

Table 2
Dp-brane charges from K-theory, classified by $K(\mathbb{S}^{9-p})$, $KO(\mathbb{S}^{9-p})$ and $KSp(\mathbb{S}^{9-p})$ [12]. A \mathbb{Z}_2 charged Dp-brane with p even or p odd represents a non-BPS Dp-brane or a bound state of a Dp and an anti-Dp brane, respectively [13].

	D(-1)	D0	D1	D2	D3	D4	D5	D6	D7	D8	D9
type IIB	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
O9 ⁻ (type I)	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
O9 ⁺	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}

Table 3
External G (left-most column) and internal \tilde{G} gauge groups for each spatial dimension d and symmetry class; U, O, Sp, represents U(1), O(1) = \mathbb{Z}_2 , and Sp(1) = SU(2), respectively.

G	class\ d	0	1	2	3	4	5	6	7
U	A	U	-	U	-	U	-	U	-
U	AIII	-	U	-	U	-	U	-	U
O	AI	O	-	-	-	Sp	-	U	O
O	BDI	O	O	-	-	-	Sp	-	U
O	D	U	O	O	-	-	-	Sp	-
O	DIII	-	U	O	O	-	-	-	Sp
Sp	AII	Sp	-	U	O	O	-	-	-
Sp	CII	-	Sp	-	U	O	O	-	-
Sp	C	-	-	Sp	-	U	O	O	-
Sp	CI	-	-	-	Sp	-	U	O	O

terms of two D-branes (Dp- and Dq-branes), possibly with an orientifold plane (O-plane). Besides the appealing mathematical similarity between TIs/TSCs and D-branes, realizing TIs/TSCs in string theory has a number of merits, since string theory and D-branes are believed to be rich enough to reproduce many types of field theories and interactions in a fully consistent and UV complete way. Indeed, our string theory realizations of TIs/TSCs give rise to massive fermion spectra, which are in one-to-one correspondence with the ten-fold classification of TIs/TSCs, and come quite naturally with gauge interactions. These systems, while interacting, are all topologically stable, as protected by the K-theory charge of D-branes. We thus make a first step toward understanding interacting TIs/TSCs [14]. We are also separately preparing a regular paper with more details and expanded results in [15].

In Dp-Dq systems, massive fermions arise as an open string excitation between the two D-branes. The distance between the branes corresponds to the mass of fermions. Open strings ending on the same D-branes give rise to a gauge field, which we call A_μ (Dp) and \tilde{A}_μ (Dq) with gauge group G and \tilde{G} , respectively, and couple to the fermions. These two gauge fields play different roles in our construction: The gauge field A_μ “measures” K-theory charge of the Dq-brane, and in that sense it can be interpreted as an “external” gauge field. In this picture, the Dq-brane charge is identified with the topological (K-theory) charge of TIs/TSCs. On the other hand, \tilde{A}_μ is an internal degree of freedom on the Dp-brane. For example, in the integer/fractional QHE, the external gauge field is the electromagnetic U(1) gauge field, which measures the Hall conductivity, while the internal gauge field is the Chern–Simons (CS) gauge field describing the dynamics of the droplet itself.

The massive fermions can be integrated out, yielding the description of the topological phase in terms of the gauge fields. The resulting effective field theory comes with terms of topological nature, such as the CS or the θ -terms. In our string theory setup, they can be read off from the Wess–Zumino (WZ) action of the D-branes, by taking one of the D-branes as a background for the other. One can view these gauge-interacting TIs/TSCs from Dp-Dq systems as an analogue of the projective (parton) construction of the (fractional) QHE [16]. Our string theory realization of TIs/TSCs sheds light on extending the projective construction of the QHE to more generic TIs/TSCs; it tells us what type of gauge field is “natu-

Table 4
Dp-Dq systems for class A and AIII where p = 5 and q = 3, 5, 7 for A, and p = 4 and q = 4, 6 for AIII. The D-branes extend in the μ -th direction denoted by “x” in the ten-dimensional space-time ($\mu = 0, \dots, 9$); d + 1 is the number of common directions of Dp- and Dq-branes; The last column shows the Dq-brane charge, together with fermion spectra per Dq-brane, where “ N_f Mj” or “ N_f Di” represents N_f flavor of Majorana and Dirac spinor, respectively.

