

Applications of a Recurrence for the Bernoulli Numbers

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We give an easy proof of a recently published recurrence for the Bernoulli numbers and we present some applications of the recurrence. One of the applications is a simple proof of the well-known Staudt–Clausen Theorem. Proofs are also given for theorems of Carlitz, Frobenius, and Ramanujan. An analogous recurrence for Genocchi numbers is proved and applications are given. In particular, theorems of Lehmer, Ramanujan, and Kummer are proved and, in some cases, extended.

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1. INTRODUCTION

The Bernoulli numbers B_m may be defined by means of the generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!} \quad (|x| < 2\pi). \tag{1}$$

It is well-known that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$ and $B_{2k+1} = 0$ for $k > 0$. Deeba and Rodriguez [4] and Gessel [6] have recently proved the following recurrence

$$B_m = \frac{1}{n(1 - n^m)} \sum_{k=0}^{m-1} n^k \binom{m}{k} B_k \sum_{j=1}^{n-1} j^{m-k}, \tag{2}$$

which is true for any positive integer m and any positive integer $n > 1$. We prove at the end of this section that Eq. (2) is not new; it is a special case of the “multiplication theorem” for Bernoulli polynomials. We give another proof of (2) in Section 3, and then we prove that the following theorems are easy consequences of (2):

THEOREM 1 (The Staudt–Clausen Theorem [1], [12, p. 257]). For $m \geq 1$,

$$B_{2m} = A_{2m} - \sum_{(p-1)|2m} \frac{1}{p},$$

where A_{2m} is an integer and the summation is over all primes p such that $(p-1)|2m$.

THEOREM 2 (Carlitz [2, 3]). Let $m > 1$. If p is any prime number and if $(p-1)p^h|2m$, then p^h divides the numerator of $B_{2m} + (1/p) - 1$. That is, $pB_{2m} \equiv p-1 \pmod{p^{h+1}}$.

THEOREM 3 (Frobenius [5, p. 821]). If $m > 2$, then 16 divides the numerator of $B_{2m} - \frac{1}{2} + 6m$. That is, $2B_{2m} \equiv 1 - 12m \pmod{32}$.

In Section 6 we are able to extend Frobenius' theorem and obtain congruences $\pmod{64}$ and $\pmod{128}$.

THEOREM 4 [14, pp. 258–260]. Let $B_{2m} = P_{2m}/Q_{2m}$ with $\gcd(P_{2m}, Q_{2m}) = 1$. Then for all positive integers m and n ,

$$[1^{2m} + 2^{2m} + \dots + (n-1)^{2m}] Q_{2m} \equiv nP_{2m} \pmod{n^2}.$$

THEOREM 5 (Ramanujan [12, p. 7; 13; 15]). If 4 divides n , then 20 divides the numerator of $B_n + \frac{1}{30}$. That is, $30B_{4m} \equiv -1 \pmod{200}$. Also, 5 divides the numerator of $B_{4m+2}/(4m+2) - 1/12$. That is, $B_{4m+2}/(4m+2) \equiv 3 \pmod{5}$.

In Section 9 we prove a formula analogous to (2) for the Genocchi numbers $G_m = 2(1-2^m)B_m$, and some applications are given. In particular the formula produces an easy proof that $(2^{m+1}(1-2^{2m})/2m)B_{2m}$ is an integer. It also furnishes proofs, and extensions, of congruences of Lehmer [10], Ramanujan [13], and Kummer [9].

Formula (2) provides an extremely simple, unified approach to some of the classical theorems and congruences for the Bernoulli numbers. For example, the writer respectfully submits that the proof of the Staudt–Clausen Theorem given in Section 4 is as simple as any proof that has been published. In some ways this paper is similar to [8], in which Wells Johnson gave a unified approach, using p -adic proofs, to many theorems for the Bernoulli numbers. The present paper, which essentially uses only formula (2) and the analogous formula for Genocchi numbers (Theorem 6), is more elementary and consequently less comprehensive than [8]. Many of the theorems of the present paper can be proved by

means of the Euler–Maclaurin summation formula, as in [8]. Our purpose, however, is to present applications of (2) and Theorem 6, so the Euler–Maclaurin formula is not used.

