Sussmann's orbit theorem and maps ☆

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Abstract

A map between manifolds which matches up families of complete vector fields is a fiber bundle mapping on each orbit of those vector fields.

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1. Introduction

Definition 1. Write $e^{tX}(m) \in M$ for the flow of a vector field X through a point m after time t. Let $\mathfrak{F}$ be a family of smooth vector fields on a manifold M. The *orbit* of $\mathfrak{F}$ through a point $m \in M$ is the set of all points $e^{t_1 X_1} e^{t_2 X_2} \ldots e^{t_k X_k}(m)$ for any vector fields $X_j \in \mathfrak{F}$ and numbers t_j (positive or negative) for which this is defined.

Example 1. The vector field $\frac{\partial}{\partial \theta}$ on the Euclidean plane (in polar coordinates) has orbits the circles around the origin, and the origin itself.

Example 2. The set of smooth vector fields supported in a disk has as orbits the open disk (a 2-dimensional orbit) and the individual points outside or on the boundary of the disk (zero dimensional orbits).

Example 3. On Euclidean space, the set of vector fields supported inside a ball, together with the radial vector field coming from the center of the ball, forms a set of vector fields with a single orbit.

Example 4. Translation in a generic direction on a flat torus has densely winding orbits.

Héctor Sussmann [2–4] proved that the orbits of any family of smooth vector fields are immersed submanifolds (also see Stefan [1]). We prove that a mapping between two manifolds which carries one family of complete vector fields into another, is a fiber bundle mapping on each orbit.

2. Proofs

For completeness, we prove Sussmann's theorem.

Theorem 1. (Sussmann [2]) *The orbit of any point under any family of smooth vector fields is an immersed submanifold (in a canonical topology). If two orbits intersect, then they are equal. Let $\tilde{\mathfrak{F}}$ be the largest family of smooth vector fields which have the same orbits as the given family $\mathfrak{F}$. Then $\tilde{\mathfrak{F}}$ is a Lie algebra of vector fields, and a module over the algebra of smooth functions.*

Remark 1. Obviously, one could localize these results, replacing globally defined vector fields with subsheaves of the sheaf of locally defined smooth vector fields.

Proof. We can replace $\mathfrak{F}$ by $\tilde{\mathfrak{F}}$ without loss of generality. Therefore, if $X, Y \in \mathfrak{F}$, we can suppose that $e_*^X Y \in \mathfrak{F}$ since the flow of $e_*^X Y$ is

$$e^{t(e_*^X Y)} = e^X e^{tY},$$

which must preserve orbits. We refer to this process as *pushing around* vector fields.

Fix attention on a specific orbit. For each point $m_0 \in M$, take as many vector fields as possible $X_1, \ldots, X_k$, out of $\mathfrak{F}$, which are linearly independent at m. Refer to the number k of vector fields as the *orbit dimension*. Pushing around convinces us that the orbit dimension is a constant throughout the orbit. Refer to the map

$$(t_1, \ldots, t_k) \in \text{ open } \subset \mathbb{R}^k \mapsto e^{t_1 X_1} \ldots e^{t_k X_k} m_0 \in M$$

(which we will take to be defined in some open set on which it is an embedding) as a *distinguished chart* and its image as a *distinguished set*. The tangent space to each point $e^{t_1 X_1} \ldots e^{t_k X_k} m_0$ of a distinguished set is spanned by the linearly independent vector fields

$$X_1, e_*^{t_1 X_1} X_2, \ldots, e^{t_1 X_1} \ldots e_*^{t_{k-1} X_{k-1}} X_k,$$

which belong to $\mathfrak{F}$, since they are just pushed around copies of the X_j. Let Ω be a distinguished set. Suppose that $Y \in \mathfrak{F}$ is a vector field, which is not tangent to Ω. Then at some point of Ω, Y is not a multiple of those pushed around vector fields, so the orbit dimension must exceed k.

Therefore all vector fields in $\mathfrak{F}$ are tangent to all distinguished sets. So any point inside any distinguished set stays inside that set under the flow of any vector field in $\mathfrak{F}$, at least for a short time. So such a point must also stay inside the distinguished set under compositions of flows of the vector fields, at least for short time. Therefore a point belonging to two distinguished sets must remain in both of them under the flows that draw out either of them, at least for short times. Therefore that point belongs to a smaller distinguished set lying inside both of them. Therefore the intersection of distinguished sets is a distinguished set.

