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Parameter-based Fisher's information of orthogonal polynomials

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Abstract

The Fisher information of the classical orthogonal polynomials with respect to a parameter is introduced, its interest justified and its explicit expression for the Jacobi, Laguerre, Gegenbauer and Grosjean polynomials found.

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1. Introduction

Let $\{\rho_\theta(x) \equiv \rho(x|\theta); x \in \Omega \subset \mathbb{R}\}$ be a family of probability densities parametrized by a parameter $\theta \in \mathbb{R}$. The Fisher information of $\rho_\theta(x)$ with respect to the parameter θ is defined [5,13] as

$$I(\rho_\theta) := \int_\Omega \left[\frac{\partial \ln \rho(x|\theta)}{\partial\theta}\right]^2 \rho(x|\theta)\,\mathrm{d}x = \int_\Omega \frac{\left[\partial\rho(x|\theta)/\partial\theta\right]^2}{\rho(x|\theta)}\,\mathrm{d}x$$
$$= 4\int_\Omega \left\{\frac{\partial[\rho(x|\theta)^{1/2}]}{\partial\theta}\right\}^2 \mathrm{d}x. \tag{1}$$

This quantity refers to information about an unknown parameter in the probability density. It is a measure of the ability to estimate the parameter θ. It gives the minimum error in estimating the parameter of the probability density $\rho_\theta(x)$ [4]. The notion of Fisher information was introduced by Sir R.A. Fisher in estimation theory [13]. Nowadays, it is being used in numerous scientific areas ranging from statistics, information theory [4] to signal analysis [42] and quantum physics

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[14,15,23]. This information-theoretic quantity has, among other characteristics, a number of important properties, beyond the mere nonnegativity, which deserve to be resembled here.

(1) Additivity for independent events. In the case that $\rho(x, y|\theta) = \rho_1(x|\theta) \cdot \rho_2(y|\theta)$, it happens that

$$I[\rho(x, y|\theta)] = I[\rho_1(x|\theta)] \cdot I[\rho_2(y|\theta)].$$

(2) Scaling invariance. The Fisher information is invariant under sufficient transformations $y = t(x)$, so that

$$I[\rho(y|\theta)] = I[\rho(x|\theta)].$$

This property is not only closely related to the Fisher maximum likelihood method but also it is very important for the theory of statistical inference.

(3) Cramer–Rao inequality [41]. It states that the reciprocal of the Fisher information $I(\rho_\theta)$ bounds from below the mean square error of an unbiased estimator *f* of the parameter θ; i.e.,

$$\sigma^2(f) \geqslant \frac{1}{I(\rho_\theta)},$$

where $\sigma^2(f)$ denotes the variance of *f*. This inequality, which lies at the heart of statistical estimation theory, shows how much information the distribution provides about a parameter. Moreover, this says that the Fisher information $I(\rho_\theta)$ is a more sensitive indicator of the localization of the probability density than the Shannon entropy power.

(4) Relation to other information-theoretic properties. The Fisher information is related to the Shannon entropy of $\rho(x|\theta)$ via the elegant de Bruijn's identity [4,24,41]

$$\frac{\partial}{\partial\theta} S(\tilde{\rho}_\theta) = \frac{1}{2} I(\tilde{\rho}_\theta),$$

where $\tilde{\rho}_\theta$ denotes the convolution probability density of any probability density $\rho(x|\theta)$ with the normal density with zero mean and variance $\theta > 0$, and $S(\tilde{\rho}_\theta) := -\int_\Omega \tilde{\rho}_\theta(x) \ln \tilde{\rho}_\theta(x)\, \mathrm{d}x$ is the Shannon entropy of $\tilde{\rho}_\theta(x)$. Moreover, the Fisher information $I(\rho_\theta)$ satisfies, under proper regularity conditions, the limiting property [14]

$$I(\rho_\theta) = \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2} D(\rho_{\theta+\varepsilon} \| \rho_\theta),$$

where the symbol $D(p\|q) := \int_\Omega p(x) \ln(p(x)/q(x))\, \mathrm{d}x$ denotes the relative entropy or Kullback–Leibler divergence of the probability densities $p(x)$ and $q(x)$. Further connections of the Fisher information with other information-theoretic properties are known; see e.g., [15,40,41].

(5) Applications in quantum physics. The classical orthogonal polynomials appear as the radial part of the wavefunctions which characterize the stationary quantum-mechanical states of numerous physical and chemical systems. It is well known, at least for quantum physicists and a large group of applied mathematicians, that the wavefunctions are the physically admissible solutions of the nonrelativistic Schrödinger equation of motion for these systems, which for polar spherical coordinates can often be separated in radial and angular parts. The square of these wavefunctions is a real probability density which, when the system is electrically charged, denotes the experimentally accessible distribution of charge of the system. Often, this density is essentially the Rakhmanov density of orthogonal polynomials as defined by Eq. (5) of this paper. The quantum-mechanical properties of the physical systems completely depend on the spreading of the charge all over the space, that is, on the distribution of the Rakhmanov density all over the corresponding orthogonality support. Furthermore the charge distribution is mostly controlled by the parameter of the involved orthogonal polynomials. The Fisher information is one of the best estimators of this parameter. Up until now the explicit computation of this information-theoretic measure has not been performed.