It should be pointed out that (2) is a special case of the multiplication theorem for the Bernoulli polynomials $B_m(x) = \sum_{k=0}^m \binom{m}{k} B_k x^{m-k}$. The multiplication theorem can be stated this way [11, p. 21]: If n and m are positive integers, with $n > 1$, then

$$n^{1-m} B_m(nx) = \sum_{j=0}^{n-1} B_m\left(x + \frac{j}{n}\right). \tag{3}$$

If we put $x = 0$ in (3), and multiply both sides by n^m , we have

$$\begin{aligned} nB_m &= n^m \sum_{j=0}^{n-1} B_m\left(\frac{j}{n}\right) = n^m \sum_{j=0}^{n-1} \sum_{k=0}^m \binom{m}{k} \left\{ B_k \cdot \left(\frac{j}{n}\right)^{m-k} \right\} \\ &= \sum_{k=0}^m n^k \binom{m}{k} B_k \sum_{j=0}^{n-1} j^{m-k}, \end{aligned}$$

which is the same as Eq. (2). Even though (2) is not really new, the writer believes that all the proofs in the present paper are new.

2. PRELIMINARIES

In this paper the letter p always signifies a prime number, and the notation $p^h \parallel t$ means p^h divides t , but p^{h+1} does not divide t . For convenience, we use the notation

$$S_k(n) = 1^k + 2^k + \dots + n^k. \tag{4}$$

The generating function for $S_k(n)$ is

$$\sum_{k=0}^{\infty} S_k(n) \frac{x^k}{k!} = e^x + e^{2x} + \dots + e^{nx} = \frac{e^{(n+1)x} - e^x}{e^x - 1}. \tag{5}$$

The following lemma is well-known and easily proved [14, p. 232]:

LEMMA 1. *If p is a prime number and k is a non-negative integer, then*

$$S_k(p-1) \equiv \begin{cases} -1 \pmod{p} & \text{if } p-1 \text{ divides } k; \\ 0 \pmod{p} & \text{if } p-1 \text{ does not divide } k. \end{cases}$$

The next lemma is also easily proved:

LEMMA 2. *If p is a prime number such that $p^h \parallel m$, and if $k > 2$, then*

$$p^k \binom{m}{k} \equiv 0 \pmod{p^{h+2}}.$$

If p is odd, the congruence also holds for $k = 2$. If $p \geq 5$ and $k \geq 4$, then the modulus can be raised to p^{h+4} .

Proof. It is clear the congruence holds for $k = 2$ if p is odd. Assume $k > 2$, and suppose $p^w \parallel k$. Since $p^k \binom{m}{k} = m \cdot p^k/k \cdot \binom{m-1}{k-1}$, we see that $p^{h+(k-w)}$ divides $p^k \binom{m}{k}$, and we will show that $(k-w) \geq 2$. If $w \geq 2$, then $(k-w) \geq (p^w - w) \geq 2$. If $w = 0$ or $w = 1$, then (since $k > 2$) we have $(k-w) \geq 2$, and the proof for modulus p^{h+2} is complete. If $p \geq 5$ and $k \geq 4$, we have $(k-w) \geq (p^w - w) \geq 4$ if $w > 0$, and we have $k - 0 \geq 4$ if $w = 0$. This completes the proof.

It is convenient to use the following definition of congruence for rational numbers [14, p. 263]. Let a, b, c, d, m be integers, with $m > 1$. If $\gcd(b, m) = 1$, we say a/b is *integral* (mod m). If a/b and c/d are integral (mod m), we define

$$\frac{a}{b} \equiv \frac{c}{d} \pmod{m}$$

if and only if m divides $(ad - bc)$; that is, if and only if $ad \equiv bc \pmod{m}$ with the usual definition of congruence. It is easily seen that the familiar properties of congruence continue to hold. We have stated Theorems 1A, 2, 3 and 5 in terms of this kind of rational congruence.

3. PROOF OF (2)

Since $S_0(n-1) = n-1$, recurrence (2) can be written

$$(n - n^m) \frac{B_m}{m!} = \sum_{k=0}^m \left(\frac{n^k B_k}{k!} \cdot \frac{S_{m-k}(n-1)}{(m-k)!} \right). \quad (6)$$

The generating function for the left side of (6) is

$$\sum_{m=0}^{\infty} (n - n^m) \frac{B_m}{m!} x^m = \frac{nx}{e^x - 1} - \frac{nx}{e^{nx} - 1} = \frac{nx}{e^{nx} - 1} \cdot \frac{e^{nx} - e^x}{e^x - 1}. \quad (7)$$

The expression on the extreme right of (7) is the generating function

$$\sum_{k=0}^{\infty} \frac{n^k B_k}{k!} x^k \cdot \sum_{j=0}^{\infty} \frac{S_j(n-1)}{j!} x^j = \sum_{m=0}^{\infty} \left(\sum_{k=0}^m \frac{n^k B_k}{k!} \cdot \frac{S_{m-k}(n-1)}{(m-k)!} \right) x^m. \tag{8}$$

Equating coefficients of x^m in the extreme left side of (7) and the right side of (8) gives us (6), and the proof is complete.

