barrier set. In general, the union or intersection of two barrier sets is not a barrier set. However, it can be shown that the intersection of two (inclusionwise) maximal barrier sets is a barrier set. The $A_0(G)$ is both a barrier set and an extreme set. It can be shown that $A_0(G)$ is in fact the intersection of all the maximal barrier sets in G. We shall extend this fact to $A_\theta(G)$ (see Theorem 3.6).

In the next section, we prove a version of the Berge–Tutte formula extended to general θ. Let the number of θ-critical components in G be denoted by $c_\theta(G)$.

Theorem 2.1 (*Generalized Berge–Tutte Formula*)**.**

$$\text{mult}(\theta, G) = \max_{X \subseteq V(G)} c_\theta(G \setminus X) - |X|.$$

Definition 1.11. Motivated by the generalized Berge–Tutte formula, we define a θ-*barrier set* to be a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G) = c_\theta(G \setminus X) - |X|$.

We define a θ-*extreme set* to be a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$.

The main theorem of this paper, which is proved in Section 3, is the following.

Theorem 3.6. *If $N_\theta(G) = \varnothing$, then $A_\theta(G)$ is the intersection of all maximal θ-barrier sets in G.*

We emphasize that this paper is built up by generalizing some of the statements given in Chapter 3 of Lovász and Plummer's book [11] to the roots of the matching polynomial. Almost all proofs here have a resemblance to those found in [11]. The novelty of this paper is to merge the tools developed by Godsil [3] with the Lovász–Plummer investigations. This paper also fits into a series of papers [6–10] by the authors about the generalization of the results of classical matching theory with respect to the roots of the matching polynomial.

2. Properties of θ-barrier sets

An immediate consequence of part (a) of Theorems 1.2 and 1.7 is the following inequality, which we use frequently.

$$\text{mult}(\theta, G) \geq c_\theta(G) \quad \text{for any graph } G. \tag{1}$$

We prove the following analogue of the Berge–Tutte formula. The proof is similar to that of the generalization of Tutte's theorem due to the authors in [7]. For the sake of completeness, we repeat the statement.

Theorem 2.1 (*Generalized Berge–Tutte Formula*)**.**

$$\text{mult}(\theta, G) = \max_{X \subseteq V(G)} c_\theta(G \setminus X) - |X|.$$

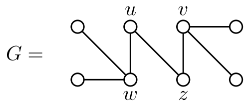

Fig. 3.

Proof. We claim that $c_\theta(G \setminus X) \leq |X| + \text{mult}(\theta, G)$ for all $X \subseteq V(G)$. If not, then $c_\theta(G \setminus X) > |X| + \text{mult}(\theta, G)$ for some $X \subseteq V(G)$. Recall that $\text{mult}(\theta, G \setminus X) \geq c_\theta(G \setminus X)$. Together with Lemma 1.3, we have $\text{mult}(\theta, G) \geq \text{mult}(\theta, G \setminus X) - |X| > \text{mult}(\theta, G)$, a contradiction.

Now it suffices to show that there is a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G) = c_\theta(G \setminus X) - |X|$. Take $X = A_\theta(G)$; by (ii) of Corollary 1.8 we are done. □

Note that the definitions of 0-extreme set and extreme set coincide, but the definitions of 0-barrier set and barrier set are different. Our next proposition shows that a 0-barrier set is a barrier set.

Proposition 2.2. *A* 0*-barrier set is a barrier set.*

Proof. If X is a 0-barrier set, then $c_0(G \setminus X) = \text{mult}(0, G) + |X|$. Note that $c_0(G \setminus X) \leq o(G \setminus X)$. Using Theorem 1.9, we conclude that $o(G \setminus X) = \text{mult}(0, G) + |X|$. Hence X is a barrier set. □

The converse of Proposition 2.2 is not true. The graph G in Fig. 3 is well known (see [11, Figure 3.3.1 on p. 105]). Note that $X = \{u, v\}$ is a barrier set in G, but it is not a 0-barrier set.

A weak converse of Proposition 2.2 can be easily proved by using part (b) of Exercise 3.3.18 on p. 109 of [11].

Proposition 2.3. *A (inclusionwise) maximal barrier set is a maximal* 0*-barrier set.* □

Now we shall study the properties of θ-barrier and θ-extreme sets.

