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Approximation from the Exterior of a Multifunction and Its Applications in the "Sweeping Process"*

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We approximate from the exterior an upper semicontinuous multifunction $C(\cdot)$ from a metric space into the closed convex subsets of a normed space by means of globally Lipschitzean multifunctions; in particular, when $C(\cdot)$ is continuous, this approximation allows us to reduce the problem of the existence of solutions of the associated evolution equation to the case in which $C(\cdot)$ is Lipschitzean.

1. INTRODUCTION

It is known that an upper semicontinuous multifunction $t \to C(t)$ from a metric space I into the closed convex subsets of a Banach space E can be approximated from the exterior, under suitable conditions, by more "regular" multifunctions. In particular, it is possible to build a decreasing sequence of locally Lipschitzean multifunctions $C_n(\cdot)$ which converge pointwise, with respect to the Hausdorff distance, to $C(\cdot)$ (see, for instance, Haddad [10], De Blasi [6], El Arni [9], and Ionescu Tulcea [11]).

In the procedure usually adopted, every C_n comes from a piecewise constant multifunction, suitably "patched" by a partition of unity; in this paper we introduce a different method, which allows us to get globally Lipschitzean approximating multifunctions (Theorem 2.1) and is marked by a quite geometric approach. For the sake of simplicity, E is assumed to be a normed space, but the given result holds, more generally, in any locally convex metric space. As in the quoted literature, the proof relies on convexity assumptions, whose contribution is wholly exploited in Lemma 2.1.

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The last section of the work is devoted to the differential inclusion (3.1), the so-called "Sweeping Process." This problem was introduced by J. J. Moreau, who solved it in the case in which the multifunction $C(\cdot)$ has "finite retraction" [15, 16]. Soon afterward other situations were studied, and now the solution is known to exist also when $C(\cdot)$ is continuous, or when E has finite dimension and $C(\cdot)$ is lower semicontinuous (Castaing [1, 2, 3, 4], Monteiro Marques [12, 13], Tanaka [17], and Valadier [18, 20]).

Since the existence of solutions can be proved more easily when $C(\cdot)$ is Lipschitzean, it is useful to reduce to that case by means of a suitable approximation of $C(\cdot)$: In this scheme of things some authors studied the connections between some kinds of convergence for multifunctions and the convergence of the corresponding solutions (see, for instance, Castaing [4]). Furthermore, in [19], Valadier provided an approximation from the interior of a lower semicontinuous multifunction by means of Lipschitzean multifunctions, and in this way obtained [20] the existence of solutions in that case.

Here we deal with a continuous multifunction and give the following application to the Sweeping Process (Theorem 3.1): The solutions coming from the approximating multifunctions $C_n(\cdot)$ of Theorem 2.1 converge uniformly, and the limit function u is actually the solution of problem (3.1).

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2. Approximation from the Exterior of an Upper Semicontinuous Multifunction

In all that follows (I, d) will be a metric space and $(E, \|\cdot\|)$ will be a normed space. We denote by $\mathscr{C}(E)$ the family of all bounded closed convex non-empty subsets of E; B will always stand for the unitary ball of E. If $x \in E$ and $F \in \mathscr{C}(E)$ we call the *distance between x and F* the non-negative number

$$\operatorname{dist}(x, F) = \inf_{y \in F} \|y - x\|.$$

If $F, G \in \mathscr{C}(E)$, we define the *excess of F over G* as

$$e(F, G) = \sup_{x \in F} \operatorname{dist}(x, G);$$

finally, the *Hausdorff distance between F and G* is defined by

$$h(F, G) = \max(e(F, G), e(G, F)).$$

The following notation will also be used:

$$\|F\| = e(F, \{0\}) = \sup_{x \in F} \|x\|.$$

We say that a multifunction

$$C: I \to \mathscr{C}(E)$$

is (metrically) upper semicontinuous [7] at $t \in I$ if

$$\lim_{\tau \to t} e(C(\tau), C(t)) = 0,$$

or, in other words, if for every $\varepsilon > 0$ there exists a neighbourhood U of t such that