	0	1	2	3	4	5	6	7	8	9	d	A
D5	x	x	x	x	x	x						
D3	x						x	x	x		0	\mathbb{Z} (2 Mj)
D5	x	x	x				x	x	x		2	\mathbb{Z} (2 Mj)
D7	x	x	x	x	x		x	x	x		4	\mathbb{Z} (1 Di)

	0	1	2	3	4	5	6	7	8	9	d	AIII
D4	x		x	x	x	x						
D4	x		x				x	x	x		1	\mathbb{Z} (2 Mj)
D6	x		x	x	x		x	x	x		3	\mathbb{Z} (2 Mj)

ral” to couple with fermions in topological phases, and guarantees the topological stability of the system.

2. Complex case

Let us start with the most familiar example of the QHE (class A in $d = 2$). We fix the value of p to be p = 5 by T-duality, and consider a D5-brane in type IIB string theory which extends in the $x^{0,1,2,3,4,5}$ directions in ten-dimensional space-time. We take the Dq-brane with q = 5 in the $x^{0,1,2,6,7,8}$ directions (Table 4). By T-duality, this setup is equivalent to the D3-D7 system studied in [19–21]. Since the number of Neumann–Dirichlet (ND) directions is six, open string excitations between the D5-branes give rise to two Majorana fermions (Mj) [= one two-component Dirac fermion (Di), ψ] and no bosons. The distance between the D-branes in x^9 direction (Δx^9) is proportional to the mass m of the fermions. The low-energy effective theory is schematically summarized by the effective Lagrangian in the (2 + 1)-dimensional common direction of the two D5-branes,

$$\mathcal{L} = \bar{\psi} [\gamma^\mu (i\partial_\mu - A_\mu - \tilde{A}_\mu) - m] \psi + \dots \tag{1}$$

Integrating the massive fermions yields the CS terms $\frac{k}{4\pi} \int A \wedge dA$ and $\frac{k}{4\pi} \int \tilde{A} \wedge d\tilde{A}$ with $k = \pm 1/2$ (parity anomaly). The Hall conductivity is read off from the CS term for A_μ as $\sigma_{xy} = k/(2\pi)$. Alternatively, the presence of the CS terms can be read off from the WZ action of either one of D5-branes, e.g.,

$$S_{D5}^{WZ} \propto \int_{D5} F \wedge F \wedge C_2 = \int_{D5} A \wedge F \wedge (dC)_3 \tag{2}$$

for the external gauge field A_μ , where C_2 is the RR 2-form from the Dq-brane. When we change the sign of m by passing the Dq-brane through the Dp-brane, the value of k jumps from $\pm 1/2$ to $\mp 1/2$. If we instead put N_f Dq-branes, we have N_f copies of massive Dirac fermions ψ_i which couple with U(N_f) gauge fields A_μ and \tilde{A}_μ (when all Dq are coincident).

This brane construction can be extended to other even space dimensions $d = 2n$ by considering D5-Dq systems with $q = 5, 7$ (Table 4). This setup gives rise to the fermion spectrum consisting

Table 5

Dp–Dq systems for eight “real” symmetry classes, where $p = 5$ for classes C, D, CI, DIII, and $p = 4$ for classes AII, AI, CII, BDI. For classes AII, AI, CII, BDI, the O8-plane extends except x^5 .

	0	1	2	3	4	5	6	7	8	9	d	C (O9 ⁻)	D (O9 ⁺)
D5	×	×	×	×	×	×							
D3	×						×	×	×		0	0	\mathbb{Z}_2 (2 Mj)
D4	×	×					×	×	×		1	0	\mathbb{Z}_2 (1 Mj)
D5	×	×	×				×	×	×		2	\mathbb{Z} (4 Mj)	\mathbb{Z} (1 Mj)
D6	×	×	×	×			×	×	×		3	0	0
D7	×	×	×	×	×		×	×	×		4	\mathbb{Z}_2 (2 Di)	0
	0	1	2	3	4	5	6	7	8	9	d	CI (O9 ⁻)	DIII (O9 ⁺)
D5	×	×	×	×	×	×							
D2	×						×	×			0	0	0
D3	×	×					×	×			1	0	\mathbb{Z}_2 (2 Mj)
D4	×	×	×				×	×			2	0	\mathbb{Z}_2 (2 Mj)
D5	×	×	×	×			×	×			3	\mathbb{Z} (4 Mj)	\mathbb{Z} (1 Mj)
	0	1	2	3	4	5	6	7	8	9	d	AII (O8 ⁻)	AI (O8 ⁺)
D4	×	×	×	×	×								
D4	×						×	×	×	×	0	\mathbb{Z} (4 Mj)	\mathbb{Z} (1 Mj)
D5	×	×					×	×	×	×	1	0	0
D6	×	×	×				×	×	×	×	2	\mathbb{Z}_2 (4 Mj)	0
D7	×	×	×	×			×	×	×	×	3	\mathbb{Z}_2 (2 Mj)	0
D8	×	×	×	×	×		×	×	×	×	4	\mathbb{Z} (1 Di)	\mathbb{Z} (1 Di)
	0	1	2	3	4	5	6	7	8	9	d	CII (O8 ⁻)	BDI (O8 ⁺)
D4	×	×	×	×	×								
D3	×						×	×	×		0	0	\mathbb{Z}_2 (2 Mj)
D4	×	×					×	×	×		1	\mathbb{Z} (4 Mj)	\mathbb{Z} (1 Mj)
D5	×	×	×				×	×	×		2	0	0
D6	×	×	×	×			×	×	×		3	\mathbb{Z}_2 (4 Mj)	0
D7	×	×	×	×	×		×	×	×		4	\mathbb{Z}_2 (2 Di)	0