Lemma 2.4. *A subset of a θ-extreme set is a θ-extreme set.*

Proof. Let X be an θ-extreme set, and consider $Y \subseteq X$. Now $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$. By Lemma 1.3, $\text{mult}(\theta, G \setminus Y) \leq \text{mult}(\theta, G) + |Y|$. If Y is not θ-extreme, then $\text{mult}(\theta, G \setminus Y) < \text{mult}(\theta, G) + |Y|$, and by Lemma 1.3 again, $\text{mult}(\theta, G \setminus X) \leq \text{mult}(\theta, G \setminus Y) + |X \setminus Y| < \text{mult}(\theta, G) + |X|$, a contradiction. Hence a subset of an θ-extreme set is θ-extreme. □

Lemma 2.5. *If X is a θ-barrier [θ-extreme] set and $Y \subseteq X$, then $X \setminus Y$ is a θ-barrier [θ-extreme] set in $G \setminus Y$.*

Proof. Note that $c_\theta(G \setminus X) = |X| + \text{mult}(\theta, G)$. By Theorem 2.1 and Lemma 1.3, $c_\theta(G \setminus X) \leq |X \setminus Y| + \text{mult}(\theta, G \setminus Y) \leq |X \setminus Y| + \text{mult}(\theta, G) + |Y| = |X| + \text{mult}(\theta, G)$. Hence $c_\theta(G \setminus X) = |X \setminus Y| + \text{mult}(\theta, G \setminus Y)$, and $X \setminus Y$ is a θ-barrier set in $G \setminus Y$. □

Lemma 2.6. *Every θ-extreme set of G lies in a θ-barrier set.*

Proof. If X is a θ-extreme set and $T = A_\theta(G \setminus X) \cup X$, then

$$\begin{aligned} c_\theta(G \setminus T) &= c_\theta(G \setminus (A_\theta(G \setminus X) \cup X)) \\ &= c_\theta((G \setminus X) \setminus A_\theta(G \setminus X)) \\ &= |A_\theta(G \setminus X)| + \text{mult}(\theta, G \setminus X) \quad \text{(by (ii) of Corollary 1.8)} \\ &= |A_\theta(G \setminus X)| + \text{mult}(\theta, G) + |X| \quad (X \text{ is } \theta\text{-extreme}) \\ &= |T| + \text{mult}(\theta, G), \end{aligned}$$

and hence T is a θ-barrier set. □

Lemma 2.7. *If X is a θ-barrier set, then X is a θ-extreme set.*

Proof. Recall from (1) that $\text{mult}(\theta, G \setminus X) \geq c_\theta(G \setminus X)$. Since $c_\theta(G \setminus X) = |X| + \text{mult}(\theta, G)$, by Lemma 1.3, we have

$$\text{mult}(\theta, G) \geq \text{mult}(\theta, G \setminus X) - |X| \geq c_\theta(G \setminus X) - |X| = \text{mult}(\theta, G).$$

Hence $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$, and X is a θ-extreme set. □

Note that in general a θ-extreme set is not a θ-barrier set. In Fig. 3, $X_1 = \{u\}$ is a 0-extreme set but is not a 0-barrier set. Furthermore, in Fig. 1, $X_2 = \{u_1, u_{10}\}$ is a 1-extreme set but is not a 1-barrier set.

Lemma 2.8. *If X is a θ-barrier set and H is a component of $G \setminus X$, then either H is θ-critical or* $\text{mult}(\theta, H) = 0$.

Proof. Note that $c_\theta(G\setminus X) = |X| + \mathrm{mult}(\theta, G)$. By Lemma 2.7, X is a θ-extreme set. Therefore $\mathrm{mult}(\theta, G\setminus X) = \mathrm{mult}(\theta, G) + |X| = c_\theta(G\setminus X)$. Now, if H is not θ-critical and $\mathrm{mult}(\theta, H) > 0$, then, by part (a) of Theorem 1.2, $\mathrm{mult}(\theta, G\setminus X) > c_\theta(G\setminus X)$, a contradiction. Hence either H is θ-critical or $\mathrm{mult}(\theta, H) = 0$. □