$$C(\tau) \subseteq C(t) + \varepsilon B, \qquad \forall \tau \in U. \tag{2.1}$$

Furthermore, $C(\cdot)$ will be said (metrically) continuous at $t \in I$ if

$$\lim_{\tau \to t} h(C(t), C(\tau)) = 0,$$

or, equivalently, if for every $\varepsilon > 0$ there exists a neighborhood U of t such that (2.1) holds, and

$$C(t) \subseteq C(\tau) + \varepsilon B, \qquad \forall \tau \in U. \tag{2.2}$$

We say that the multifunction $C(\cdot)$ is bounded if, for some $M > 0$,

$$\|C(t)\| \leqslant M, \qquad \forall t \in I. \tag{2.3}$$

THEOREM 2.1. *Let $C: I \to \mathscr{C}(E)$ be bounded and upper semicontinuous; then there exist constants $L_n \in]0, +\infty[$ and multifunctions $C_n: I \to \mathscr{C}(E)$ $(n = 1, 2, ...)$ which enjoy the following properties:*

$$\text{(a)} \quad C_n(t) \supseteq C_{n+1}(t) \supseteq C(t), \qquad \forall t \in I;$$

$$\text{(b)} \quad \lim_{n \to +\infty} h(C_n(t), C(t)) = 0, \qquad \forall t \in I;$$

$$\text{(c)} \quad h(C_n(t), C_n(t')) \leqslant L_n d(t, t'), \qquad \forall t, t' \in I.$$

We put forward the following result:

LEMMA 2.1. *Let $(\Gamma_i)_{i \in I}$ be a family in $\mathscr{C}(E)$; suppose that, for some $x_0 \in E$, $h \in \mathbf{R}^+$*

$$x_0 + hB \subseteq \Gamma_i, \qquad \forall i \in I,$$

and put

$$L=\inf_i \|\Gamma_i - x_0\| = \inf_i \sup_{x\in\Gamma_i} \|x-x_0\|.$$

Then, for every $\varepsilon>0$,

$$\bigcap_{i\in I}(\Gamma_i+\varepsilon B)\subseteq\left(\bigcap_{i\in I}\Gamma_i\right)+\frac{L}{h}\varepsilon B.$$

Proof. Let us put, for convenience, $\Gamma=\bigcap_{i\in I}\Gamma_i$, $\Gamma_\varepsilon=\bigcap_{i\in I}(\Gamma_i+\varepsilon B)$. We must show that, whenever $x\in\Gamma_\varepsilon$, there exists some $x'\in\Gamma$ with $\|x-x'\|\leqslant L\varepsilon/h$. To this aim we put

$$x'=x+\frac{\varepsilon}{h+\varepsilon}(x_0-x)$$

and deduce at once the inequality $\|x'-x\|=\varepsilon\,\|x_0-x\|/(\varepsilon+h)\leqslant \varepsilon(L+\varepsilon)/(h+\varepsilon)\leqslant L\varepsilon/h$. Now we only have to prove that $x'\in\Gamma$. To this end we consider the homothetic mapping $\rho\colon E\to E$ defined by

$$\rho(y)=x'-\frac{h}{\varepsilon}(y-x'),\qquad y\in E.$$

It is easy to check that

$$\rho(x+\varepsilon B)=x'-\frac{h}{\varepsilon}(x-x'+\varepsilon B)=x_0+hB.$$

Now, since $x\in\Gamma_\varepsilon$, for every $i\in I$ there is some $x_i\in\Gamma_i\cap(x+\varepsilon B)$. Then the point

$$z_i=\rho(x_i)=x'-\frac{h}{\varepsilon}(x_i-x')$$

lies in x_0+hB and, even more so, in Γ_i. On the other hand, the definition of z_i allows us to express x' in the following way:

$$x'=\frac{\varepsilon}{h+\varepsilon}z_i+\frac{h}{h+\varepsilon}x_i.$$

Then x' is a convex combination of points of Γ_i, so that $x'\in\Gamma_i$. Since the index i is arbitrary, $x'\in\Gamma$, as claimed. ∎

