Functional codes arising from quadric intersections with Hermitian varieties

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1. Introduction

We study the functional code $C_2(X)$ in $\text{PG}(N,q^2)$, where $X$ is a non-singular Hermitian variety $H(N,q^2)$. Let $X = \{P_1, \ldots, P_n\}$, where we normalize the coordinates of these points with respect to the leftmost non-zero coordinate. Let $\mathcal{F}$ be the set of all homogeneous quadratic polynomials $f(X_0, \ldots, X_N)$ defined by $N+1$ variables with coefficients in $\mathbb{F}_{q^2}$. The functional code $C_2(X)$ [10] is the linear code

$$C_2(X) = \left\{ (f(P_1), \ldots, f(P_n)) \mid f \in \mathcal{F} \cup \{0\} \right\}.$$ 

This linear code has length $n = |X|$ and dimension $k = \binom{N+2}{2}$ over $\mathbb{F}_{q^2}$. The third fundamental parameter of this linear code is its minimum distance $d$. Since the code is linear, this minimum distance corresponds to the minimum weight of the code. The small weight codewords, i.e., the codewords

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having the minimum weight or a weight close to the minimum weight, arise from the quadrics having the (almost) largest intersections with X.

Sørensen [11] conjectured that the maximum size for the intersection of a quadric Q with the Hermitian variety \( H(3, q^2) \) in \( \text{PG}(3, q^2) \) is equal to \( 2q^3 + 2q^2 - q + 1 \). The correctness of this conjecture was proven by Edoukou in [3,4].

More precisely, Edoukou not only proved that the maximum size for the intersection of a quadric Q with the Hermitian variety \( H(3, q^2) \) in \( \text{PG}(3, q^2) \) is equal to \( 2q^3 + 2q^2 - q + 1 \); he also proved that the second largest intersection size of a quadric Q with the Hermitian variety \( H(3, q^2) \) in \( \text{PG}(3, q^2) \) is at most \( 2q^3 + q^2 + 1 \).

Regarding the largest intersection sizes of a quadric Q with the Hermitian variety \( H(4, q^2) \) in \( \text{PG}(4, q^2) \), Edoukou [5] determined the five largest intersection sizes, leading to the 5 smallest weights for the code \( C_2(X), X = H(4, q^2) \).

In [5, Conjecture 2, p. 145], he also stated that the five smallest weights for the code \( C_2(X), X = H(N, q^2) \), arise from the intersections of X with the quadrics which are the union of two distinct hyperplanes.

We determine the 5 smallest weights of \( C_2(X), X = H(N, q^2), N < O(q^2) \), via geometrical arguments, and prove the validity of the conjecture of Edoukou for \( N < O(q^2) \). These 5 smallest weights will be the small weights of the code \( C_2(X), X = H(N, q^2) \), on which we will concentrate.

First of all, we will investigate the different intersections of quadrics Q in \( \text{PG}(4, q^2) \) with \( H(4, q^2) \); leading to a lower bound on the intersection size guaranteeing that any quadric having more than this number of points in common with \( H(4, q^2) \) must be the union of two hyperplanes. We use this result to find a bound on the intersection sizes of absolutely irreducible quadrics with the non-singular Hermitian variety \( H(N, q^2) \). Here this lower bound on the intersection size guarantees that Q is the union of 2 hyperplanes. Using this bound, we prove that the small weight codewords correspond to quadrics which are the union of 2 hyperplanes. There are several possibilities for the intersection of such a quadric with a non-singular Hermitian variety X. So we can construct tables with the 5 smallest weights of the functional code \( C_2(X), X \) a non-singular Hermitian variety in \( \text{PG}(N, q^2), N < O(q^2) \).

The results of this article continue the research on the small weight codewords of functional codes performed in [6,7]. In [6], we determined the smallest weights of the non-zero codewords of the functional codes \( C_2(Q) \), which are defined by the intersections of all quadrics with a non-singular quadric Q in \( \text{PG}(N, q) \), and in [7], we determined the smallest weights of the non-zero codewords of the functional codes \( C_{\text{herm}}(X) \), which are defined by the intersections of all Hermitian varieties with a non-singular Hermitian variety in \( \text{PG}(N, q^2) \). In these cases, the smallest weight codewords arise in [6] from the intersections of Q with the quadrics which are the union of two hyperplanes, and in [7] from the intersections of X with the Hermitian varieties which are the union of \( q + 1 \) hyperplanes through a common \((N - 2)\)-dimensional space of \( \text{PG}(N, q^2) \).