of one Dirac fermion per Dq-brane, and the CS terms of the form $\propto k \int A \wedge F^n$. All of these brane configurations are identified with class A TIs, which are characterized by the absence of any discrete symmetries.

Now let us turn to AIII, which is characterized by the presence of SLS. We argue that SLS is equivalent to an invariance of the brane configurations under inversion of a coordinate in the Dirichlet–Dirichlet (DD) directions of open strings between the D5- and Dq-branes. One way to realize this is to assume two DD directions, say, x^1 and x^9 , and impose $x^1 = 0$. Indeed such a configuration is obtained by taking T-dual of class A configurations (Table 4). Again, the fermion spectrum consists of two Majorana fermions (= one Dirac fermion) for all dimensions, and the mass of the fermion is, again, proportional to Δx^9 .

In our setup in general, the number of Neumann–Neumann (NN) directions #NN is equal to the space–time dimensions $d + 1$ of TIs/TSCs. On the other hand, #DD represents the number of possible mass deformations: it is one if there is no SLS (class A), while it is two in the presence of SLS. Finally, #ND is determined by #NN and #DD via the relation $\#ND = 10 - \#NN - \#DD$. Note also that the T-dual in any ND directions does not change #NN, #DD and #ND and thus is a redundant operation.

3. Real case

To realize eight “real” symmetry classes in string theory, we need to implement TRS and PHS. To preserve PHS, we require the internal gauge field \tilde{A}_μ is not independent of its complex conjugate. This is the same as the orientation (Ω) projection in string theory.

To realize TRS, we recall that it can be viewed as a product of PHS and SLS [9]. As SLS can be imposed as a parity symmetry

in string theory, we can interpret TRS as the orientifold projection.

Let us start with class C and D, which are characterized by the presence of PHS but lack of TRS. We take an Ω projection of the class A setup. Note that there are two types of Ω projections, represented by two types of O9-plane, i.e., O9⁻ (orthogonal) and O9⁺ (symplectic). While only O9⁻ leads to supersymmetric type I string theory, here we consider both because the T-dual of O9⁺ is equivalent to O9⁺ planes ($p \leq 8$) in type II string theory. We realize class C and class D TSCs by considering a D5-brane which extends in the $x^{0,1,2,3,4,5}$ directions, in the presence of an O9-plane. As before, we put a Dq-brane with $q = d + 3$ so that there are $d + 1$ common directions (Table 5). For class AII and AI, characterized by the presence of TRS but lack of PHS, we take the orientifold projection which leads to an O8-plane in type IIA theory [22]. By choosing $(p, q) = (4, d + 4)$, we obtain the brane configuration given in the third table in Table 5. Though the D-brane charges with an O p -plane for $p \leq 8$ are classified by KR-theory, the same result can be obtained from KO-theory via T-duality for our purpose [22]. Finally, the remaining four classes, CII, BDI, CI and DIII can be obtained by taking DD directions to be two instead of one. SLS is imposed by requiring $x^9 = 0$ for all of these classes. These are O8- or O9-projection of the class AIII setup (Table 5).

For these D-brane configurations, we chose a Dp-brane to be a standard one with the integer K-theory charge so that it is regarded as the background (bulk) material itself. Then, we find that the K-theory charge of the Dq-branes (shown in the last two columns in Table 5) agrees precisely with the corresponding classification of TIs/TSCs (Table 1). Moreover, the fermion content of these string theory realizations (denoted in the last two columns in Table 5 either by “ N_f Mj” or “ N_f Di” with N_f an integer) can be compared with the Dirac representative of TIs/TSCs constructed

Table 6

D_p – D_q systems in the presence of an O-plane; “ $d \geq 2$ ” means the constraint of possible spatial dimensions d in the brane systems due to the existence of open string tachyons. “Chiral” denotes the existence of chiral fermions and is interpreted as boundary (edge) states. The horizontal direction is shifted by a T-duality in the NN direction.