If $t\to C(t)$ is an upper semicontinuous multifunction and $F\in\mathscr{C}(E)$, it is easy to see that the function $t\to e(C(t),F)$ is upper semicontinuous in turn;

in particular, if $t \to C(t)$ fulfills the assumptions of the foregoing theorem and we put

$$r(t, \tau) = e(C(t), C(\tau)) \tag{2.4}$$

r turns out to be upper semicontinuous with respect to t, and bounded above by $2M$, with M given by (2.3). Hence it can be approximated from above, in a standard way, by functions r_n which are Lipschitzean with respect to t. For technical reasons we need the strict inequality $r_n > r$; then we define

$$r_n(t, \tau) = \frac{1}{n} + \sup_{t' \in I} (r(t', \tau) - nd(t, t')). \tag{2.5}$$

We sum up the main properties of the functions r_n we shall need later:

(a′) $r_n(t, \tau) \geqslant r_{n+1}(t, \tau), \qquad 1 + 2M \geqslant r_n(t, \tau) \geqslant 1/n + r(t, \tau), \qquad \forall t, \tau \in I$

(b′) $\lim_{n \to +\infty} r_n(t, \tau) = e(C(t), C(\tau)), \qquad \forall, \tau \in I$

(c′) $|r_n(t, \tau) - r_n(t', \tau)| \leqslant nd(t, t'), \qquad \forall t, t', \tau \in I.$

Furthermore, the definition of $r(t, \tau)$ implies immediately, whenever $\rho > r(t, \tau)$,

$$C(t) \subseteq C(\tau) + \rho B. \tag{2.6}$$

Now we are ready to prove the foregoing theorem.

Proof of Theorem 2.1. We put

$$C_n(t) = \mathrm{cl}\left(\bigcap_{\tau \in I} (C(\tau) + r_n(t, \tau) B) \right).$$

Of course, for every $n \in \mathbf{Z}^+$, $t \in I$, $C_n(t) \in \mathscr{C}(E)$; now we are going to show that $C_n(\cdot)$ fulfills conditions (a), (b), and (c). Since r_n is a non-increasing sequence, it is obvious that $C_n(t) \supseteq C_{n+1}(t)$. Moreover, (2.6) and the last inequality in (a′) entail the inclusion $C_n(t) \supseteq C(t)$.

In order to prove (b), fix $t \in I$, $\varepsilon > 0$ and choose $\bar{n} \in \mathbf{Z}^+$ such that $r_n(t, t) < \varepsilon$ whenever $n \geqslant \bar{n}$. Then $C_n(t) \subseteq \mathrm{cl}(C(t) + r_n(t, t) B) \subseteq C(t) + \varepsilon B$, so that $h(C_n(t), C(t)) \leqslant \varepsilon$.

Now we only have to prove (c). To this end put, for convenience,

$$\Gamma_n(t, \tau) = C(\tau) + r_n(t, \tau) B,$$

and fix $n \in \mathbf{Z}^+$, $t, t' \in T$. Since r_n enjoys property (c′), we deduce at once that

$$\Gamma_n(t', \tau) \subseteq C(\tau) + (r_n(t, \tau) + nd(t, t')) B = \Gamma_n(t, \tau) + nd(t, t') B.$$

Furthermore, the family $\{\Gamma_n(t, \tau); \tau \in I\}$ fulfills the assumptions of Lemma 2.1, where x_0 is any point of $C(t)$ and $h < 1/n$ (take, for instance, $h = 1/2n$). Indeed, thanks to (2.6) and the last inequality in (a′), we have

$$\begin{aligned}
x_0 + \frac{1}{2n} B &\subseteq C(t) + \frac{1}{2n} B \\
&\subseteq C(\tau) + \left(r(t, \tau) + \frac{1}{2n} \right) B + \frac{1}{2n} B \\
&\subseteq C(\tau) + r_n(t, \tau) B = \Gamma_n(t, \tau).
\end{aligned}$$