4. PROOF OF THEOREM 1

We shall prove the following, which is equivalent to Theorem 1:

THEOREM 1A. *If p is a prime number and if $r > 0$, then*

$$pB_{2r} \equiv \begin{cases} 0 \pmod{p} & \text{if } p-1 \text{ does not divide } 2r, \\ -1 \pmod{p} & \text{if } p-1 \text{ divides } 2r. \end{cases}$$

Proof. The proof is by induction on r . Since $B_2 = \frac{1}{6}$, the theorem is true for $r = 1$; assume it is true for $r = 1, \dots, m-1$. If we let $n = p$ in (2), we have

$$p(1 - p^{2m}) B_{2m} = S_{2m}(p-1) - mpS_{2m-1}(p-1) + \sum_{k=1}^{m-1} p^{2k} \binom{2m}{2k} B_{2k} S_{2m-2k}(p-1). \tag{9}$$

By the induction hypothesis, we see that the right side of (9) is integral $(\text{mod } p)$. In fact, by the induction hypothesis we have

$$p^{2k} B_{2k} \equiv 0 \pmod{p} \quad (1 \leq k \leq m-1).$$

Thus pB_{2m} is integral $(\text{mod } p)$, and from (9) and Lemma 1 we have

$$pB^{2m} \equiv S_{2m}(p-1) \equiv \begin{cases} 0 \pmod{p} & \text{if } (p-1) \text{ does not divide } 2m, \\ -1 \pmod{p} & \text{if } (p-1) \text{ divides } 2m. \end{cases}$$

This completes the proof.

Theorem 1A tells us that if $m \geq 1$, then the denominator of B_{2m} (in lowest terms) is exactly the product of those primes p for which $p-1$ divides $2m$.

It is easy to see that Theorem 1A is equivalent to Theorem 1. It is clear that Theorem 1 implies Theorem 1A. Conversely, if Theorem 1A is true, we have for $m \geq 1$

$$B_{2m} = \frac{N}{p_1 p_2 \cdots p_k}; \quad N \equiv -p_2 \cdots p_k \pmod{p_1},$$

where p_1, \dots, p_k are those primes p for which $(p-1)$ divides $2m$. Thus

$$B_{2m} = -\frac{1}{p_1} + \frac{N_1}{p_2 \cdots p_k}; \quad N_1 \equiv -p_3 \cdots p_k \pmod{p_2},$$

since $p_2 B_{2m} \equiv -1 \pmod{p_2}$. Thus we have

$$B_{2m} = -\frac{1}{p_1} - \frac{1}{p_2} + \frac{N_2}{p_3 \cdots p_k}; \quad N_2 \equiv -p_4 \cdots p_k \pmod{p_3}.$$

Continuing in this manner, we eventually have

$$B_{2m} = -\frac{1}{p_1} - \frac{1}{p_2} - \cdots - \frac{1}{p_k} + N_k,$$

where N_k is an integer. Letting $N_k = A_{2m}$, we obtain Theorem 1.

5. PROOF OF THEOREM 2

Suppose $p^h \parallel 2m$, and note that $h > 0$ if $p = 2$. From (9) we have

$$\begin{aligned} pB_{2m} &\equiv S_{2m}(p-1) - mpS_{2m-1}(p-1) \\ &\quad + \sum_{k=1}^{m-1} p^{2k} \binom{2m}{2k} B_{2k} S_{2m-2k}(p-1) \pmod{p^{h+1}}, \end{aligned}$$

and by Lemma 2 we have, for $p > 2$,

$$p^{2k} \binom{2m}{2k} B_{2k} \equiv 0 \pmod{p^{h+1}} \quad (1 \leq k \leq m-1). \quad (10)$$