In addition, the Fisher information $I(\rho_\theta)$ plays a fundamental role in the quantum-mechanical description of physical systems [8,11,14,15,17,19,20,24,25,28–33,40]. It has been shown

(a) to be a measure of both disorder and uncertainty [14,15] as well as a measure of nonclassicality for quantum systems [19,20],

(b) to describe, some factor apart, various macroscopic quantities such as, for example, the kinetic energy [24,40] and the Weiszäcker energy [28,29,31],
(c) to derive the Schrödinger and Klein–Gordon equations of motion [15] as well as the Euler equation of the density functional theory [25], from the principle of minimum Fisher information,
(d) to predict the most distinctive nonlinear spectral phenomena, known as avoided crossings, encountered in atomic and molecular systems under strong external fields [17], and
(e) to be involved in numerous uncertainty inequalities [10,20,21,23,24,28,32,33,41].

These applications are most apparent when θ is a parameter of locality, so that $\rho_\theta(x) = \rho(x+\theta)$. Then, the Fisher information for locality, also called intrinsic accuracy [40], does not depend on the parameter θ; so, without loss of generality, we may set the location at the origin and the Fisher information of the density $\rho_0(x) \equiv \rho(x)$ becomes

$$I(\rho) = \int \frac{[\rho'(x)]^2}{\rho(x)}\,\mathrm{d}x,$$

where $\rho'(x)$ denotes the first derivative of $\rho(x)$. The locality Fisher information or Fisher information associated with translations of an one-dimensional observable x with probability density $\rho(x)$ has been calculated in the literature [4,42] for some simple families of probability densities. In particular, it is well known that the locality Fisher information of the Gaussian density is equal to the reciprocal of its variance, what illustrates the spreading character of these information-theoretic measures.

Recently, Dehesa and Sánchez-Ruiz [39] have exactly derived the locality Fisher information of a wider and much more involved class of probability densities, the Rakhmanov densities, defined by

$$\rho_n(x) = \frac{1}{d_n^2} p_n^2(x)\omega(x)\chi_{[a,b]}(x), \tag{2}$$

where $\chi_{[a,b]}(x)$ is the characteristic function for the interval $[a,b]$, and $\{p_n(x)\}$ denotes a sequence of real polynomials orthogonal with respect to the nonnegative definite weight function $\omega(x)$ on the interval $[a,b] \subseteq \mathbb{R}$, that is

$$\int_a^b p_n(x)p_m(x)\omega(x)\,\mathrm{d}x = d_n^2\delta_{n,m} \tag{3}$$

with deg $p_n(x) = n$. As first pointed out by Rakhmanov [27], these densities play a fundamental role in the analytic theory of orthogonal polynomials. In particular, he has shown that these probability densities govern the asymptotic behavior of the ratio $p_{n+1}(x)/p_n(x)$ as $n \to \infty$. On the other hand, these two fundamental and applied reasons have motivated an increasing interest for the determination of the spreading of the classical orthogonal polynomials $\{p_n(x)\}$ throughout its interval of orthogonality by means of the information-theoretic measures of their corresponding Rakhmanov densities $\rho(x)$ [3,6–11,39].

The Shannon information entropy of these densities has been examined numerically [3]. On the theoretical side, let us point out that its asymptotics ($n \to \infty$) is well known for all classical orthogonal polynomials, but its exact value for every fixed n is only known for Chebyshev polynomials [43] and some Gegenbauer polynomials [2,37]. To this respect see [9] which reviews the knowledge up to 2001. The variance and Fisher information entropy of the Rakhmanov densities have been found in a closed and compact form for all classical orthogonal polynomials [11,39]. For other functionals of these Rakhmanov densities see Ref. [38].

In this paper we shall calculate the Fisher information $I(\rho_\theta)$ for the real and continuous classical orthogonal polynomials which are parameter dependent, namely the Jacobi and Laguerre polynomials. In addition, we have considered the Fisher quantities of two specific families of Jacobi polynomials: the Gegenbauer or ultraspherical polynomials and the Grosjean polynomials, not only because of its intrinsic interest but also because, in particular, the Gegenbauer polynomials control the bulky shape of the physical systems with spherically symmetric potentials. Needless to say that we do not consider the Hermite polynomials $H_n(x)$ because it is well known that they do not depend on any specific parameter.

The structure of the paper is the following. We begin in Section 2 with the definition of this notion and, as well, we collect here some basic properties of the classical orthogonal polynomials which will be used later on. Then, the Fisher information with respect to a parameter is fully determined for Jacobi and Laguerre polynomials in Section 3, and for Gegenbauer and Grosjean polynomials in Section 4. Finally, conclusions and some open problems are given.