Lemma 2.9. *Let X be a maximal θ-barrier set. If H is a component of $G\setminus X$ and* $\mathrm{mult}(\theta, H) = 0$*, then, for all $u \in V(H)$, u is θ-neutral in H. Furthermore, if $Y \subseteq V(H)$ and $Y \neq \varnothing$, then $c_\theta(H\setminus Y) \leq |Y| - 1$.*

Proof. If H has a θ-positive vertex, say u, then $\mathrm{mult}(\theta, H\setminus u) = 1$. By (ii) of Corollary 1.8, $c_\theta((H\setminus u)\setminus A_\theta(H\setminus u)) = |A_\theta(H\setminus u)| + \mathrm{mult}(\theta, H\setminus u) = |A_\theta(H\setminus u)| + 1$. Now

$$\begin{aligned} c_\theta(G\setminus(X\cup\{u\}\cup A_\theta(H\setminus u))) &= c_\theta(G\setminus X) + c_\theta((H\setminus u)\setminus A_\theta(H\setminus u)) \\ &= |X| + \mathrm{mult}(\theta, G) + |A_\theta(H\setminus u)| + 1 \\ &= |X\cup\{u\}\cup A_\theta(H\setminus u)| + \mathrm{mult}(\theta, G), \end{aligned}$$

and so $X\cup\{u\}\cup A_\theta(H\setminus u)$ is a θ-barrier in G, a contradiction to the maximality of X. Hence, for all $u\in V(H)$, u is θ-neutral in H.

Since $Y\neq\varnothing$, we may choose $y\in Y$. Let $Y' = Y\setminus y$ and $H' = H\setminus y$. Note that $\mathrm{mult}(\theta, H\setminus y) = 0$ since y is θ-neutral in H. By Theorem 2.1, $c_\theta(H'\setminus Y') \leq |Y'|$. Since $H\setminus Y = H'\setminus Y'$, we have $c_\theta(H\setminus Y)\leq |Y| - 1$. □

Lemma 2.10. *If G is θ-critical, then, for all $Y\subseteq V(G)$ and $Y\neq\varnothing$, $c_\theta(G\setminus Y)\leq |Y| - 1$.*

Proof. Since $Y\neq\varnothing$, we may choose $y\in Y$. Let $Y' = Y\setminus y$ and $G' = G\setminus y$. Note that $\mathrm{mult}(\theta, G\setminus y) = 0$ since y is θ-essential in G. By Theorem 2.1, $c_\theta(G'\setminus Y')\leq |Y'|$. Since $G\setminus Y = G'\setminus Y'$, we have $c_\theta(G\setminus Y)\leq |Y| - 1$. □

In general, the union of two θ-barrier sets is not necessarily a θ-barrier set. In Fig. 3, $X_3 = \{u, v, w\}$ and $X_4 = \{v, w, z\}$ are two 0-barrier sets, but $X_3\cup X_4$ is not a 0-barrier set. In Fig. 1, $X_5 = \{u_1, u_7\}$ and $X_6 = \{u_1, u_{10}\}$ are 1-barrier sets and $X_5\cup X_6$ is a 1-barrier set. Let C_3 be a cycle with three vertices. Every set containing a single vertex of C_3 is a 1-barrier set, but the union of two such sets is not 1-barrier set.

However, the intersection of two θ-barrier sets is a θ-barrier set. We shall prove this fact in Theorem 3.10. At present, let us use the results in this section to prove a weaker version.

Theorem 2.11. *The intersection of two maximal θ-barrier sets is a θ-barrier set.*

Proof. Let X and Y be two maximal θ-barrier sets. Let $G_1, G_2, \ldots, G_k$ be the θ-critical components of $G\setminus X$ and $H_1, H_2, \ldots, H_m$ be the components of $G\setminus Y$. Note that $k = |X| + \mathrm{mult}(\theta, G)$. Let $X_i = X\cap V(H_i)$, $Y_i = Y\cap V(G_i)$, and $Z = X\cap Y$. By relabelling if necessary, we may assume that $X_1, \ldots, X_{m_1}\neq\varnothing$ and $Y_1, \ldots, Y_{k_1}\neq\varnothing$, but $X_{m_1+1} = \cdots = X_m = Y_{k_1+1} = \cdots = Y_k = \varnothing$, and also that $k_1\leq m_1$. Note that $G_{k_1+1}, \ldots, G_k$ are θ-critical components in $(G\setminus X)\setminus Y$, so each is contained in a component of $G\setminus Y$.