Thus if $p > 2$ and $m \geq 1$, we have

$$pB_{2m} \equiv S_{2m}(p-1) \pmod{p^{h+1}}.$$

If $p = 2$, Lemma 2 tells us that (10) holds for $2 \leq k \leq m-1$. Thus if $2^h \parallel 2m$, we have

$$\begin{aligned} 2B_{2m} &\equiv S_{2m}(1) - 2mS_{2m-1}(1) \\ &\quad + \left(\frac{1}{6}\right) 4m(2m-1) S_{2m-2}(1) \pmod{2^{h+1}} \quad (m > 1). \end{aligned}$$

Since $S_k(1) = 1$ for all k , we have, after simplifying,

$$2B_{2m} \equiv 1 + \left(\frac{1}{3}\right) 2m(2m-4) \equiv 1 \pmod{2^{h+1}} \quad (m > 1).$$

Thus for all primes p we have the following useful result: If $p^h \mid\mid 2m$, then

$$pB_{2m} \equiv S_{2m}(p-1) \pmod{p^{h+1}} \quad (m > 1). \tag{11}$$

Now we note that $\theta(p^{h+1}) = p^h(p-1)$, where θ is Euler's phi function. By Euler's generalization of Fermat's Little Theorem [14, p. 146], we know that if $\gcd(a, p) = 1$ and if $p^h(p-1)$ divides $2m$, then $a^{2m} \equiv 1 \pmod{p^{h+1}}$. Thus

$$\begin{aligned} S_{2m}(p-1) &= 1^{2m} + 2^{2m} + \dots + (p-1)^{2m} \\ &\equiv 1 + 1 + \dots + 1 \equiv p-1 \pmod{p^{h+1}}. \end{aligned} \tag{12}$$

By (11) and (12) we have for all primes p : If $(p-1)p^h$ divides $2m$, then

$$pB_{2m} \equiv p-1 \pmod{p^{h+1}} \quad (m > 1). \tag{13}$$

Formula (13) is equivalent to Theorem 2, and the proof is complete.

We note here that if $p \geq 5$ the modulus in (10) can be raised to p^{h+2} , by Lemma 2. This fact and Lemma 1 imply the modulus in (11) can be raised to p^{h+2} if $p \geq 5$. We can go further: If $p \geq 5$ and $p-1$ does not divide $2m-2$, then

$$pB_{2m} \equiv S_{2m}(p-1) \pmod{p^{h+3}}. \tag{14}$$

As Johnson [8] points out, congruence (14) is useful in proving some congruences of Lehmer. We shall state (14) as Theorem 12 and prove it in Section 10.

6. PROOF OF THEOREM 3

More generally, it follows immediately from (2) and Theorem 1 that

$$(1 - 2^{2m}) 2B_{2m} \equiv 1 - 2m + \sum_{k=1}^j 2^{2k} \binom{2m}{2k} B_{2k} \pmod{2^{2j+1}}. \tag{15}$$

Thus if $j = 2$, we have, for $m > 2$,

$$\begin{aligned} 2B_{2m} &\equiv 1 - 2m + \sum_{k=1}^2 2^{2k} \binom{2m}{2k} B_{2k} \\ &\equiv 1 - 2m + \frac{2}{3} \binom{2m}{2} - \frac{8}{15} \binom{2m}{4} \pmod{32}. \end{aligned}$$

After simplifying, we have

$$2B_{2m} \equiv 16m^4 + 4m + 1 \equiv 1 - 12m \pmod{32},$$

which is equivalent to Theorem 3. This completes the proof.

Note that (15) allows us to extend Frobenius' congruence to the modulus 2^{2j+1} for arbitrary j . It follows from (15), for example, that if $m > 3$ then

$$2B_{2m} \equiv 1 - 12m + 16m(1 + m^3) \pmod{64},$$

$$2B_{2m} \equiv 1 - 12m + 16m(1 + m^3) - 32m^2(1 + m) \pmod{128}.$$

7. PROOF OF THEOREM 4

From (2) we have, with m replaced by $2m$:

$$n(1 - n^{2m}) B_{2m} = \sum_{k=0}^{2m-1} n^k \binom{2m}{k} B_k S_{2m-k}(n-1). \quad (16)$$

By the Staudt-Clausen Theorem we know that nB_k is integral (mod n), and we know that 6 divides Q_{2m} , if $m > 0$. Thus from (16) we have

$$nP_{2m} \equiv S_{2m}(n-1) Q_{2m} - nmS_{2m-1}(n-1) Q_{2m} \pmod{n^2}. \quad (17)$$

Now we examine $S_{2m-1}(n-1)$. From (2) we have (with m replaced by $2m-1$)

$$S_{2m-1}(n-1) = - \sum_{k=1}^{2m-2} n^k \binom{2m-1}{k} B_k S_{2m-1-k}(n-1),$$

so

$$nS_{2m-1}(n-1) Q_{2m} \equiv 0 \pmod{n^2}. \quad (18)$$

Returning to (17), we use (18) to obtain

$$nP_{2m} \equiv S_{2m}(n-1) Q_{2m} \pmod{n^2},$$

and the proof of this "remarkable congruence" [14, p. 260] is complete.