Now we can apply Lemma 2.1 with $\varepsilon = nd(t, t')$ and L controlled by the evaluation

$$\begin{aligned}
L &= \inf_{\tau \in I} \|\Gamma_n(t, \tau) - x_0\| \leqslant \|\Gamma_n(t, t) - x_0\| \\
&\leqslant r_n(t, t) + \|C(t) - x_0\| \leqslant 2M + 1 + \|C(t)\| + \|x_0\| \\
&\leqslant 4M + 1.
\end{aligned}$$

We get

$$\begin{aligned}
\bigcap_{\tau \in I} \Gamma_n(t', \tau) &\subseteq \bigcap_{\tau \in I} (\Gamma_n(t, \tau) + nd(t, t') B) \\
&\subseteq \left(\bigcap_{\tau \in I} \Gamma_n(t, \tau) \right) + 2n^2(4M + 1) d(t, t') B \\
&\subseteq C_n(t) + L_n d(t, t') B.
\end{aligned}$$

Hence $e(C_n(t'), C_n(t)) \leqslant L_n d(t, t')$, and in the same way we can get an analogous evaluation on $e(C_n(t), C_n(t'))$. Then

$$h(C_n(t), C_n(t')) \leqslant L_n d(t, t')$$

as claimed. ∎

Remark 2.2. For the sake of simplicity, Theorem 2.1 is stated under the assumption (2.3); some slight changes in the proof, however, show that the assertion still holds if the multifunction $C(\cdot)$ has only a "controlled growth"; that is it satisfies, for some $M > 0$, $t_0 \in I$ and every $t \in I$, the condition

$$\|C(t)\| \leqslant M(1 + d(t, t_0)).$$

3. Applications to the Sweeping Process

In this section E will be a separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$; $C: I \to \mathscr{C}(E)$ is now a continuous multifunction, and the metric space I is nothing but a closed interval $[0, T]$ of the real line. $\mathscr{B}(I)$ will be the Borel σ-field in I, and λ the Lebesgue measure on I. If $\nu: \mathscr{B}(I) \to [0, +\infty[$ is a measure, $L^1(\nu)$ will denote the space of all strongly ν-measurable functions $u: I \to E$ such that $\int_I \|u\| \, d\nu < +\infty$. A countably additive set fuction $\mu: \mathscr{B}(I) \to E$ will be simply called a *vector measure* on I, and $M(I)$ will stand for the space of such measures; if $\mu \in M(I)$, we denote by $|\mu|: \mathscr{B}(I) \to [0, +\infty[$ its *variation measure*, and by $d\mu/d|\mu|$ the Radon–Nikodym derivative of μ with respect to $|\mu|$ [8, Theorem 6, p. 64]. We denote by BV the space of all functions $u: I \to E$ with bounded variation. As is known, if $u \in BV$, the limits

$$u^-(t) = \lim_{\tau \to t^-} u(\tau) \qquad (t > 0), \qquad u^+(t) = \lim_{\tau \to t^+} u(\tau) \qquad (t < T)$$

exist, and the mapping $]s, t] \to u^+(t) - u^+(s)$ can be extended to a unique measure $\mu: \mathscr{B}(I) \to E$ which is called the *derivative measure* of u, and simply denoted by du, then $|du|$ will stand for the total variation of du, and the Radon–Nikodym derivative of du with respect to $|du|$ will be denoted, improperly, by $du/|du|$.

Definition 3.1. If $C \in \mathscr{C}(E)$ and $u \in C$ we define the *exterior normal cone* to C at u as

$$N(u; C) = \{p \in E \mid \langle p, x - u \rangle \leqslant 0, x \in C\};$$

if $u \notin C$, we put $N(u; C) = \varnothing$.