In the article [6], the crucial element was the fact that the intersection V of two quadrics Q and Q’ lies in all the \( q + 1 \) quadrics \( \lambda Q + \mu Q’ \), \( (\lambda, \mu) \in \mathbb{F}_q^2 \setminus \{ (0, 0) \} \), of the pencil of quadrics defined by Q and Q’ and similarly for the second article [7], the crucial element was the fact that the intersection V of two Hermitian varieties X and X’ in \( \text{PG}(N, q^2) \) lies in all the \( q + 1 \) Hermitian varieties \( \lambda X + \mu X’ \), \( (\lambda, \mu) \in \mathbb{F}_q^2 \setminus \{ (0, 0) \} \), of the pencil of Hermitian varieties defined by X and X’. This enabled us to obtain results for general dimensions N.

We cannot use this fact in this article. A quadric and a Hermitian variety do not define together a pencil of quadrics or of Hermitian varieties. This implied that different arguments had to be used, which enabled us to obtain results up to dimension \( N < O(q^2) \) for the Hermitian variety X in \( \text{PG}(N, q^2) \).

2. Quadrics and Hermitian varieties

By \( \pi_i \), we denote a projective subspace of dimension i in \( \text{PG}(N, q^2) \). We will often use the term space instead of projective subspace. The space generated by two spaces \( \pi_i \) and \( \pi_i’ \) is denoted by \( \langle \pi_i, \pi_i’ \rangle \).

For the fundamental properties of quadrics and Hermitian varieties, we refer to [9, Chapters 22 and 23]. We repeat the relevant properties for the arguments in this article.
The non-singular quadrics in $\operatorname{PG}(N, q^2)$ are equal to:

- the non-singular parabolic quadrics $Q(N, q^2)$ in $\operatorname{PG}(N = 2N', q^2)$ having standard equation $X_0^2 + X_1X_2 + \cdots + X_{2N'-1}X_{2N'} = 0$. These quadrics contain $q^{4N'-2} + \cdots + q^2 + 1$ points, and the largest dimensional spaces contained in a non-singular parabolic quadric of $\operatorname{PG}(2N', q^2)$ have dimension $N' - 1$.

- the non-singular hyperbolic quadrics $Q^+(N, q^2)$ in $\operatorname{PG}(N = 2N' + 1, q^2)$ having standard equation $X_0X_1 + X_2X_3 + \cdots + X_{2N'}X_{2N'+1} = 0$. These quadrics contain $(q^{2N'} + 1)(q^{2N''+2} - 1)/(q^2 - 1) = q^{4N'} + q^{4N''} + \cdots + q^{2N''} + q^{2N'-2} + q^{2N'-4} + \cdots + q^2 + 1$ points, and the largest dimensional spaces contained in a non-singular hyperbolic quadric of $\operatorname{PG}(N = 2N' + 1, q^2)$ have dimension $N'$.

- the non-singular elliptic quadrics $Q^-(N, q^2)$ in $\operatorname{PG}(N = 2N' + 1, q^2)$ having standard equation $f(X_0, X_1) + X_2X_3 + \cdots + X_{2N'}X_{2N'+1} = 0$, where $f(X_0, X_1)$ is an irreducible quadratic polynomial over $\mathbb{F}_q$. These quadrics contain $(q^{2N''+2} + 1)(q^{2N'} - 1)/(q^2 - 1) = q^{4N'} + q^{4N''} + \cdots + q^{2N''} + q^{2N'-2} + \cdots + q^2 + 1$ points, and the largest dimensional spaces contained in a non-singular elliptic quadric of $\operatorname{PG}(2N' + 1, q^2)$ have dimension $N' - 1$.

The non-singular Hermitian variety $H(N, q^2)$ in $\operatorname{PG}(N, q^2)$ has standard equation $X_0^q + X_1^q + \cdots + X_N^q = 0$. This variety contains $q^{\frac{(N+1)(N-1)}{2}}$ points, and the largest dimensional spaces contained in a non-singular Hermitian variety of $\operatorname{PG}(N, q^2)$ have dimension $\lfloor \frac{N-1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to $x$.

All the quadrics and Hermitian varieties of $\operatorname{PG}(N, q^2)$, including the non-singular ones, can be described as a quadric/Hermitian variety having an $s$-dimensional vertex $\pi_s$ of singular points, $s \geq 1$, and having a non-singular base $X_{N-s-1}$ in an $(N-s-1)$-dimensional space skew to $\pi_s$. We denote such a quadric or Hermitian variety in $\operatorname{PG}(N, q^2)$ with vertex $\pi_s$ and base $X_{N-s-1}$ by $\pi_sX_{N-s-1}$. The points $P$ of the vertex $\pi_s$ of a quadric or Hermitian variety $\pi_sX_{N-s-1}$ are called the singular points of $\pi_sX_{N-s-1} \setminus \pi_s$, are called non-singular. A quadric or Hermitian variety $\pi_sX_{N-s-1}$ is called singular when it has a vertex $\pi_s$ of dimension $s > 0$.