8. PROOF OF THEOREM 5

To prove the first congruence of Theorem 5, we first note that by Theorem 2 we have $2B_{4m} \equiv 1 \pmod{8}$ for $m > 0$. Thus

$$30B_{4m} \equiv 15 \equiv -1 \pmod{8}. \quad (19)$$

From (14) we have

$$5B_{4m} \equiv S_{4m}(4) \pmod{25}, \tag{20}$$

where

$$\begin{aligned} S_{4m}(4) &= 1^{4m} + 2^{4m} + 3^{4m} + 4^{4m} = 1 + 16^m + 81^m + 256^m \\ &\equiv 1 + 16^m + 6^m + 6^m \equiv 1 + (1 + 15)^m + (1 + 5)^m + (1 + 5)^m \\ &\equiv 1 + (1 + 15m) + (1 + 5m) + (1 + 5m) \equiv 4 \pmod{25}. \end{aligned}$$

Thus (20) gives us

$$30B_{4m} \equiv 24 \equiv -1 \pmod{25}. \tag{21}$$

By (19) and (21) we can conclude that $30B_{4m} \equiv -1 \pmod{200}$, i.e., 20 divides the numerator of $B_{4m} + \frac{1}{30}$. The proof of this fact in [15] is not quite correct; more than a simple application of Theorem 4 is required. In [15] Theorem 4 is applied to give $30P_{4m} \equiv -Q_{4m} \pmod{8}$. Both sides are divided by $30Q_{4m}$, which does not give $B_{4m} + \frac{1}{30} \equiv 0 \pmod{4}$, as stated. Since $2 \parallel Q_{4m}$, the modulus should be 2. Since $5 \parallel Q_{4m}$, the same mistake occurs when proving $B_{4m} + \frac{1}{30} \equiv 0 \pmod{5}$.

To prove the second congruence of Theorem 5, we suppose that $5^h \parallel (2m + 1)$. By (11) and the remarks preceding (14), we have

$$5B_{4m+2} \equiv S_{4m+2}(4) \equiv 1 + (5 - 1)^{2m+1} + (10 - 1)^{2m+1} + (15 + 1)^{2m+1} \pmod{5^{h+2}}. \tag{22}$$

By Lemma 2 and (22), we have

$$\begin{aligned} 5B_{4m+2} &\equiv 1 + [-1 + (2m + 1) 5] + [-1 + (2m + 1) 10] \\ &\quad + [1 + (2m + 1) 15] \\ &\equiv 30(2m + 1) \pmod{5^{h+2}}. \end{aligned}$$

Thus we have

$$\frac{B_{4m+2}}{4m+2} \equiv 3 \equiv \frac{1}{12} \pmod{5},$$

and the proof is complete.

9. A FORMULA FOR GENOCCHI NUMBERS

The Genocchi numbers G_m may be defined by the generating function

$$\frac{2x}{e^x + 1} = \sum_{m=0}^{\infty} G_m \frac{x^m}{m!} \quad (|x| < \pi). \tag{23}$$

It follows from (23) that

$$G_m = 2(1 - 2^m) B_m, \tag{24}$$

and it follows from (24) and the Staudt-Clausen Theorem that the Genocchi numbers are integers.

In this section we derive a formula analogous to (2) for the Genocchi numbers. It is a special case of the multiplication theorem for the Euler polynomials $E_m(x)$, which can be defined [11, pp. 23-29] by

$$E_m(x) = \sum_{k=0}^m \binom{m}{k} \frac{G_{k+1}}{k+1} x^{m-k}. \tag{25}$$

We define the alternating sum $Z_m(n)$ by

$$Z_m(n) = 1^m - 2^m + 3^m - \dots (-1)^{n+1} n^m. \tag{26}$$

Then we have the generating function

$$\sum_{k=0}^{\infty} Z_k(n) \frac{x^k}{k!} = e^x - e^{2x} + e^{3x} - \dots (-1)^{n+1} e^{nx} = \frac{e^x + (-1)^{n+1} e^{(n+1)x}}{e^x + 1}. \tag{27}$$