2. Some properties of the parameter-dependent classical orthogonal polynomials

Let $\{\tilde{y}_n(x;\theta)\}_{n\in\mathbb{N}_0}$ stand for the sequence of polynomials orthonormal with respect to the nonnegative definite weight function $\omega(x;\theta)$ on the real support (a,b), so that

$$\int_a^b \tilde{y}_n(x;\theta)\tilde{y}_m(x;\theta)\omega(x;\theta)\,\mathrm{d}x = \delta_{nm}, \tag{4}$$

with deg $\tilde{y}_n = n$. Here we shall consider the celebrated classical families of Laguerre $L_n^{(\alpha)}(x)$ and Jacobi $J_n^{(\alpha,\beta)}(x)$ polynomials. The normalized-to-unity density functions $\tilde{\rho}_n(x;\theta)$ defined as

$$\tilde{\rho}_n(x;\theta) = \omega(x;\theta)\tilde{y}_n^2(x;\theta) \tag{5}$$

are called for Rakhmanov densities [27].

Here we gather various properties of the parameter-dependent classical orthogonal polynomials in a real and continuous variable (i.e., Laguerre and Jacobi) in the form of two lemmas, which shall be used later on. The weight function of these polynomials can be written as

$$\omega(x;\theta) = h(x)[t(x)]^\theta \tag{6}$$

with

$$h_L(x) = \mathrm{e}^{-x} \quad \text{and} \quad t_L(x) = x \tag{7}$$

for the Laguerre case, $L_n^{(\theta)}(x)$, and

$$h_J(x) = (1+x)^\beta \quad \text{and} \quad t_J(x) = 1-x \tag{8}$$

for the Jacobi case, $P_n^{(\theta,\beta)}(x)$.

Lemma 1. *The derivative of the orthonormal polynomial $\tilde{y}_n(x;\theta)$ with respect to the parameter θ is given by*

$$\frac{\partial}{\partial\theta}\tilde{y}_n(x;\theta) = \sum_{k=0}^{n} \tilde{A}_k(\theta)\tilde{y}_k(x;\theta) \tag{9}$$

with

$$\tilde{A}_k(\theta) = \frac{d_k(\theta)}{d_n(\theta)}A_k(\theta) \quad \text{for } k = 0,1,\ldots,n-1, \tag{10}$$

$$\tilde{A}_n(\theta) = A_n(\theta) - \frac{1}{d_n(\theta)}\frac{\partial}{\partial\theta}[d_n(\theta)], \tag{11}$$

where $d_m^2(\theta)$ is, according to Eq. (3), *the normalization constant of the orthogonal polynomial $p_m(x) = y_m(x;\theta)$, and $A_k(\theta)$ with $k = 0,1,\ldots$ are the expansion coefficients of the derivative of $y_m(x;\theta)$ in terms of the system $\{y_m(x;\theta)\}$; i.e.,*

$$\frac{\partial}{\partial\theta}y_m(x;\theta) = \sum_{k=0}^{m} A_k(\theta)y_k(x;\theta). \tag{12}$$

Both quantities $d_m(\theta)$ and $A_m(\theta)$ are known in the literature for the Laguerre and Jacobi cases. Indeed, the norms for the Laguerre $L_k^{(\alpha)}(x)$ and the Jacobi $P_k^{(\alpha,\beta)}(x)$ polynomials [26] are

$$[d_k^{(L)}(\alpha)]^2 = \frac{\Gamma(k+\alpha+1)}{k!}, \tag{13}$$

$$[d_k^{(J)}(\alpha,\beta)]^2 = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{k!(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}, \tag{14}$$

respectively.

On the other hand the expansion coefficients in Eq. (12) are known [16,22,36] to have the form

$$A_k^{(L)} = \frac{1}{n-k} \quad \text{for } k=0,1,\ldots,n-1 \quad \text{and} \quad A_n^{(L)} = 0 \tag{15}$$

for the Laguerre polynomials $L_n^{(\alpha)}$, and

$$A_k^{(J_\alpha)} = \frac{\alpha+\beta+1+2k}{(n-k)(\alpha+\beta+1+n+k)}\frac{(\beta+k+1)_{n-k}}{(\alpha+\beta+k+1)_{n-k}}; \quad k=0,1,\ldots,n-1, \tag{16}$$

$$A_n^{(J_\alpha)} = \sum_{k=0}^{n-1}\frac{1}{\alpha+\beta+1+n+k} = \psi(1+\alpha+\beta+2n) - \psi(1+\alpha+\beta+n), \tag{17}$$

for the Jacobi expansion of $(\partial/\partial\alpha)P_n^{(\alpha,\beta)}(x)$, and

$$A_k^{(J_\beta)} = (-1)^{n-k}\frac{\alpha+\beta+1+2k}{(n-k)(\alpha+\beta+1+n+k)}\frac{(\alpha+k+1)_{n-k}}{(\alpha+\beta+k+1)_{n-k}}; \quad k=0,1,\ldots,n-1, \tag{18}$$