#DD	(09 ⁻ , 09 ⁺)	(08 ⁻ , 08 ⁺)	(07 ⁻ , 07 ⁺)	(06 ⁻ , 06 ⁺)
0	Chiral			
1	(C, D)	(AII, AI)		
2	(CI, DIII)	(CII, BDI)	(DIII, CI)	
3	$d \leq 2$	$d \leq 2$	$d \leq 2$	$d \leq 2$
4	$d \leq 1$	$d \leq 1$	$d \leq 1$	$d \leq 1$

in Ref. [9]. Indeed, they agree completely. It is also interesting to note that, for the Dirac representative of TIs/TSCs, the *momentum* dependence of the projection operator, which is one of key ingredients in the classification of TIs/TSCs [9], looks quite similar to the spatial profile of the tachyon field in string theory in *real* space [12].

We now describe the field theory content of the D_p – D_q systems charges in more details. First, for the \mathbb{Z} TIs/TSCs on the diagonal in Table 1 (“primary series”), the internal gauge group is $O(1) = \mathbb{Z}_2$. In particular, for class D in $d = 2$, our string theory realization corresponds to (a proper supersymmetric generalization of) the honeycomb lattice Kitaev model in the weak pairing (non-Abelian) phase [17]. Similarly, for class DIII in $d = 3$, it corresponds to an interacting bosonic model on the diamond lattice [18].

For \mathbb{Z}_2 TIs/TSCs of the first descendant of the primary series, i.e., BDI ($d = 0$), D ($d = 1$), DIII ($d = 2$), and AII ($d = 3$), the internal gauge group is $O(1) = \mathbb{Z}_2$ (Table 3).

For \mathbb{Z}_2 TIs/TSCs of the second descendant of the primary series, i.e., D ($d = 0$), DIII ($d = 1$), AII ($d = 2$), and CII ($d = 3$), the internal gauge group is $U(1)$. For D and DIII, the $U(1)$ unitary gauge field couples to two real fermions which can be combined into a single complex field. For AII and CII, the fermion spectrum consists of 4 Mj. In this case, the external gauge field is $Sp(1) = SU(2)$, which couples to a doublet, $\psi_{\uparrow/\downarrow}$. Each has 2 Mj (= 1 Di) degrees of freedom and couples to a $U(1)$ internal gauge field as follows:

$$\mathcal{L} = \bar{\psi} [\gamma^\mu (i\partial_\mu - A_\mu - \tilde{A}_\mu) - mM] \psi + \dots, \quad (3)$$

where $\psi = (\psi_\uparrow, \psi_\downarrow)^T$, and M is a diagonal mass matrix whose eigenvalue is ± 1 for $\psi_{\uparrow/\downarrow}$, respectively.

Finally, for TIs/TSCs labeled by $2\mathbb{Z}$, i.e., AII ($d = 0$), CII ($d = 1$), C ($d = 2$), and CI ($d = 3$), the gauge group is $G \times \tilde{G} = SU(2) \times SU(2)$, with 4 Mj fermions in bi-fundamental.

Even though the ten-dimensional string theories in the bulk are supersymmetric, our brane setups are not in general. When $\#ND = 4$ with \mathbb{Z} charge, they exceptionally preserve a quarter of supersymmetries. In general, when $\#ND = 4$, there exist massive bosons in addition to the massive fermions. This happens when $d = 4$ for A, C, AI, AII, and when $d = 3$ for AIII, CII, CI and DIII. Since $\#NN > 4$ for all the other branes systems, we only have fermions from open strings between the D_p - and D_q -branes and there are no tachyons.

We can take the T-duality further in NN directions. However, this lead to theories with different properties than TIs/TSCs (Table 6). Note that we have succeeded to realize all TIs/TSCs in space dimensions $d \leq 4$.

4. Boundary of TIs/TSCs

A defining property of TIs/TSCs is the appearance of stable gapless degrees of freedom, when the system is terminated by a $(d - 1)$ -dimensional boundary. In our brane construction, the sample boundary can be constructed by bending the D_p -brane to-

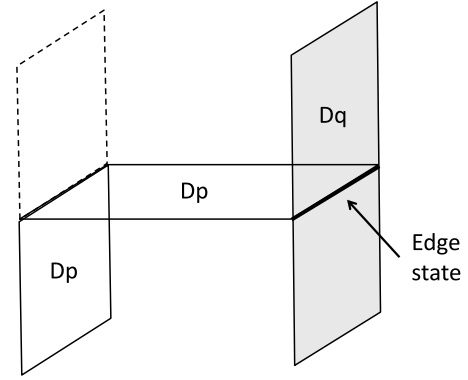


Fig. 1. A boundary of a TI/TSC as a brane intersection.

ward the D_q -brane, to create an intersection between these branes (Fig. 1). This leads to a position-dependent fermion mass, which changes its sign at the intersection. This increases $\#ND$ by two and the correct number of massless fermions appears at the intersection.