THEOREM 6. *If n and m are integers, with $n > 1$, $m > 1$, and n odd, then*

$$(n^m - n) G_m = \sum_{k=1}^{m-1} \binom{m}{k} n^k G_k Z_{m-k}(n-1).$$

Proof. Since $G_0 = 0$ and $Z_0(n-1) = 0$ if n is odd, the formula in Theorem 6 is

$$(n^m - n) \frac{G_m}{m!} = \sum_{k=0}^m \left(\frac{n^k G_k}{k!} \cdot \frac{Z_{m-k}(n-1)}{(m-k)!} \right). \tag{28}$$

The generating function for the left side of (28) is

$$\sum_{m=0}^{\infty} (n^m - n) \frac{G_m}{m!} x^m = \frac{2nx}{e^{nx} + 1} - \frac{2nx}{e^x + 1} = \frac{2nx}{e^{nx} + 1} \cdot \frac{e^x - e^{nx}}{e^x + 1}. \tag{29}$$

The expression on the extreme right of (29) is the generating function

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{n^k G_k}{k!} x^k \cdot \sum_{j=0}^{\infty} \frac{Z_j(n-1)}{j!} x^j \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^m \frac{n^k G_k}{k!} \cdot \frac{Z_{m-k}(n-1)}{(m-k)!} \right) x^m. \end{aligned} \tag{30}$$

Equating coefficients of x^m in the extreme left side of (29) and the right side of (30) gives us Theorem 6, and the proof is complete.

The multiplication theorem for $E_m(x)$, for n odd, is [11, p. 24]:

$$\sum_{j=0}^{n-1} (-1)^j E_m \left(x + \frac{j}{n} \right) = n^{-m} E_m(nx). \tag{31}$$

Using (25), it is easily proved that letting $x=0$ in (31) will yield Theorem 6. The multiplication theorem for n even is

$$\sum_{j=0}^{n-1} (-1)^j B_{m+1} \left(x + \frac{j}{n} \right) = -\frac{(m+1)n^{-m}}{2} E_m(nx). \tag{32}$$

Letting $x=0$ in (32) gives us the formula

$$n(1-2^m) B_m = \sum_{k=0}^{m-1} n^k \binom{m}{k} B_k Z_{m-k}(n-1). \tag{33}$$

We can now prove the following, which corresponds to Theorem 4.

THEOREM 7. *Let m and n be positive integers, with $m > 1$, $n > 1$, and n odd. Then*

$$G_{2m} \equiv -2mZ_{2m-1}(n-1) \pmod{n^2}.$$

Proof. From Theorem 6 we have

$$G_{2m} \equiv -2mZ_{2m-1}(n-1) - m(2m-1)nZ_{2m-2}(n-1) \pmod{n^2}. \tag{34}$$

Now we show that $(2m-1)nZ_{2m-2}(n-1) \equiv 0 \pmod{n^2}$. If m is odd, $m > 1$, then it follows from Theorem 6 that

$$mZ_{m-1}(n-1) \equiv -G_m \equiv 0 \pmod{n}. \tag{35}$$

Returning to (34) and using (35) with m replaced by $2m-1$, we have

$$G_{2m} \equiv -2mZ_{2m-1}(n-1) \pmod{n^2},$$

and the proof is complete.

THEOREM 8. *If $m \geq 1$, then $(2^{m+1}(1-2^{2m})/2m) B_{2m}$ is an integer.*

Proof. Let p be an odd prime and let $p^h \parallel 2m$. First we prove that

$$G_{2m} \equiv 0 \pmod{p^h}. \tag{36}$$

By Lemma 2, we know that if $k \geq 2$ then $p^k \binom{2m}{k} \equiv 0 \pmod{p^{h+2}}$. If $k = 1$, we have $p \binom{2m}{1} \equiv 0 \pmod{p^{h+1}}$. Thus by Theorem 6, with $n = p$ and m replaced by $2m$, we have $pG_{2m} \equiv 0 \pmod{p^{h+1}}$, which is equivalent to (36). Thus $2^m G_{2m}/2m$ is integral \pmod{p} for any odd prime p . We know $2B_{2m}$ is integral $\pmod{2}$ by the Staudt–Clausen Theorem, and we know if $2^w \parallel 2m$ then $m \geq w$. Thus $2^m G_{2m}/2m$ is integral $\pmod{2}$. Hence for all primes p we have proved that $2^{m+1}(1 - 2^{2m}) B_{2m}/2m = 2^m G_{2m}/2m$ is integral \pmod{p} , and the proof is complete.