$$A_n^{(J_\beta)} = \sum_{k=0}^{n-1}\frac{1}{\alpha+\beta+1+n+k} = \psi(1+\alpha+\beta+2n) - \psi(1+\alpha+\beta+n), \tag{19}$$

for the Jacobi expansion of $(\partial/\partial\beta)P_n^{(\alpha,\beta)}(x)$. The symbol $\psi(x)=\Gamma'(x)/\Gamma(x)$ denotes the well-known digamma function. It is worth to remark that the superindices J_α and J_β indicate the expansion coefficients for the derivatives of the Jacobi polynomial with respect to the first and second parameter, respectively.

Then, taking into account Eqs. (10)–(11) and Eqs. (13) and (15), Lemma 1 says that the expansion coefficients $\tilde{A}_k(\alpha)$ for the orthonormal Laguerre polynomials $\tilde{L}_n^{(\alpha)}(x)$ are

$$\tilde{A}_k^{(L)} = \frac{1}{n-k}\left[\frac{(k+1)_{n-k}}{(k+\alpha+1)_{n-k}}\right]^{1/2}, \quad k=0,1,\ldots,n-1, \tag{20}$$

$$\tilde{A}_n^{(L)} = -\frac{\psi(n+\alpha+1)}{2}. \tag{21}$$

In a similar way the same lemma together with Eqs. (10)–(11) and (14)–(17) has allowed us to find the expressions

$$\tilde{A}_k^{(J_\alpha)} = \left[\frac{(k+\beta+1)_{n-k}(k+1)_{n-k}}{(k+\alpha+1)_{n-k}(k+\alpha+\beta+1)_{n-k}}\frac{2n+\alpha+\beta+1}{2k+\alpha+\beta+1}\right]^{1/2}\frac{2k+\alpha+\beta+1}{(n-k)(n+k+\alpha+\beta+1)},$$
$$k=0,1,\ldots,n-1, \tag{22}$$

$$\tilde{A}_n^{(J_\alpha)} = \frac{1}{2}\left[2\psi(2n+\alpha+\beta+1) - \psi(n+\alpha+\beta+1) - \psi(n+\alpha+1) - \ln 2 + \frac{1}{2n+\alpha+\beta+1}\right], \tag{23}$$

for the expansion coefficients $\tilde{A}_k^{(J_\alpha)}$ of the derivative with respect to the parameter α of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Finally, the same procedure with Eqs. (10)–(11), (14), (18) and (19) leads to the values

$$\tilde{A}_k^{(J_\beta)} = \left[\frac{(k+\alpha+1)_{n-k}(k+1)_{n-k}}{(k+\beta+1)_{n-k}(k+\alpha+\beta+1)_{n-k}}\frac{2n+\alpha+\beta+1}{2k+\alpha+\beta+1}\right]^{1/2} \times (-1)^{n-k}\frac{2k+\alpha+\beta+1}{(n-k)(n+k+\alpha+\beta+1)}; \quad k=0,1,\ldots,n-1, \tag{24}$$

$$\tilde{A}_n^{(J_\beta)} = \frac{1}{2}\left[2\psi(2n+\alpha+\beta+1) - \psi(n+\alpha+\beta+1) - \psi(n+\beta+1) - \ln 2 + \frac{1}{2n+\alpha+\beta+1}\right], \tag{25}$$

for the expansion coefficients $\tilde{A}_k^{(J_\beta)}$ $(k=0,1,\ldots,n)$ in Eq. (9) of the β-derivative of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

Lemma 2. *The parameter-dependent classical orthonormal polynomials* $\tilde{y}_n(x;\theta)$ *satisfy*

(a) $$\int_a^b \frac{\partial\omega(x;\theta)}{\partial\theta}[\tilde{y}_n(x;\theta)]^2\,\mathrm{d}x = -2\tilde{A}_n(\theta),$$

(b) $$\int_a^b \frac{\partial\omega(x;\theta)}{\partial\theta}\tilde{y}_n(x;\theta)\tilde{y}_k(x;\theta)\,\mathrm{d}x = -\tilde{A}_k(\theta), \quad k=0,1,\ldots,n-1,$$

(c) $$\int_a^b \frac{\partial^2\omega(x;\theta)}{\partial\theta^2}[\tilde{y}_n(x;\theta)]^2\,\mathrm{d}x = 2\sum_{k=0}^{n}(\tilde{A}_k(\theta))^2 + 2(\tilde{A}_n(\theta))^2 - 2\frac{\partial\tilde{A}_n(\theta)}{\partial\theta}.$$

Proof. To prove integrals (a) and (b) we have to derive with respect to the parameter θ the orthonormalization condition (4) for $m=n$ and $m=k\neq n$, respectively. Then one has to use Lemma 1 and again Eq. (4), and the results follow.