5. Conclusions

The main conclusion of this Letter is that there is a one-to-one correspondence between the tachyon-free D_p – D_q systems and the ten classes of topological insulators in $d \leq 4$ dimensions. Indeed, we explicitly constructed the corresponding ten classes of D_p – D_q brane configurations in superstring theory. Two out of ten are realized in type II string theory without orientifolds, while the other eight require orientifolds. The K-theory charges of the D_q -branes agree with that of the topological insulators.

One may wonder if there are other tachyon-free D_p – D_q systems which have not been considered in this Letter. However, it turns out that their low-energy theories just correspond to multiple copies of the ten classes of topological insulators (for details refer to [15]).

Since a topological insulator has a mass gap in the bulk, the distance between the D_p and D_q is taken to be non-vanishing in its corresponding D-brane system. When we discuss the boundary of a TI/TSC, however, the D_p -brane is bent toward the D_q -brane and thus D_p and D_q are intersecting with each other. Therefore massless fermions appear at the intersection and they are identified with the boundary modes (or edge modes).

We can also consider holographic descriptions of these systems by extending the constructions in [20,21] in principle, though it is not possible to take the large- N limit of \mathbb{Z}_2 charged D-branes.

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References

- [1] C.L. Kane, E.J. Mele, Phys. Rev. Lett. 95 (2005) 146802;

- C.L. Kane, E.J. Mele, *Phys. Rev. Lett.* 95 (2005) 226801.
- [2] R. Roy, *Phys. Rev. B* 79 (2009) 195321;
R. Roy, *Phys. Rev. B* 79 (2009) 195322.
- [3] J.E. Moore, L. Balents, *Phys. Rev. B* 75 (2007) 121306(R).
- [4] B.A. Bernevig, T.L. Hughes, S.-C. Zhang, *Science* 314 (2006) 1757.
- [5] M. König, et al., *Science* 318 (2007) 766.
- [6] L. Fu, C.L. Kane, E.J. Mele, *Phys. Rev. Lett.* 98 (2007) 106803.
- [7] X.-L. Qi, T. Hughes, S.-C. Zhang, *Phys. Rev. B* 78 (2008) 195424.
- [8] D. Hsieh, et al., *Nature* 452 (2008) 970.
- [9] A. Schnyder, S. Ryu, A. Furusaki, A. Ludwig, *Phys. Rev. B* 78 (2008) 195125;
A. Schnyder, S. Ryu, A. Furusaki, A. Ludwig, *AIP Conf. Proc.* 1134 (2009) 10;
S. Ryu, A. Schnyder, A. Furusaki, A. Ludwig, arXiv:0912.2157.
- [10] A. Kitaev, *AIP Conf. Proc.* 1134 (2009) 22.
- [11] The importance of K-theory in the physics of Fermi surfaces has been first pointed out in P. Horava, *Phys. Rev. Lett.* 95 (2005) 016405.
- [12] E. Witten, *JHEP* 9812 (1998) 019;
P. Horava, *Adv. Theor. Math. Phys.* 2 (1999) 1373.
- [13] A. Sen, *Int. J. Mod. Phys. A* 20 (2005) 5513.
- [14] For a different approach to TIs from string theory, see, for example, A. Karch, *Phys. Rev. Lett.* 103 (2009) 171601.
- [15] S. Ryu, T. Takayanagi, arXiv:1007.4234.
- [16] X.-G. Wen, *Mod. Phys. Lett. B* 5 (1991) 39;
X.-G. Wen, *Phys. Rev. B* 60 (1999) 8827.
- [17] Alexei Kitaev, *Ann. Phys.* 321 (2006) 2.
- [18] Ryu Shinsei, *Phys. Rev. B* 79 (2009) 075124.
- [19] S.J. Rey, *Prog. Theor. Phys.* 177 (2009) 128.
- [20] J.L. Davis, P. Kraus, A. Shah, *JHEP* 0811 (2008) 020.
- [21] M. Fujita, W. Li, S. Ryu, T. Takayanagi, *JHEP* 0906 (2009) 066.
- [22] For a KR theory analysis in similar brane systems, see T. Imoto, T. Sakai, S. Sugimoto, arXiv:0907.2968, and references therein.