It is well known that the “tangent coefficients” t_{2m} , defined by

$$\tan x = \sum_{m=1}^{\infty} (-1)^{m-1} t_{2m} \frac{x^{2m-1}}{(2m-1)!} \quad (|x| < \pi/2),$$

are closely related to the Bernoulli numbers, i.e., [11, p. 35]

$$t_{2m} = 2^{2m}(2^{2m} - 1) B_{2m}/2m. \tag{37}$$

We note that Theorem 8 provides an easy proof that the tangent coefficients are integers.

Theorem 8 is not new. Ramanujan [12, p. 5; 13] observed that $2^n(2^n - 1) B_n/n$ is an integer, and in fact it is known that if k is an arbitrary integer and $m \geq 1$, then $k^{m+1}(1 - k^{2m}) B_{2m}/2m$ is an integer [7].

THEOREM 9. *If $p \geq 5$ and $m > 1$, then*

$$\frac{2(2^{2m} - 1) B_{2m}}{2m} \equiv -\frac{G_{2m}}{2m} \equiv Z_{2m-1}(p-1) \pmod{p^2}.$$

If $p \geq 5$ and $p - 1$ does not divide $(2m - 2)$, then

$$\frac{(2^{2m} - 1) B_{2m}}{2m} \equiv \sum_{0 < a < p/2} (p - 2a)^{2m-1} \pmod{p^2}. \tag{38}$$

Proof. It follows from Theorem 6 and Lemma 2 that

$$pG_{2m} \equiv -2mpZ_{2m-1}(p-1) + m(2m-1)p^2Z_{2m-2}(p-1) \pmod{p^{h+3}}. \tag{39}$$

Since p is odd, it follows from (35) that $(2m - 1)Z_{2m-2}(p - 1) \equiv 0 \pmod{p}$. Hence, returning to (39), we have

$$\frac{G_{2m}}{2m} \equiv -Z_{2m-1}(p-1) \pmod{p^2}. \tag{40}$$

If $p - 1$ does not divide $(2m - 2)$, we can prove by means of (2) (see Eq. (42) below) that

$$S_{2m-1}(p-1) \equiv 0 \pmod{p^2}.$$

In this case

$$\begin{aligned} Z_{2m-1}(p-1) &= -S_{2m-1}(p-1) + 2 \sum_{0 < a < p/2} (p-2a)^{2m-1} \\ &\equiv 2 \sum_{0 < a < p/2} (p-2a)^{2m-1} \pmod{p^2}, \end{aligned}$$

and congruence (38) follows from (40). This completes the proof.

Congruence (38) was evidently first proved by Lehmer [10]. Theorem 9 can be considered a slight extension of Lehmer's result, since it furnishes a congruence (mod p^2) for $(2^{2m}-1)B_{2m}/2m$ regardless of whether or not $p-1$ divides $(2m-2)$.

THEOREM 10 (Ramanujan [12, p. 7; 13; 15]). *If m is any nonnegative integer, then*

$$\frac{2(2^{4m+2}-1)}{2m+1} B_{4m+2}, \quad \frac{-2(2^{8m+4}-1)}{2m+1} B_{8m+4}$$

are integers of the form $30k+1$.

Proof. It follows from Theorem 8 and the fact that $2B_{4m+2}$ is integral (mod 2) that $2(2^{4m+2}-1)B_{4m+2}/(2m+1)$ is an odd integer. From Theorem 9 we have

$$\begin{aligned} \frac{2(2^{4m+2}-1)}{2m+1} B_{4m+2} &\equiv 2Z_{4m+1}(4) \equiv 2(1^{4m+1} - 2^{4m+1} + 3^{4m+1} - 4^{4m+1}) \\ &\equiv 2(1-2+3-4) \equiv 1 \pmod{5}. \end{aligned}$$

Also

$$\begin{aligned} \frac{2(2^{4m+2}-1)}{2m+1} B_{4m+2} &\equiv 2Z_{4m+1}(2) \equiv 2(1^{4m+1} - 2^{4m+1}) \\ &\equiv 2(1-2) \equiv 1 \pmod{3}. \end{aligned}$$

Thus we have

$$\frac{2(2^{4m+2}-1)}{2m+1} B_{4m+2} \equiv 1 \pmod{30},$$

and the first part of the theorem is proved. The proof for $-2(2^{8m+4}-1)B_{8m+4}/(2m+1)$ is entirely similar. This completes the proof.