A line intersecting the quadric or Hermitian variety $X$ in a unique point is called a tangent line. A tangent hyperplane through a point $P \in X$ is a hyperplane such that all lines through $P$ in this hyperplane are either tangent lines or either contained in $X$. Such a hyperplane is denoted by $T_P(X)$. A non-singular point of a quadric or Hermitian variety $X$ has a unique tangent hyperplane; for a singular point $P$ of $X$, every hyperplane through $P$ is a tangent hyperplane to $X$.

Consider a non-singular quadric or Hermitian variety $X$ in $N$ dimensions, then a non-tangent hyperplane intersects $X$ in a non-singular quadric or non-singular Hermitian variety, and a tangent hyperplane intersects this non-singular quadric or Hermitian variety $X$ in a cone $\pi_0X'$, with $X'$ a quadric or Hermitian variety in $N - 2$ dimensions of the same type as $X$; see [1, 2] for these properties in the case of Hermitian varieties.

We call the largest dimensional spaces contained in a quadric or Hermitian variety the generators of this quadric or Hermitian variety.

The quadrics having the largest size are the union of two distinct hyperplanes of $\operatorname{PG}(N, q^2)$, and have size $2q^{2N-2} + q^{2N-4} + \cdots + q^2 + 1$.

As we mentioned in the introduction, the smallest weight codewords of the code $C_2(X)$ correspond to the quadrics $Q$ having the largest intersections with the Hermitian variety $X$ of $\operatorname{PG}(N, q^2)$. We will show that the largest intersections arise from the quadrics $Q$ that are the union of two distinct hyperplanes of $\operatorname{PG}(N, q^2)$, when $N < O(q^2)$. This proves the conjecture of F.A.B. Edoukou [5] in small dimensions $N$.

Finally, the set of $q + 1$ transversals of three pairwise skew lines in $\operatorname{PG}(3, q)$ is called a regulus. Three lines of a regulus define again a regulus, called the opposite regulus. A hyperbolic quadric $Q^+(3, q)$ is a pair of complementary reguli.
3. Dimension 4

The goal is to look for a bound \( W_4 \) on the intersection size of an absolutely irreducible quadric \( Q \) with the Hermitian variety \( X (= H(4, q^2)) \), in such a way that if the intersection size of \( Q \cap X \) is larger than this bound, then the quadric \( Q \) has to be the union of 2 hyperplanes. Therefore we search for the largest intersection size of an absolutely irreducible quadric with \( X \). This problem was first investigated by Edoukou [5]. We present here an alternative approach.

3.1. The quadric \( Q \) is the non-singular quadric \( Q(4, q^2) \)

Lemma 3.1. If \( Q^+(3, q^2) \cap H(3, q^2) \) contains 3 skew lines, then the intersection consists of \( 2(q + 1) \) lines forming a hyperbolic quadric \( Q^+(3, q) \) and \(|Q^+(3, q^2) \cap H(3, q^2)| = 2q^3 + q^2 + 1\).

Proof. This is [8, Lemma 19.3.1].

This implies that
\[
|Q^+(3, q^2) \cap H(3, q^2)| = (q + 1)(q^2 + 1) + (q^2 - q)(q + 1) = 2q^3 + q^2 + 1.
\]

Lemma 3.2. If \( Q^+(3, q^2) \cap H(3, q^2) \) contains at most 2 skew lines, then \(|Q^+(3, q^2) \cap H(3, q^2)| \leq q^3 + 3q^2 - q + 1\).

Proof. (See also [3].) We count according to the lines of one regulus of \( Q^+(3, q^2) \):
\[
|Q^+(3, q^2) \cap H(3, q^2)| \leq 2(q^2 + 1) + (q^2 - 1)(q + 1) \leq q^3 + 3q^2 - q + 1.
\]

Lemma 3.3. Let \( L \) be a line of \( Q(4, q^2) \) containing at most \( q \) points of \( Q(4, q^2) \cap H(4, q^2) \), then \(|Q(4, q^2) \cap H(4, q^2)| \leq q^5 + 3q^4 + 2q^2 + q + 1\).