Next we count the indices i with $k_1 + 1\leq i\leq k$ such that G_i is contained in some H_j. If $m_1 + 1\leq j\leq m$, then H_j is a component in $(G\setminus X)\setminus Y$. So, if $G_i\subseteq H_j$, then $G_i = H_j$. Furthermore, G_i is a component of $G\setminus Z$. By Theorem 2.1, the number of such G_i's is at most $c_\theta(G\setminus Z)\leq |Z| + \mathrm{mult}(\theta, G)$.

Suppose that $1\leq j\leq m_1$. If $G_{i_1}, \ldots, G_{i_t}$ are contained in H_j, then they are θ-critical components in $H_j\setminus X_j$. By Lemma 2.8, either H_j is θ-critical or $\mathrm{mult}(\theta, H) = 0$. If $\mathrm{mult}(\theta, H) = 0$, then, by Lemma 2.9, $c_\theta(H_j\setminus X_j)\leq |X_j| - 1$. If H_i is θ-critical, then, by Lemma 2.10, $c_\theta(H_j\setminus X_j)\leq |X_j| - 1$. Therefore, in either case, $t\leq |X_j| - 1$.

The number of G_i's where $k_1 + 1\leq i\leq k$ that are disjoint from Y is at most

$$\begin{aligned} c_\theta(G\setminus Z) + \sum_{j=1}^{m_1}(|X_j| - 1) &\leq |Z| + \mathrm{mult}(\theta, G) + |X\setminus Z| - m_1 \\ &= |X| + \mathrm{mult}(\theta, G) - m_1 \\ &= k - m_1 \\ &\leq k - k_1. \end{aligned}$$

Since this number is exactly $k - k_1$, we infer that equality must hold throughout. Hence $c_\theta(G\setminus Z) = |Z| + \mathrm{mult}(\theta, G)$, and Z is a θ-barrier set. □

3. Characterizations of $A_\theta(G)$

A characterization of $A_\theta(G)$ is that it is the unique inclusion-minimal θ-barrier set (see Theorem 3.5). If $N_\theta(G) = \varnothing$, then another characterization of $A_\theta(G)$ is that it is the intersection of all maximal θ-barrier sets in G (see Theorem 3.6).

Lemma 3.1. *If X is a θ-barrier set or a θ-extreme set, then $X\subseteq A_\theta(G)\cup P_\theta(G)$.*

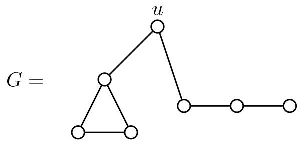

Fig. 4.

Proof. By Lemma 2.7, we may assume that X is θ-extreme. Let $x \in X$. By Lemma 2.4, $\{x\}$ is a θ-extreme set. Therefore $\mathrm{mult}(\theta, G \setminus x) = \mathrm{mult}(\theta, G) + 1$, and x is θ-positive. Hence $x \in A_\theta(G) \cup P_\theta(G)$, and $X \subseteq A_\theta(G) \cup P_\theta(G)$. □

Lemma 3.2. *Let X be a θ-barrier set. If $X \subseteq A_\theta(G)$, then $X = A_\theta(G)$.*

Proof. Note that $c_\theta(G \setminus X) = \mathrm{mult}(\theta, G) + |X|$. By Lemma 2.8, we conclude that $A_\theta(G \setminus X) = \varnothing$. By Theorem 1.6, $A_\theta(G \setminus X) = A_\theta(G) \setminus X$. Hence $X = A_\theta(G)$. □

We shall need the following result of Godsil [3].

Theorem 3.3 (*Theorem 4.2 of [3]*). *If θ is a root of $\mu(G, x)$ with non-zero multiplicity k and we let u be a θ-positive vertex in G, then*

(a) *if v is θ-essential in G, then it is θ-essential in $G \setminus u$;*
(b) *if v is θ-positive in G, then it is θ-essential or θ-positive in $G \setminus u$;*
(c) *if u is θ-neutral in G, then it is θ-essential or θ-neutral in $G \setminus u$.*

Lemma 3.4. *If $u \in P_\theta(G)$, then $A_\theta(G) \subseteq A_\theta(G \setminus u)$.*