Integral (c) is obtained by deriving integral (a) with respect to θ and taking into account the values of the two previous integrals (a) and (b). □

3. Parameter-based Fisher information of Jacobi and Laguerre polynomials

The distribution of the orthonormal polynomials $\tilde{y}_n(x;\theta)$ on their orthonormality interval and the spreading of the associated Rakhmanov densities can be most appropriately measured by means of their information-theoretic measures, the Shannon entropy and the Fisher information [13]. The former has been theoretically [9,6] and numerically [3] examined for general orthogonal polynomials, while for the latter it has been studied the Fisher information associated with translations of the variable (i.e., the locality Fisher information) both analytically [39] and numerically [11]. Here we extend this study by means of the computation of a more general concept, the parameter-based Fisher information of the polynomials $\tilde{y}_n(x;\theta)$. This quantity is defined as the Fisher information of the associated Rakhmanov density (5) with respect to the parameter θ; that is, according to Eq. (1), by

$$I_n(\theta) = 4\int_a^b \left\{\frac{\partial}{\partial\theta}[\tilde{\rho}_n(x;\theta)]^{1/2}\right\}^2\mathrm{d}x \tag{26}$$

with

$$[\tilde{\rho}_n(x;\theta)]^{1/2} = [\omega(x;\theta)]^{1/2}\tilde{y}_n(x;\theta).$$

Theorem 1. *The parameter-based Fisher information* $I_n(\theta)$ *of the parameter-dependent classical orthonormal polynomials* $\tilde{y}_n(x;\theta)$ (*i.e., Jacobi and Laguerre*) *defined by Eq.* (26) *has the value*

$$I_n(\theta) = 2\sum_{k=0}^{n-1}[\tilde{A}_k(\theta)]^2 - 2\frac{\partial\tilde{A}_n(\theta)}{\partial\theta}, \tag{27}$$

where $\tilde{A}_k(\theta)$, $k=0,1,\ldots,n$ *are the expansion coefficients of the derivative with respect to* θ *of* $\tilde{y}_n(x;\theta)$ *in terms of the polynomials* $\{\tilde{y}_k(x;\theta)\}_{k=0}^{n}$, *which are given by Lemma* 1. *See Eqs.* (20)–(21) *and* (22)–(25) *for the Laguerre and Jacobi families*, *respectively*.

Remark. Let us underline that the Fisher quantities of orthogonal, monic orthogonal and orthonormal polynomials have the same value because of Eqs. (5) and (26), keeping in mind the probabilistic character of the Rakhmanov density and the fact that all these polynomials are orthogonal with respect to the same weight function.

Proof. To prove this theorem we begin with Eq. (26). Then, the derivative with respect to θ and Lemma 1 leads to

$$\frac{\partial}{\partial\theta}[\tilde{\rho}_n(x;\theta)]^{1/2}=[\omega(x;\theta)]^{1/2}\sum_{k=0}^{n}\tilde{A}_k(\theta)\tilde{y}_k(x;\theta)+\frac{\partial[\omega(x;\theta)]^{1/2}}{\partial\theta}\tilde{y}_n(x;\theta).$$

The substitution of this expression into Eq. (26) and the consideration of the orthonormalization condition (4) have led us to

$$I_n(\theta)=J_1+J_2+J_3,$$

where

$$J_1=4\int_a^b\omega(x;\theta)\left(\sum_{k=0}^{n}\tilde{A}_k(\theta)\tilde{y}_k(x;\theta)\right)^2\mathrm{d}x=4\sum_{k=0}^{n}[\tilde{A}_k(\theta)]^2,$$

$$J_2=4\int_a^b\left(\frac{\partial[\omega(x;\theta)]^{1/2}}{\partial\theta}\right)^2[\tilde{y}_n(x;\theta)]^2\,\mathrm{d}x,$$

and

$$J_3=8\sum_{k=0}^{n}\tilde{A}_k(\theta)\int_a^b[\omega(x;\theta)]^{1/2}\frac{\partial[\omega(x;\theta)]^{1/2}}{\partial\theta}\tilde{y}_n(x;\theta)\tilde{y}_k(x;\theta)\,\mathrm{d}x.$$

Now we take into account that the weight function of the parameter-dependent families of classical orthonormal polynomials in a real and continuous variable (i.e., Laguerre and Jacobi) has the form $\omega(x;\theta)=h(x)[t(x)]^{\theta}$, so that

$$\left\{\frac{\partial[\omega(x;\theta)]^{1/2}}{\partial\theta}\right\}^2=\frac{1}{4}\omega(x;\theta)[\ln t(x)]^2=\frac{1}{4}\frac{\partial^2\omega(x;\theta)}{\partial\theta^2},$$

$$[\omega(x;\theta)]^{1/2}\frac{\partial[\omega(x;\theta)]^{1/2}}{\partial\theta}=\frac{1}{2}\omega(x;\theta)\ln t(x)=\frac{1}{2}\frac{\partial\omega(x;\theta)}{\partial\theta}.$$