Proof. Let \( P \in L \) with \( P \notin Q(4, q^2) \cap H(4, q^2) \). Take a line \( M \) of \( Q(4, q^2) \) intersecting \( L \) in \( P \). Consider the plane \( (L, M) \). Then \( (L, M) \) lies in the tangent hyperplane \( T_P(Q(4, q^2)) \) to \( Q(4, q^2) \) and on \( q^2 \) 3-dimensional spaces sharing a hyperbolic quadric \( Q^+(3, q^2) \) with \( Q(4, q^2) \). No \( Q^+(3, q^2) \) can intersect \( H(4, q^2) \) in \( q + 1 \) lines of both reguli, since \( L \) has only \( q \) points of the intersection \( Q(4, q^2) \cap H(4, q^2) \). So \(|Q(4, q^2) \cap H(4, q^2)| \leq q^4(q^2 + 3q^2 - q + 1) + |T_P(Q(4, q^2)) \cap Q(4, q^2) \cap H(4, q^2)|
\]
\[
|Q(4, q^2) \cap H(4, q^2)| \leq q^5 + 3q^4 + 2q^2 + q + 1.
\]

Remark 3.4. From now on, we assume that every line of \( Q(4, q^2) \) shares at least \( q + 1 \) points with \( H(4, q^2) \). So all lines of \( Q(4, q^2) \) share \( q + 1 \) or \( q^2 + 1 \) points with \( H(4, q^2) \), since a line having more than \( q + 1 \) points of \( H(4, q^2) \) is contained in \( H(4, q^2) \).

Lemma 3.5. Let \( P \in Q(4, q^2) \cap H(4, q^2) \), then \( T_P(Q(4, q^2)) \neq T_P(H(4, q^2)) \).

Proof. Assume that \( T_P(Q(4, q^2)) = T_P(H(4, q^2)) \). Let \( Q(2, q^2) \) be the base of \( T_P(Q(4, q^2)) \cap Q(4, q^2) \) and let \( H(2, q^2) \) be the base of \( T_P(H(4, q^2)) \cap H(4, q^2) \). Take a line \( L \) through \( P \) to a point of \( Q(2, q^2) \) \( \cap H(2, q^2) \). This line \( L \) only shares \( P \) with \( H(4, q^2) \), while it should contain at least \( q + 1 \) points of \( H(4, q^2) \). □

Lemma 3.6. Assume that all lines of \( Q(4, q^2) \) share \( q + 1 \) or \( q^2 + 1 \) points with \( H(4, q^2) \), then \(|Q(4, q^2) \cap H(4, q^2)| \leq q^5 + 3q^4 - 4q^2 + 3q + 1\).
Proof. Let $P$ be a point of $Q(4, q^2)$ not lying in the intersection $Q(4, q^2) \cap H(4, q^2)$, and take 2 lines $L$ and $M$ of $Q(4, q^2)$ through $P$. All $q^2 + 1$ lines of $Q(4, q^2)$ through $P$ contain $q + 1$ points of $Q(4, q^2) \cap H(4, q^2)$, so $|T_P(Q(4, q^2)) \cap Q(4, q^2) \cap H(4, q^2)| = (q + 1)(q^2 + 1)$.

Consider the $q + 1$ points $P_1, \ldots, P_{q+1}$ of $L \cap Q(4, q^2) \cap H(4, q^2)$. They lie on at most 2 lines contained in $Q(4, q^2) \cap H(4, q^2)$ (Lemma 3.5). For, such a line through a point $P_i$ lies in the tangent hyperplanes $T_P(Q(4, q^2))$ and $T_{P_i}(H(4, q^2))$. But these tangent hyperplanes only have a plane in common and this plane has at most two lines through $P_i$ contained in $Q(4, q^2) \cap H(4, q^2)$. So at most two of the $q^2$ distinct hyperbolic quadrics $Q^+(3, q^2)$ of $Q(4, q^2)$ through $(L, M)$ can intersect $H(4, q^2)$ in $2(q + 1)$ lines, so we get at most twice $q^3 + q^2 + 1 - 2(q + 1) = 2q^3 + q^2 - 2q - 1$ extra intersection points. At least $q^2 - 2$ times, we get at most $q^3 + 3q^2 - q + 1 - 2(q + 1) = q^3 + 3q^2 - 3q - 1$ extra intersection points.

So in total there are at most $q^3 + 3q^2 - 4q^2 + 3q + 1$ intersection points. □

3.2. The quadric cone $Q = \pi_0 Q^-(3, q^2)$

Case I. $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ does not contain a line.