Proof. If $A_\theta(G) = \varnothing$, then we are done. Suppose that $A_\theta(G) \neq \varnothing$. If $v \in A_\theta(G)$, then v is adjacent to a θ-essential vertex w. By Theorem 3.3, w is θ-essential in $G \setminus u$, and v is either θ-positive or θ-essential in $G \setminus u$. If v is θ-essential in $G \setminus u$, then $\mathrm{mult}(\theta, G \setminus uv) = \mathrm{mult}(\theta, G)$. By Theorem 1.6, $u \in P_\theta(G) = P_\theta(G \setminus v)$. Since v is θ-special in G, v is θ-positive in G (see Corollary 4.3 of [3]). Hence $\mathrm{mult}(\theta, G \setminus uv) = \mathrm{mult}(\theta, G) + 2$, a contradiction. Therefore v is θ-positive in $G \setminus u$. Since v is adjacent to w, we must have $v \in A_\theta(G \setminus u)$. Hence $A_\theta(G) \subseteq A_\theta(G \setminus u)$. □

Theorem 3.5. *If X is a θ-barrier set in G, then $A_\theta(G) \subseteq X$. In particular, $A_\theta(G)$ is the unique minimal θ-barrier set.*

Proof. By Lemma 3.1, $X \subseteq A_\theta(G) \cup P_\theta(G)$. We shall prove the result by induction on $|X \cap P_\theta(G)|$. If $|X \cap P_\theta(G)| = 0$, then $X \subseteq A_\theta(G)$, and, by Lemma 3.2, $X = A_\theta(G)$. Suppose that $|X \cap P_\theta(G)| \geq 1$. We may assume that, if X' is a θ-barrier set in G' with $|X' \cap P_\theta(G')| < |X \cap P_\theta(G)|$, then $A_\theta(G') \subseteq X'$.

Let $x \in X \cap P_\theta(G)$. By Lemma 2.5, $X' = X \setminus x$ is a θ-barrier set in $G' = G \setminus x$. By Lemmas 3.1 and 3.4, we have $X' \subseteq A_\theta(G') \cup P_\theta(G')$ and $A_\theta(G) \subseteq A_\theta(G')$. Therefore $|X' \cap P_\theta(G')| < |X \cap P_\theta(G)|$. By the induction hypothesis, $A_\theta(G') \subseteq X'$. Hence $A_\theta(G) \subseteq X$. □

In general, $A_\theta(G)$ is not the intersection of all maximal θ-barrier sets in G. For instance, in Fig. 4, $\mathrm{mult}(\sqrt{3}, G) = 0$ and $A_{\sqrt{3}}(G) = \varnothing$. Now $\{u\}$ is the only maximal $\sqrt{3}$-barrier set, but $A_{\sqrt{3}}(G) \neq \{u\}$. However, we can show that $A_\theta(G)$ is the intersection of all maximal θ-barrier sets in G if $N_\theta(G) = \varnothing$.

Theorem 3.6. *If $N_\theta(G) = \varnothing$, then $A_\theta(G)$ is the intersection of all maximal θ-barrier sets in G.*

Proof. By Theorem 3.5, $A_\theta(G)$ is contained in the intersection of all maximal θ-barriers in G. It is sufficient to show that for each $x \in V(G) \setminus A_\theta(G)$ there is a maximal barrier that does not contain x. If $x \in D_\theta(G)$, then, by Lemma 3.1, x is not contained in any θ-barriers and thus any maximal θ-barriers. If $x \in P_\theta(G)$, then x is contained in a component H in $G \setminus A_\theta(G)$ with $\mathrm{mult}(\theta, H) = 0$. Note that $|V(H)| \geq 2$ for $x \in P_\theta(G) = P(G \setminus A_\theta(G))$, and $\mathrm{mult}(\theta, H \setminus x) = 1$ (see Theorem 1.6). By (c) of Theorem 1.2 and the fact that $\mathrm{mult}(\theta, H) = 0$, we deduce that there is a vertex $y \in V(H \setminus x)$ for which $\mathrm{mult}(\theta, H \setminus xy) = 0$. Now $y \in P_\theta(G)$ for $N_\theta(G) = \varnothing$. Furthermore, x is θ-essential in $H \setminus y$. Therefore $x \notin A_\theta(H \setminus y)$ and, by (ii) of Corollary 1.8, $c_\theta((H \setminus y) \setminus A_\theta(H \setminus y)) = |A_\theta(H \setminus y)| + 1$. Hence