Definition 3.2. If $u \in BV$ and $a \in C(0)$, we say that u is a solution of the sweeping process $(\mathscr{P})$ associated to the multifunction $C(\cdot)$ issuing from $(0, a)$ if:

$$\begin{aligned} &\text{(i)} \quad u(0) = a, \\ &\text{(ii)} \quad u(t) \in C(t), \qquad \forall t \in I \qquad\qquad (3.1) \\ &\text{(iii)} \quad -\frac{du}{|du|}(t) \in N(u(t); C(t)) \qquad |du| \text{ — a.e. in } I. \end{aligned}$$

Remark 3.1. If the solution u exists, and is required to be right-continuous, then it is necessarily unique (see, for instance, [14, 18]). Furthermore, if u is absolutely continuous, condition (iii) in problem $(\mathscr{P})$ can more simply be stated as

$$-u'(t) \in N(u(t); C(t)) \qquad \lambda\text{—a.e. in } I.$$

Remark 3.2. If u is continuous, condition (iii) holds if and only if, on every half-closed interval $J =]s, t]$, and for every continuous selection ϕ of $C(\cdot)$, the following inequality is satisfied (see [12 or 18, Prop. 6]):

$$\tfrac{1}{2}(\|u(t)\|^2 - \|u(s)\|^2) \leqslant \int_J \langle \phi, du \rangle. \tag{3.2}$$

The following result provides, in particular, an application of Theorem 2.1 to problem ($\mathscr{P}$), and can be related to Theorem 2.3 of [4], which deals with a more general case. Here, the particular approximation given by Theorem 2.1 allows us to prove something more, that is the convergence of the whole sequence of the approximating solutions, not only of a suitable subsequence.

THEOREM 3.1. *Let $C: I \to \mathscr{C}(E)$ be a continuous multifunction, and suppose that, for every $t \in I$, $C(t)$ has a non-empty interior. Let $C_n: I \to \mathscr{C}(E)$ satisfy conditions* (a), (b) *and* (c) *of Theorem* 2.1; *given $a \in C(0)$, for every $n \in \mathbf{Z}^+$ call u_n the solution of the sweeping process ($\mathscr{P}_n$) associated to $C_n(\cdot)$ issuing from* $(0, a)$. *Then the sequence $(u_n)_n$ converges uniformly on I to a continuous BV-function u which is the solution of problem* ($\mathscr{P}$).

Proof. We divide the proof in three steps.

Step 1: The sequence $(u_n)_n$ is bounded in variation.

To this aim we can proceed as in [13]: Since for every $t \in [0, T]$ the interior of $C(t)$ is not empty and $C(\cdot)$ is continuous and convex-valued, we can divide I in a finite number of intervals

$$J_1 = [t_0, t_1], \qquad J_2 = [t_1, t_2], ..., \qquad J_s = [t_{s-1}, t_s],$$

with $t_0 = 0 < t_1 < \cdots < t_s = T$, in such a way that for $k = 1, ..., s$ the set $\bigcap \{C(t); t \in J_k\}$ contains a ball $x_k + r_k B$. Now, by virtue of an evaluation given by Valadier (see, for instance, [18, Lemma 1]) on each J_k the variation of u_n does not exceed the value $M_k = \|u_n(t_{k-1}) - x_k\|^2/2r_k$; hence an easy inductive argument on k shows the existence of a constant M such that

$$\mathscr{V}(u_n; I) = \int_0^T \|u_n'(\tau)\| \, d\tau \leqslant M. \tag{3.3}$$

Step 2: The sequence $(u_n)_n$ converges uniformly to a BV-function.