Then the $q^4 + 1$ lines through $\pi_0$ on $Q^-(3, q^2)$ have at most $q + 1$ points of $H(4, q^2)$. So

$$|H(4, q^2) \cap \pi_0 Q^-(3, q^2)| \leq (q + 1)(q^4 + 1)$$

(1)

$$\leq q^5 + q^4 + q + 1.$$  (2)

This upper bound is also determined in [5, Subsection 3.3.1].

Case II. $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ contains at least one line.

Lemma 3.7. If $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ contains at least one line $L$, then $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ contains at most $2(q + 1)$ lines.

Proof. Since $L \subseteq H(4, q^2) \cap \pi_0 Q^-(3, q^2)$, necessarily $\pi_0 \subseteq H(4, q^2) \cap \pi_0 Q^-(3, q^2)$. Every line $L'$ of $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ passes through $\pi_0$, so lies in the tangent hyperplane $T_{\pi_0}(H(4, q^2))$. This hyperplane intersects $\pi_0 Q^-(3, q^2)$ in a cone $\pi_0 Q(2, q^2)$ if there are at least two lines contained in $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$. Since $L \subseteq H(4, q^2) \cap \pi_0 Q^-(3, q^2)$, it defines a point of $H(2, q^2) \cap Q(2, q^2)$, with $H(2, q^2)$ and $Q(2, q^2)$ the basis of the tangent cone $T_{\pi_0}(H(4, q^2))$ and of $\pi_0 Q^-(3, q^2) \cap T_{\pi_0}(H(4, q^2))$. By Bézout's theorem, $|H(2, q^2) \cap Q(2, q^2)| \leq 2(q + 1)$. So at most $2(q + 1)$ lines of $\pi_0 Q^-(3, q^2)$ lie completely on $H(4, q^2)$. □

By the previous lemma, we have

$$|H(4, q^2) \cap \pi_0 Q^-(3, q^2)| \leq 2(q + 1)(q^2 + 1) + (q^4 - 2q - 1)(q + 1)$$

(3)

$$\leq q^5 + q^4 + 2q^3 - q + 1.$$  (4)

3.3. The quadric cone $Q = \pi_0 Q^+(3, q^2)$

(See also [5, Section 3.1].) We can describe $\pi_0 Q^+(3, q^2)$ by $q^2 + 1$ planes defined by $\pi_0$ and the lines of one regulus of $Q^+(3, q^2)$. No plane lies completely on $H(4, q^2)$, so every plane shares at most $q^3 + q^2 + 1$ points, of a cone $PH(1, q^2)$, with $H(4, q^2)$. Hence,

$$|H(4, q^2) \cap \pi_0 Q^+(3, q^2)| \leq (q^2 + 1)(q^3 + q^2 + 1)$$

(5)

$$\leq q^5 + q^4 + q^3 + 2q^2 + 1.$$  (6)
3.4. The quadric cone $Q = \pi_1 Q(2, q^2)$

(See also [5, Section 3.1].) Also this quadric can be described by $q^2 + 1$ planes, so as above

$$|H(4, q^2) \cap \pi_1 Q(2, q^2)| \leq q^5 + q^4 + q^3 + 2q^2 + 1.$$ 

3.5. The quadric cone $Q = \pi_2 Q^-(1, q^2)$

Then we have in fact the intersection of a plane with $H(4, q^2)$. So this intersection size will be smaller than the previous bounds.

3.6. Conclusion

Let $Q$ be a quadric in PG($4, q^2$).

**Theorem 3.8.** If $|Q \cap H(4, q^2)| > q^5 + 3q^4 + 2q^2 + q + 1$, then $Q$ is the union of $2$ hyperplanes.

**Proof.** From Lemmata 3.3 and 3.6, we know that the intersection size of the non-singular quadric $Q(4, q^2)$ with $H(4, q^2)$ is at most $q^5 + 3q^4 + 2q^2 + q + 1$. For the different intersection sizes of other quadrics with $H(4, q^2)$, (2), (4), and (6) learn us that they are smaller than the previous one. So this proves the theorem. 

From now on, we will denote this bound by $W_4 = q^5 + 3q^4 + 2q^2 + q + 1$.

### 4. General case

Let $Q$ be a quadric in PG($N, q^2$).

**Theorem 4.1.** If $|Q \cap H(N, q^2)| > (q^2 + 2)^{N-4} W_4$, then $Q$ is the union of two hyperplanes, for dimension $N < O(q^2)$.