$$\begin{aligned} c_\theta(G \setminus (A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y))) &= c_\theta(G \setminus A_\theta(G)) + c_\theta((H \setminus y) \setminus A_\theta(H \setminus y)) \\ &= |A_\theta(G)| + \mathrm{mult}(\theta, G) + |A_\theta(H \setminus y)| + 1 \\ &= |A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)| + \mathrm{mult}(\theta, G), \end{aligned}$$

and so $A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)$ is a θ-barrier set not containing x. Let Z be a maximal θ-barrier set containing $Y = A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)$. By Lemma 2.5, $Z \setminus Y$ is a θ-barrier set in $G \setminus Y$. Using Theorem 1.6 and the fact that x is θ-essential in $H \setminus y$, we can deduce that $x \in D_\theta(G \setminus Y)$. By Lemma 3.1, we conclude that $x \notin Z \setminus Y$, and hence $x \notin Z$. The proof of the theorem is completed. □

Since $N_0(G) = \varnothing$, by Theorem 3.6 and Proposition 2.3, we deduce the following classical result.

Corollary 3.7 (*Theorem 3.3.15 of [11]*). *$A_0(G)$ is the intersection of all maximal barrier sets in G.*

Finally, we prove that the intersection of two θ-barrier sets is a θ-barrier set. We shall need the following two lemmas.

Lemma 3.8. *A set $X \subseteq V(G)$ is a θ-barrier set in G if and only if $X \cap H$ is a θ-barrier set in H for each component H of G.*

Proof. Let $H_1, \ldots, H_m$ be the components of G. Note that $c_\theta(G \setminus X) = \sum_{i=1}^m c_\theta(H_i \setminus X)$. By part (a) of Theorem 1.2, $\text{mult}(\theta, G) = \sum_{i=1}^m \text{mult}(\theta, H_i)$ and $\text{mult}(\theta, G \setminus X) = \sum_{i=1}^m \text{mult}(\theta, H_i \setminus X)$.
($\Leftarrow$) If $X \cap H_i$ is a θ-barrier set in H_i for all i, then $\text{mult}(\theta, H_i) = c_\theta(H_i \setminus X) - |H_i \cap X|$. Therefore $\text{mult}(\theta, G) = \sum_{i=1}^m (c_\theta(H_i \setminus X) - |H_i \cap X|) = c_\theta(G \setminus X) - |X|$, and X is a θ-barrier set in G.
($\Rightarrow$) If X is a θ-barrier set in G, then $\text{mult}(\theta, G) = c_\theta(G \setminus X) - |X|$. So $\sum_{i=1}^m \text{mult}(\theta, H_i) = \sum_{i=1}^m (c_\theta(H_i \setminus X) - |H_i \cap X|)$ and $\sum_{i=1}^m (\text{mult}(\theta, H_i) - (c_\theta(H_i \setminus X) - |H_i \cap X|)) = 0$. By Theorem 2.1, each summand on the left in the last equation must be non-negative. We thus conclude that $\text{mult}(\theta, H_i) = c_\theta(H_i \setminus X) - |H_i \cap X|$, and $X \cap H_i$ is a θ-barrier set in H_i for all i. $\square$

Lemma 3.9. *If B is a θ-barrier set in G with $X = B \cup T$ for some $T \subseteq V(G \setminus B)$, then X is a θ-barrier set in G if and only if T is a θ-barrier set in $G \setminus B$.*

Proof. First note that, by Lemma 2.7, $\text{mult}(\theta, G \setminus B) = \text{mult}(\theta, G) + |B|$.
($\Leftarrow$) If T is a θ-barrier set in $G \setminus B$, then $\text{mult}(\theta, G \setminus B) = c_\theta(G \setminus (B \cup T)) - |T|$, and so $\text{mult}(\theta, G) = c_\theta(G \setminus (B \cup T)) - |B \cup T|$. Hence X is a θ-barrier set in G.
($\Rightarrow$) If X is a θ-barrier set in G, then $\text{mult}(\theta, G) = c_\theta(G \setminus (B \cup T)) - |B \cup T|$, and therefore $\text{mult}(\theta, G \setminus B) = c_\theta(G \setminus (B \cup T)) - |T|$. Hence T is a θ-barrier set in $G \setminus B$. $\square$