Fix $t \in I$ and $m, n \in \mathbf{N}$, with $m < n$: Since u_m and u_n are Lipschitzean, and $u_m(0) = u_n(0) = a$, we get

$$\int_0^t \langle u_n(\tau) - u_m(\tau), u_n'(\tau) - u_m'(\tau) \rangle \, d\tau = \tfrac{1}{2} \|u_n(t) - u_m(t)\|^2.$$

Now, let $u_{m,n}(\tau)$ be the projection of $u_m(\tau)$ on $C_n(\tau)$; then the left-hand side of the previous equality can be written as

$$\int_0^t \langle u_n(\tau)-u_{m,n}(\tau), u_n'(\tau)\rangle\, d\tau + \int_0^t \langle u_{m,n}(\tau)-u_m(\tau), u_n'(\tau)\rangle\, d\tau$$
$$+\int_0^t \langle u_m(\tau)-u_n(\tau), u_m'(\tau)\rangle\, d\tau.$$

Now, u_n is the solution of problem $(\mathscr{P}_n)$, and, by definition, $u_{m,n}(\tau)\in C_n(\tau)$; hence the first term of the foregoing sum is non-positive. On the other hand, u_m is the solution of problem $(\mathscr{P}_m)$, and from the inequality $n>m$ we deduce that $u_n(\tau)\in C_m(\tau)$; then the third term does not exceed zero as well, and we can get the following chain of inequalities:

$$\begin{aligned}\tfrac{1}{2}\,\|u_n(t)-u_m(t)\|^2 &\leqslant \int_0^t \langle u_{m,n}(\tau)-u_m(\tau), u_n'(\tau)\rangle\, d\tau\\ &\leqslant \mathscr{V}(u_n; I)\max\{\|u_{m,n}(\tau)-u_m(\tau)\|; 0\leqslant\tau\leqslant t\}\\ &\leqslant M\max\{h(C_m(\tau), C(\tau)); 0\leqslant\tau\leqslant T\}.\end{aligned}$$

Now, the functions $\tau\to h(C_k(\tau), C(\tau))$ are continuous, and converge monotonically to zero as $k\to+\infty$. By virtue of Dini's Theorem they converge uniformly on I: we can conclude that

$$\lim_{m,n\to+\infty}\max\{\|u_n(\tau)-u_m(\tau)\|; 0\leqslant\tau\leqslant T\}=0.$$

Hence $(u_n)_n$ is a Cauchy sequence with respect to the norm of uniform convergence, and therefore converges to a continuous function $u\colon I\to E$. Furthermore, the variation of a function on an interval is lower semicontinuous with respect to pointwise convergence, so that $u\in BV$ because of the inequality

$$\int_I |du|\leqslant \liminf_{n\to+\infty}\int_I \|u_n'\|\, d\tau\leqslant M.$$

Step 3: u is the solution of problem $(\mathscr{P})$.

Condition (i) is obviously satisfied; in order to prove (ii) it is enough to recall that, for every $t\in I$, $C(t)$ is closed, and

$$\operatorname{dist}(u(t), C(t)) = \lim_{n\to+\infty}\operatorname{dist}(u_n(t), C(t))\leqslant \lim_{n\to+\infty} h(C_n(t), C(t))=0.$$

Now, by virtue of Remark 3.2, we only have to prove (3.2). Since $(u_n)_n$ is bounded in variation and converges uniformly to u, it can be shown that, on every interval $J=[s, t]$ and for every continuous function $\phi: I \to E$,

$$\lim_{n \to +\infty} \int_J \langle \phi, u_n' \rangle \, d\tau = \int_J \langle \phi, du \rangle. \tag{3.4}$$

Indeed, (3.4) is certainly true for any step function ϕ, but (3.3) allows us to extend it to the closure of such functions with respect to uniform convergence, then, in particular, to all continuous functions.

Now, let ϕ be a continuous selection of $C(\cdot)$; then, for every $n \in \mathbf{Z}^+$, ϕ is a selection of $C_n(\cdot)$, and since u_n is a solution of $(\mathscr{P}_n)$, it is

$$\tfrac{1}{2}(\|u_n(t)\|^2 - \|u_n(s)\|^2) \leqslant \int_J \langle \phi, u_n' \rangle \, d\tau.$$

Now, let $n \to +\infty$; thanks to (3.4), (3.2) follows at once, and the proof is complete. ∎

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