**Proof.** Part 1. Denote $(q^2 + 2)^{N-4} W_4$ by $W_N$. The bound is valid for $N = 4$ (Theorem 3.8).

Suppose that the lemma holds for dimension $N - 1$. By induction, we show that the bound is true for dimension $N$.

Select $(q^2 + 2)^{N-4} W_4$ points $P$ of $Q \cap H(N, q^2)$ and count the incidences $(P, H)$, with $P \in Q \cap H(N, q^2)$ and $H$ a tangent hyperplane to $H(N, q^2)$. This gives

$$((q^2 + 2)^{N-4} W_4) |PH(N - 2, q^2)| = |H(N, q^2)| X_N,$n

with $X_N$ the average number of those $(q^2 + 2)^{N-4} W_4$ points of $Q \cap H(N, q^2)$ in a tangent hyperplane to $H(N, q^2)$.

So some tangent hyperplane $T_P (H(N, q^2))$, $P \in H(N, q^2)$, contains at most

$$X_N \leq \frac{((q^2 + 2)^{N-4} W_4)((q^N - 1) + (-1)^{N-2})q^{N-2} + (-1)^{N-1}q^2 + q^2 - 1)}{(q^{N+1} + (-1)^N)q^N + (-1)^{N+1}} \leq W_{N-1} \left(1 + \frac{3}{q^2 - 1}\right),$$

of those points.

There remain more than $(q^2 + 2)W_{N-1} - W_{N-1}(1 + \frac{3}{q^2 - 1}) = (q^2 + 1 - \frac{3}{q^2 - 1}) W_{N-1}$ points in $Q \cap H(N, q^2)$, not lying in this tangent hyperplane $T_P (H(N, q^2))$. Take an arbitrary $H(N - 3, q^2)$ on the
base $H(N - 2, q^2)$ of $T_P(H(N, q^2)) \cap H(N, q^2)$. We do not know $|H(N - 3, q^2) \cap Q \cap H(N, q^2)|$, but we know that the $q^2 + 1$ hyperplanes through $P(H(N - 3, q^2))$ are $T_P(H(N, q^2))$, the only tangent hyperplane through $(P, H(N - 3, q^2))$, and $q^2$ hyperplanes intersecting $H(N, q^2)$ in a non-singular Hermitian variety $H(N - 1, q^2)$.

So one of them, denoted by $\pi$, contains more than $\frac{(q^2 + 1 - 2)}{q^2}W_{N-1} \geq W_{N-1}$ points of the intersection. Then in this hyperplane $\pi$, since $|\pi \cap Q \cap H(N - 1, q^2)| > W_{N-1}$, $\pi \cap Q$ is the union of two $(N - 2)$-dimensional spaces.

**Part 2.** The only quadrics containing $(N - 2)$-dimensional spaces are $\pi_{N-4}Q^+(3, q^2)$, $\pi_{N-2}Q^+(1, q^2)$, and $\pi_{N-3}Q(2, q^2)$.

We want to eliminate the quadrics $\pi_{N-4}Q^+(3, q^2)$ and $\pi_{N-3}Q(2, q^2)$: they both can be described as the union of $q^2 + 1$ $(N - 2)$-dimensional spaces $\pi_{N-2}$. The largest intersection of $\pi_{N-2} \cap H(N, q^2)$ comes from a Hermitian variety which is the union of $q + 1$ distinct $(N - 3)$-dimensional spaces sharing an $(N - 4)$-dimensional space and this has size

$$(q + 1)q^{2N-6} + q^{2N-8} + \ldots + q^2 + 1 = q^{2N-5} + q^{2N-6} + q^{2N-8} + \ldots + q^2 + 1.$$ 

If this would be the case for all these $q + 1$ distinct $\pi_{N-2}$, we would get at most an intersection size $(q^2 + 1)(q^{2N-5} + q^{2N-6} + q^{2N-8} + \ldots + q^2 + 1)$ of these quadrics with $H(N, q^2)$. Since $(q^2 + 2)^{N-4}W_A > (q^2 + 1)(q^{2N-5} + q^{2N-6} + q^{2N-8} + \ldots + q^2 + 1)$, these quadrics cannot occur.

So $Q = \pi_{N-2}Q^+(1, q^2)$ which is the union of two hyperplanes. □

**Remark 4.2.** The condition $N < O(q^2)$ arises from the fact that only for $N < O(q^2)$, the value $(q^2 + 2)^{N-4}W_A$ is smaller than or equal to the intersection size of two hyperplanes with a non-singular Hermitian variety $H(N, q^2)$. Here, necessarily $N < q^2/3$.