Ramanujan also asserted that $(-2(2^{8m+4}-1)/(2m+1))B_{16m+8}$ is an integer of the form $30k+1$. This can be proved in much the same way that

Theorem 10 is proved, though more argument is needed. The extra details that are necessary are furnished by Wagstaff in [15].

THEOREM 11. *Let t_{2m} be the tangent coefficient defined by (37), and let $p \geq 3$ be prime. Then $t_{2m+p-1} \equiv t_{2m} \pmod{p}$ for all $m \geq 1$. If $p \geq 5$, then for all $m > 1$ we have $t_{2m+p(p-1)} \equiv t_{2m} \pmod{p^2}$.*

Proof. The proof of the first congruence is similar to the proof of Theorem 9. It follows from Theorem 6, Lemma 2, and Fermat's Little Theorem that

$$\begin{aligned} t_{2m} &\equiv 2^{2m-1} Z_{2m-1}(p-1) \\ &\equiv 2^{2m+(p-1)-1} Z_{2m+(p-1)-1}(p-1) \equiv t_{2m+p-1} \pmod{p}, \end{aligned}$$

and the first congruence is proved. By Theorem 9 and Euler's generalization of Fermat's Little Theorem [14, p. 146] we have, if $p \geq 5$,

$$\begin{aligned} t_{2m} &\equiv 2^{2m-1} Z_{2m-1}(p-1) \\ &\equiv 2^{2m+p(p-1)-1} Z_{2m+p(p-1)-1}(p-1) \equiv t_{2m+p(p-1)} \pmod{p^2}, \end{aligned}$$

and the proof is complete.

The initial congruence of Theorem 11 was first proved by Kummer [9].

10. FINAL COMMENTS

A special case of (2) that is of interest is the following. If we replace m by $2m+1$ and let $n=2$ in (2), we have

$$\sum_{j=0}^m 2^{2j} \binom{2m+1}{2j} B_{2j} = 2m+1 \quad (m \geq 0),$$

which was proved in another way by Ramanujan [12, p. 1; 13].

A well known property of B_{2m} that does not follow directly from (2) is: If p^h divides $2m$ and $(p-1)$ does not divide $2m$, then p^h divides the numerator of B_{2m} . This follows from (11) if we know that p^{h+1} divides $S_{2m}(p-1)$. It also follows from Theorem 8 if we know that p does not divide $1-2^{2m}$. One consequence of the second congruence of Theorem 5 is that if 5^h divides $4m+2$, then 5^h divides the numerator of B_{4m+2} .

We conclude with a proof of congruence (14) that depends only on (2) and Lemmas 1 and 2.

THEOREM 12. *Let p be prime, $p \geq 5$. If $p^h \parallel 2m$ and $p - 1$ does not divide $2m - 2$, then*

$$pB_{2m} \equiv S_{2m}(p - 1) \pmod{p^{h+3}} \quad (m > 1).$$

Proof. From Eq. (9) and Lemma 2, we have

$$pB_{2m} \equiv S_{2m}(p - 1) - mpS_{2m-1}(p - 1) + \frac{1}{6}p^2m(2m - 1)S_{2m-2}(p - 1) \pmod{p^{h+3}}.$$

Since $p - 1$ does not divide $2m - 2$, we know that $S_{2m-2}(p - 1) \equiv 0 \pmod{p}$. Thus

$$pB_{2m} \equiv S_{2m}(p - 1) - mpS_{2m-1}(p - 1) \pmod{p^{h+3}}. \quad (41)$$

Now from (2) we have, with m replaced by $2m - 1$ and n replaced by p :

$$\begin{aligned} S_{2m-1}(p - 1) &= - \sum_{k=1}^{2m-2} p^k \binom{2m-1}{k} B_k S_{2m-1-k}(p - 1) \\ &\equiv \frac{1}{2} p(2m - 1) S_{2m-2}(p - 1) \equiv 0 \pmod{p^2}. \end{aligned} \quad (42)$$

Returning to (41) and using (42), we have

$$pB_{2m} \equiv S_{2m-1}(p - 1) \pmod{p^{h+3}},$$

and the proof is complete.

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