Theorem 3.10. *The intersection of two θ-barrier sets is a θ-barrier set.*

Proof. Let B_1 and B_2 be two θ-barrier sets. By Theorem 3.5, $B_1 = A_\theta(G) \cup T_1$ for some $T_1 \subseteq V(G \setminus A_\theta(G))$. By Lemma 3.9, T_1 is a θ-barrier set in $G \setminus A_\theta(G)$. Similarly, $B_2 = A_\theta(G) \cup T_2$ for some θ-barrier set T_2 in $G \setminus A_\theta(G)$. Now $B_1 \cap B_2 = A_\theta(G) \cup (T_1 \cap T_2)$. By Lemma 3.9, it suffices to show that $T_1 \cap T_2$ is a θ-barrier set in $G \setminus A_\theta(G)$.

If $|T_1 \cap T_2| = 0$, then $T_1 \cap T_2 = \varnothing$ and we are done (for an empty set is a θ-barrier set). Suppose $|T_1 \cap T_2| \geq 1$. Assume that, if T_3 and T_4 are θ-barrier sets in $G \setminus A_\theta(G)$ and $|T_3 \cap T_4| < |T_1 \cap T_2|$, then $T_3 \cap T_4$ is a θ-barrier set in $G \setminus A_\theta(G)$.

Let $x \in T_1 \cap T_2$. Since T_1 is a θ-barrier set in $G \setminus A_\theta(G)$, by part (iii) of Corollary 1.8 and Lemma 3.1, we deduce that $x \in H$, where H is a component of $G \setminus A_\theta(G)$ with $\text{mult}(\theta, H) = 0$. Furthermore, $\text{mult}(\theta, H \setminus x) = 1$, and we deduce that $c_\theta((H \setminus x) \setminus A_\theta(H \setminus x)) = 1 + |A_\theta(H \setminus x)| = |\{x\} \cup A_\theta(H \setminus x)|$ (by Corollary 1.8 and part (a) of Theorem 1.2). Therefore $\{x\} \cup A_\theta(H \setminus x)$ is a θ-barrier set in H.

On the other hand, $T_1 \cap H$ is a θ-barrier set in H by Lemma 3.8. By Lemma 2.5, $(T_1 \cap H) \setminus x$ is a θ-barrier set in $H \setminus x$, which yields $A_\theta(H \setminus x) \subseteq (T_1 \cap H) \setminus x$ (Theorem 3.5). Therefore, we may let $T_1 \cap H = T_3 \cup (\{x\} \cup A_\theta(H \setminus x))$ for some $T_3 \subseteq V(H \setminus (\{x\} \cup A_\theta(H \setminus x)))$. Moreover, since $T_1 \cap H$ is a θ-barrier set in H, T_3 is a θ-barrier set in H (Lemma 3.9). In fact, it is not hard to see that T_3 is a θ-barrier set in $G \setminus A_\theta(G)$ (because H is a component of $G \setminus A_\theta(G)$). Similarly, $T_2 \cap H = T_4 \cup (\{x\} \cup A_\theta(H \setminus x))$ for some θ-barrier set T_4 in H that is also a θ-barrier set in $G \setminus A_\theta(G)$. Clearly $|T_3 \cap T_4| < |T_1 \cap T_2|$. By the induction hypothesis, we conclude that $T_3 \cap T_4$ is a θ-barrier set in $G \setminus A_\theta(G)$, and thus is also a θ-barrier set in H. Since $T_1 \cap T_2 \cap H = (T_3 \cap T_4) \cup (\{x\} \cup A_\theta(H \setminus x))$, we deduce from Lemma 3.9 that $T_1 \cap T_2 \cap H$ is a θ-barrier set in H.

Now if H' is a θ-critical component of $G \setminus A_\theta(G)$, then $T_1 \cap T_2 \cap H' = \varnothing$, and so $T_1 \cap T_2 \cap H'$ is a θ-barrier set in H'. Thus $T_1 \cap T_2 \cap H''$ is θ-barrier set in H'' for any component H'' of $G \setminus A_\theta(G)$, and so $T_1 \cap T_2$ is a θ-barrier set in $G \setminus A_\theta(G)$ (Lemma 3.8). $\square$

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