**5. Structure of small weight codewords**

We proved in Theorem 4.1 that the small weight codewords of $C_2(X)$, $X$ a non-singular Hermitian variety in $PG(N, q^2)$, $O(q^2) > N \geq 4$, correspond to the intersections of $X$ with the quadrics consisting of the union of two hyperplanes. We now count the number of codewords obtained via the intersections of $X$ with the union of two hyperplanes.

Consider a quadric $Q$ which is a union of two hyperplanes, then $Q$ defines $q^2 - 1$ codewords of $C_2(X)$, equal to each up to a non-zero scalar multiple.

It could be that a quadric $Q'$ which also is a union of two hyperplanes, but different from $Q$, defines the same $q^2 - 1$ codewords of $C_2(X)$. However, this can be excluded for $N \geq 4$ in the following way.

If the quadric $Q$, which is the union of the two hyperplanes $\Pi_1$ and $\Pi_2$, and the quadric $Q'$, which is the union of the two hyperplanes $\Pi'_1$ and $\Pi'_2$, define the same codewords of $C_2(X)$, then $(\Pi_1 \cup \Pi_2) \cap X = (\Pi'_1 \cup \Pi'_2) \cap X$. Assume that $\Pi'_1 \neq \Pi_1, \Pi_2$. Then the hyperplane intersection $\Pi'_1 \cap X$ must be contained in the two $(N - 2)$-dimensional intersections $\Pi'_1 \cap \Pi_1 \cap X$ and $\Pi'_1 \cap \Pi_2 \cap X$. But the smallest possible intersection size of a hyperplane with $X$ is larger than twice the largest possible intersection size of an $(N - 2)$-dimensional space with $X$. So this case does not occur.

Hence, to calculate the number of codewords arising from the union of two hyperplanes, we simply check which unions of two hyperplanes determine codewords of a particular weight (Tables 1, 2 and 3); we then count how many such pairs of hyperplanes there are in $PG(N, q^2)$, and then we multiply this number by $q^2 - 1$ since a union of two hyperplanes defines $q^2 - 1$ non-zero codewords which are a scalar multiple of each other. For $N \geq 4$, this determines the precise number of codewords of the smallest weights in $C_2(X)$ (Table 3).

We determine the geometrical construction of the smallest weight codewords. They correspond to the intersection of $H(N, q^2)$ with $\pi_{N-2}Q^+(1, q^2)$. The quadric $\pi_{N-2}Q^+(1, q^2)$ consists of two hyperplanes, which we will denote by $\Pi_1$ and $\Pi_2$, through an $(N - 2)$-dimensional space $\pi_{N-2}$. We recall that a hyperplane intersects $H(N, q^2)$ either in a non-singular Hermitian variety $H(N - 1, q^2)$
Table 1

|       | \(\pi_{N-2} \cap H(N,q^2)\) | \(|X \cap (\mathcal{I}_1 \cup \mathcal{I}_2)|\) |
|-------|---------------------------|-----------------------------|
| (1)   | H(N - 2, q^2)            | 2|H(N - 1, q^2)| - |H(N - 2, q^2)| |
| (1.1) |                           |                             |
| (1.2) |                           |                             |
| (1.3) |                           |                             |
| (2)   | \(\pi_0 H(N - 3, q^2)\)  | 2|\|\pi_0 H(N - 3, q^2)| - |\|\pi_0 H(N - 3, q^2)| |
| (2.1) |                           |                             |
| (2.2) |                           |                             |
| (3)   | \(LH(N - 4, q^2)\)      | 2|\|\pi_0 H(N - 2, q^2)| - |\|LH(N - 4, q^2)| |
| (3.1) |                           |                             |

Table 2(a)

\(N\) even.

|       | \(|X \cap (\mathcal{I}_1 \cup \mathcal{I}_2)|\) |
|-------|-----------------------------|
| (1)   | \(2q^{2N-3} + q^{3N-5} + q^{2N-7} + \ldots + q^{N+2} + q^{N-1} + q^{N-2} + q^{N-3} + \ldots + q^2 + 1\) |
| (1.1) |                             |
| (1.2) |                             |
| (1.3) |                             |
| (2)   | \(2q^{2N-3} + q^{3N-5} + q^{2N-7} + \ldots + q^{N+1} + q^{N-1} + q^{N-2} + q^{N-3} + \ldots + q^2 + 1\) |
| (2.1) |                             |
| (2.2) |                             |
| (3)   | \(2q^{2N-3} + q^{3N-5} + q^{2N-7} + \ldots + q^{N+2} + q^{N-1} + q^{N-2} + q^{N-3} + \ldots + q^2 + 1\) |
| (3.1) |                             |