The use of these two expressions in the integrals J_2 and J_3 together with Lemma 2 leads to Eq. (27). □

Corollary 1. *The Fisher information with respect to the parameter* α, $I_n^{(L)}(\alpha)$, *of the Laguerre polynomial* $\tilde{L}_n^{(\alpha)}(x)$ *is*

$$\begin{aligned}I_n^{(L)}(\alpha)&=2\sum_{k=0}^{n-1}[\tilde{A}_k^{(L)}]^2-2\frac{\partial\tilde{A}_n^{(L)}}{\partial\alpha}\\&=\psi^{(1)}(n+\alpha+1)+\frac{2n}{n+\alpha}{}_4F_3\left(\begin{matrix}1&1&1&1-n\\2&2&1-\alpha-n&\end{matrix}\middle|\,1\right),\end{aligned}\tag{28}$$

where $\psi^{(1)}(x)=(\mathrm{d}/\mathrm{d}x)\psi(x)$ *is the trigamma function*.

Corollary 2. *The Fisher information with respect to the parameter* α, $I_n^{(J_\alpha)}(\alpha,\beta)$, *of the Jacobi polynomial* $\tilde{P}_n^{(\alpha,\beta)}(x)$ *has the value*

$$\begin{aligned} I_n^{(J_\alpha)}(\alpha,\beta) &= 2\sum_{k=0}^{n-1}[\tilde{A}_k^{(J_\alpha)}]^2 - 2\frac{\partial \tilde{A}_n^{(J_\alpha)}}{\partial\alpha} \\ &= 2\frac{\Gamma(n+\beta+1)n!(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)} \\ &\quad\times\sum_{k=0}^{n-1}\frac{\Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)(2k+\alpha+\beta+1)}{\Gamma(k+\beta+1)k!(n-k)^2(n+k+\alpha+\beta+1)^2} \\ &\quad - 2\psi^{(1)}(2n+\alpha+\beta+1)+\psi^{(1)}(n+\alpha+\beta+1) \\ &\quad + \psi^{(1)}(n+\alpha+1)+\frac{1}{(2n+\alpha+\beta+1)^2}, \end{aligned} \tag{29}$$

and the Fisher information with respect to the parameter β, $I_n^{(J_\beta)}(\alpha,\beta)$, *of the Jacobi polynomial* $\tilde{P}_n^{(\alpha,\beta)}(x)$ *has the value*

$$\begin{aligned} I_n^{(J_\beta)}(\alpha,\beta) &= 2\sum_{k=0}^{n-1}[\tilde{A}_k^{(J_\beta)}]^2 - 2\frac{\partial \tilde{A}_n^{(J_\beta)}}{\partial\beta} \\ &= 2\frac{\Gamma(n+\alpha+1)n!(2n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)} \\ &\quad\times\sum_{k=0}^{n-1}\frac{\Gamma(k+\beta+1)\Gamma(k+\alpha+\beta+1)(2k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)k!(n-k)^2(n+k+\alpha+\beta+1)^2} \\ &\quad - 2\psi^{(1)}(2n+\alpha+\beta+1)+\psi^{(1)}(n+\alpha+\beta+1) \\ &\quad + \psi^{(1)}(n+\beta+1)+\frac{1}{(2n+\alpha+\beta+1)^2}. \end{aligned} \tag{30}$$

Both corollaries follow from Theorem 1 in a straightforward manner by taking into account expressions (20)–(25) for the expansion coefficients $\tilde{A}_k$ of the corresponding families. Let us underline that J_α and J_β indicate Fisher information with respect to the first and second parameter, respectively, of the Jacobi polynomial.

4. Parameter-based Fisher information of the Gegenbauer and Grosjean polynomials

In this section we describe the Fisher information of two important subfamilies of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$: the ultraspherical or Gegenbauer polynomials [1,9,26], which have $\alpha=\beta$, and the Grosjean polynomials of the first and second kind [12,18,34,35], which have $\alpha+\beta=\mp 1$, respectively. Let us remark that the parameter-based Fisher information for these subfamilies cannot be obtained from the expressions of the similar quantity for the Jacobi polynomials (given by Corollary 2) by means of a mere substitution of the parameters, because it depends on the derivative with respect to the parameter(s) and now α and β are correlated.

The Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are Jacobi-like polynomials satisfying the orthogonality condition (3) with the weight function $\omega_C(x;\lambda)=(1-x^2)^{\lambda-1/2}$, $\lambda>-\frac{1}{2}$, and the normalization constant

$$[d_k^{(C)}(\lambda)]^2 = \frac{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)}{k!(k+\lambda)\Gamma^2(\lambda)}$$

so that they can be expressed as

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x).$$

It is known [22] that expansion (12) for the derivative of $C_n^{(\lambda)}(x)$ with respect to the parameter λ has the coefficients

$$A_k^{(C)}(\lambda) = \frac{2(1+(-1)^{n-k})(k+\lambda)}{(k+n+2\lambda)(n-k)} \quad \text{for } k=0,1,\ldots,n-1,$$

$$A_n^{(C)}(\lambda) = \sum_{k=0}^{n-1} \frac{2(k+1)}{(2k+2\lambda+1)(k+2\lambda)} + \frac{2}{k+n+2\lambda}$$
$$= \psi(n+\lambda) - \psi(\lambda).$$

Then, according to Eqs. (10)–(11) of Lemma 1, expansion (9) for the derivative of the orthonormal Gegenbauer polynomials has the following coefficients:

$$\tilde{A}_k^{(C)}(\lambda) = \left[\frac{\Gamma(k+2\lambda)n!(n+\lambda)}{\Gamma(n+2\lambda)k!(k+\lambda)}\right]^{1/2} \frac{2(1+(-1)^{n-k})(k+\lambda)}{(k+n+2\lambda)(n-k)} \quad \text{for } k=0,1,\ldots,n-1,$$

$$\tilde{A}_n^{(C)}(\lambda) = \psi(n+\lambda) - \psi(n+2\lambda) + \ln 2 + \frac{1}{2(n+\lambda)}.$$

Theorem 1 provides, according to Eq. (27), the following value for the Fisher information of the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ with respect to the parameter λ:

$$I_n^{(C)}(\lambda) = \frac{16n!(n+\lambda)}{\Gamma(n+2\lambda)} \sum_{k=0}^{n-1} \frac{(1+(-1)^{n-k})\Gamma(k+2\lambda)(k+\lambda)}{k!(k+n+2\lambda)^2(n-k)^2} - 2\psi^{(1)}(n+\lambda) + 4\psi^{(1)}(n+2\lambda) + \frac{1}{(n+\lambda)^2}.$$

Let us now do the same job for the Grosjean polynomials of the first and second kind, which are the monic Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ with $\alpha+\beta=\mp 1$, respectively. So, we have [18,34]

$$G_n^{(\alpha)}(x) = c_n P_n^{(\alpha,-1-\alpha)}(x), \quad -1<\alpha<0$$

and

$$g_n^{(\alpha)}(x) = e_n P_n^{(\alpha,1-\alpha)}(x), \quad -1<\alpha<2,$$

for the Grosjean polynomials of first and second kind, respectively, with the values

$$c_n = 2^n \binom{2n-1}{n}^{-1}, \quad e_n = 2^n \binom{2n+1}{n}^{-1}.$$

The Grosjean polynomials of the first kind $G_n^{(\alpha)}(x)$ satisfy the orthogonality condition (3) with the weight function

$$\omega_G(x;\alpha) = \left(\frac{1-x}{1+x}\right)^{\alpha} \frac{1}{1+x},$$

and the normalization constant

$$[d_n^{(G)}(\alpha)]^2 = \frac{2^{2n-1}\Gamma^2(n)}{\Gamma^2(2n)} \Gamma(n+\alpha+1)\Gamma(n-\alpha).$$

These polynomials, together with the Chebyshev polynomials of the first, second, third and fourth kind, are the only Jacobi polynomials for which the associated polynomials are again Jacobi polynomials [18].

Now, expansion (12) for the derivative of the polynomials $G_n^{(\alpha)}(x)$ with respect to the parameter α can be obtained as

$$\frac{\partial G_n^{(\alpha)}(x)}{\partial\alpha}=\frac{\partial \hat{P}_n^{(\alpha,-1-\alpha)}(x)}{\partial\alpha}=\left.\frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial\alpha}\right|_{\beta=-1-\alpha}-\left.\frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial\beta}\right|_{\beta=-1-\alpha}$$
$$=\sum_{k=0}^{n-1}A_k^{(G)}(\alpha)G_n^{(\alpha)}(x)$$

with

$$A_k^{(G)}(\alpha)=\frac{2^{n-k+1}k}{n^2-k^2}\frac{\Gamma(2k)\Gamma(n+1)}{\Gamma(2n)\Gamma(k+1)}[(k-\alpha)_{n-k}-(-1)^{n-k}(k+\alpha+1)_{n-k}]$$

for $k=0,1,\ldots,n-1$ and $A_n^{(G)}(\alpha)=0$. Then, Lemma 1 provides expansion (9) for the derivative of the orthonormal Grosjean polynomials with the coefficients

$$\tilde{A}_k^{(G)}(\alpha)=\frac{2n}{n^2-k^2}\frac{(k-\alpha)_{n-k}-(-1)^{n-k}(k+\alpha+1)_{n-k}}{[(k+\alpha+1)_{n-k}(k-\alpha)_{n-k}]^{1/2}}\quad \text{for } k=0,1,\ldots,n-1,$$

$$\tilde{A}_n^{(G)}(\alpha)=\frac{1}{2}[\psi(n-\alpha)-\psi(n+\alpha+1)].$$

Finally, Eq. (27) of Theorem 1 allows us to find the following value for the Fisher information of the Grosjean polynomials of the first kind:

$$I_n^{(G)}(\alpha)=8n^2\sum_{k=0}^{n-1}\frac{1}{(n^2-k^2)^2}\frac{[(k-\alpha)_{n-k}-(-1)^{n-k}(k+\alpha+1)_{n-k}]^2}{(k+\alpha+1)_{n-k}(k-\alpha)_{n-k}}$$
$$+\psi^{(1)}(n-\alpha)+\psi^{(1)}(n+\alpha+1).$$