Table 2(b)

\(N\) odd.

|       | \(|X \cap (\mathcal{I}_1 \cup \mathcal{I}_2)|\) |
|-------|-----------------------------|
| (1)   | \(2q^{2N-3} + q^{3N-5} + q^{2N-7} + \ldots + q^{N+2} + q^{N-1} + q^{N-2} + q^{N-3} + \ldots + q^2 + 1\) |
| (1.1) |                             |
| (1.2) |                             |
| (1.3) |                             |
| (2)   | \(2q^{2N-3} + q^{3N-5} + q^{2N-7} + \ldots + q^{N+1} + q^{N-1} + q^{N-2} + q^{N-3} + \ldots + q^2 + 1\) |
| (2.1) |                             |
| (2.2) |                             |
| (3)   | \(2q^{2N-3} + q^{3N-5} + q^{2N-7} + \ldots + q^{N+2} + q^{N-1} + q^{N-2} + q^{N-3} + \ldots + q^2 + 1\) |
| (3.1) |                             |

Table 3(a)

\(N\) even, \(N < O(q^2)\).

<table>
<thead>
<tr>
<th>Weight</th>
<th>Number of codewords for (N \geq 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.1)</td>
<td>(w_1 = q^{N-2}(q^{N+1} - q^{N+1} - q - 1))</td>
</tr>
<tr>
<td>(1.2)</td>
<td>(w_1 + q^{N-1})</td>
</tr>
<tr>
<td>(2.1)</td>
<td>(w_1 + q^{N-1} + q^{N-2})</td>
</tr>
<tr>
<td>(1.3)</td>
<td>(w_1 + 2q^{N-1})</td>
</tr>
</tbody>
</table>

Table 3(b)

\(N\) odd, \(N < O(q^2)\).

<table>
<thead>
<tr>
<th>Weight</th>
<th>Number of codewords for (N \geq 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.1)</td>
<td>(w_1 = q^{N-2}(q^{N+1} - q^{N+1} - q + 1))</td>
</tr>
<tr>
<td>(2.2)</td>
<td>(w_1 + q^{N-2})</td>
</tr>
<tr>
<td>(1.2)</td>
<td>(w_1 + q^{N-1})</td>
</tr>
<tr>
<td>(2.1)</td>
<td>(w_1 + 2q^{N-1} - q^{N-2})</td>
</tr>
<tr>
<td>(1.3)</td>
<td>(w_1 + 2q^{N-1})</td>
</tr>
</tbody>
</table>
or, in case it is a tangent hyperplane, in a cone $\pi_0H(N - 2, q^2)$. This $(N - 2)$-dimensional space $\pi_{N-2}$ can intersect $H(N, q^2)$ in different ways and this gives us different weight codewords. Starting from the intersection of $\pi_{N-2} \cap H(N, q^2)$, we determine the different intersection sizes and small weights of $C_2(X)$.

For the intersection of $\pi_{N-2}$ with $H(N, q^2)$, there are three possibilities. This intersection is either a non-singular Hermitian variety $H(N - 2, q^2)$, a singular Hermitian variety $\pi_0H(N - 3, q^2)$ with vertex the point $\pi_0$ and base the non-singular Hermitian variety $H(N - 3, q^2)$, or a singular Hermitian variety $LH(N - 4, q^2)$ with vertex the line $L$ and base the non-singular Hermitian variety $H(N - 4, q^2)$.

In Table 1, we denote the different possibilities for the intersection of $X = H(N, q^2)$ with the union of two hyperplanes $\Pi_1$ and $\Pi_2$.

In Table 2, we give the intersection sizes: we split the table up into the cases $N$ even and $N$ odd. From the intersection sizes listed in Table 2, we now determine the smallest weights for $C_2(X)$ by subtracting the size of the intersection $Q \cap X$ from the length of the code $C_2(X)$. In the same table, we list the number of such codewords. We again split up the table into the cases $N$ even and $N$ odd.

To conclude this article, we restate the conjecture of Edoukou [5] regarding the smallest weights of the functional codes $C_2(X)$, $X$ a non-singular Hermitian variety of $\text{PG}(N, q^2)$; a conjecture which we have proven to be true for small dimensions $N$.

**Conjecture.** The smallest weights of the functional codes $C_2(X)$, $X$ a non-singular Hermitian variety of $\text{PG}(N, q^2)$, arise from the quadrics $Q$ which are the union of two hyperplanes of $\text{PG}(N, q^2)$.

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