On the other hand, the Grosjean polynomials of the second kind $g_n^{(\alpha)}(x)$ satisfy the orthogonality property (3) with the weight function

$$\omega_g(x;\alpha)=\left(\frac{1-x}{1+x}\right)^{\alpha}(1+x),$$

and the normalization constant

$$[d_n^{(g)}(\alpha)]^2=\frac{2^{2n+1}\Gamma^2(n)}{\Gamma^2(2n+2)}\Gamma(n+\alpha+1)\Gamma(n-\alpha+2).$$

Moreover, the derivative of these polynomials with respect to the parameter α can be expanded in the form (9) as

$$\frac{\partial g_n^{(\alpha)}(x)}{\partial\alpha}=\frac{\partial \hat{P}_n^{(\alpha,1-\alpha)}(x)}{\partial\alpha}=\left.\frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial\alpha}\right|_{\beta=1-\alpha}-\left.\frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial\beta}\right|_{\beta=1-\alpha}$$
$$=\sum_{k=0}^{n-1}A_k^{(g)}(\alpha)g_n^{(\alpha)}(x)$$

with

$$A_k^{(g)}(\alpha)=\frac{2^{n-k+1}(k+1)}{(n-k)(n+k+2)}\frac{\Gamma(2k+2)\Gamma(n+1)}{\Gamma(2n+2)\Gamma(k+1)}[(k+2-\alpha)_{n-k}-(-1)^{n-k}(k+\alpha+1)_{n-k}]$$

for $k=0, 1, \ldots, n-1$ and $A_n^{(g)}(\alpha)=0$. Then, Lemma 1 is able to provide the analogous expansion (9) for the orthonormal polynomials with the coefficients

$$\tilde{A}_k^{(g)}(\alpha) = \frac{2(k+1)}{(n-k)(n+k+2)} \frac{(k+2-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}}{[(k+\alpha+1)_{n-k}(k+2-\alpha)_{n-k}]^{1/2}}, \quad \text{for } k = 0, 1, \ldots, n-1,$$

$$\tilde{A}_n^{(g)}(\alpha) = \frac{1}{2}[\psi(n+2-\alpha) - \psi(n+\alpha+1)].$$

Finally, we obtain by means of Eq. (27) of Theorem 1 the Fisher information of the Grosjean polynomials of the second kind, which turns out to have the value

$$I_n^{(g)}(\alpha) = 8\sum_{k=0}^{n-1} \frac{(k+1)^2}{(n-k)^2(n+k+2)^2} \frac{[(k+2-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}]^2}{(k+\alpha+1)_{n-k}(k+2-\alpha)_{n-k}} + \psi^{(1)}(n+2-\alpha) + \psi^{(1)}(n+\alpha+1).$$

5. Conclusions and open problems

In summary, we have calculated the parameter-based Fisher information for the classical orthogonal polynomials of a continuous and real variable with a parameter dependence; namely, the Jacobi and Laguerre polynomials. Then we have evaluated the corresponding Fisher quantity for the two most relevant parameter-dependent Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$: the Gegenbauer ($\alpha=\beta$) and the Grosjean ($\alpha+\beta=\pm 1$) polynomials.

This paper, together with Ref. [39], opens the way for the development of the Fisher estimation theory of the Rakhmanov density for continuous and discrete orthogonal polynomials in and beyond the Askey scheme. This fundamental task in approximation theory includes the determination of the spreading of the orthogonal polynomials throughout its orthogonality domain by means of the Fisher information with a locality property. All these mathematical questions have a straightforward application to quantum systems because their wavefunctions are often controlled by orthogonal polynomials, so that the probability densities which describe the quantum-mechanical states of these physical systems are just the Rakhmanov densities of the corresponding orthogonal polynomials. In particular, they correspond to the ground and excited states of the physical systems with an exactly solvable spherically symmetric potential [26], including the most common prototypes (harmonic oscillator, hydrogen atom,...), in both position and momentum spaces.

Among the open problems let us first mention the computation of the parameter-based Fisher information of the generalized Hermite polynomials, the Bessel polynomials and the Pollaczek polynomials. A much more ambitious problem is the evaluation of the Fisher quantity for the general Wilson orthogonal polynomials.

On the other hand, nothing is known for discrete orthogonal polynomials. In this case, however, the very notion of the parameter-based Fisher information is a subtle question.

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