We define an open set of an orbit to be any union of distinguished sets; so the orbit is locally homeomorphic to Euclidean space. We can pick a countable collection of distinguished sets as a basis for the topology. Every open subset of M intersects every distinguished set in a distinguished set, so intersects every open set of the orbit in an open set of the orbit. Thus the inclusion mapping of the orbit into M is continuous. Since M is metrizable, the orbit is also metrizable, so a submanifold of M. The distinguished charts give the orbit a smooth structure. They are smoothly mapped into M, ensuring that the inclusion is a smooth map. $\quad\square$

Example 5. Let $\alpha = dy - z\,dx$ in $\mathbb{R}^3$. The vector fields on which $\alpha = 0$ have one orbit: all of $\mathbb{R}^3$, since they include $\partial_z, \partial_x + z\partial_y$, and therefore include the bracket:

$$[\partial_z, \partial_x + z\partial_y] = \partial_y.$$

Definition 2. Take a map $\phi : M_0 \to M_1$, and vector fields X_j on M_j, $j = 0, 1$. Write $\phi_* X_0 = X_1$ to mean that for all $m_0 \in M_0$, $\phi'(m_0) X_0(m_0) = X_1(\phi(m_0))$. For families of vector fields, write $\phi_* \mathfrak{F}_0 = \mathfrak{F}_1$ to mean that

(1) for any $X_0 \in \mathfrak{F}_0$ there is an $X_1 \in \mathfrak{F}_1$ so that $\phi_* X_0 = X_1$ and
(2) for any $X_1 \in \mathfrak{F}_1$ there is a vector field $X_0 \in \mathfrak{F}_0$ so that $\phi_* X_0 = X_1$.

Example 6. The vector field ∂_x on $\mathbb{R}$ has $\mathbb{R}$ as orbit. Consider the inclusion $(0, 1) \subset \mathbb{R}$ of some open interval. The orbit of ∂_x on $(0, 1)$ is $(0, 1)$. The orbits are mapped to each other by the inclusion, but not surjectively.

Example 7. If $M_0 = \mathbb{R}^2_{x,y}$ and $M_1 = \mathbb{R}^1_x$, and $\phi(x, y) = x$, and $\mathfrak{F}_0 = \{\partial_x, \partial_y\}$ and $\mathfrak{F}_1 = \{\partial_x, 0\}$, then clearly $\phi_* \mathfrak{F}_0 = \mathfrak{F}_1$.

Example 8. The group $SO(3)$ of rotations acts on the sphere S^2, and we can map $SO(3) \to S^2$, taking a rotation g to gn where n is the north pole. This map takes the left invariant vector fields to the infinitesimal rotations, and clearly is a fiber bundle, the Hopf fibration.

Theorem 2. *If $\mathfrak{F}_j$ are sets of vector fields on manifolds M_j, for $j = 0, 1$, and $\phi : M_0 \to M_1$ satisfies $\phi_* \mathfrak{F}_0 = \mathfrak{F}_1$, then ϕ takes $\mathfrak{F}_0$-orbits into $\mathfrak{F}_1$-orbits. On each orbit, ϕ has constant rank. If the vector fields in both families are complete, then ϕ is a fiber bundle mapping on each orbit.*

Proof. By restricting to an orbit in M_0, we may assume that there is only one orbit. The map ϕ is invariant under the flows of the vector fields, so must have constant rank.

Henceforth, suppose that the vector fields are complete. Given a path

$$e^{t_1 X_1} \ldots e^{t_k X_k} m_0$$

down in M_1, we can always lift it to one in M_0, so ϕ is onto. It might not be true that $\phi_* \bar{\mathfrak{F}}_0 = \bar{\mathfrak{F}}_1$, but nonetheless we can still push around vector fields, because the pushing upstairs in M_0 corresponds to pushing downstairs in M_1. So without loss of generality, both $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are closed under "pushing around".

As in the above proof, for each point $m_1 \in M_1$, we can construct a distinguished chart

$$(t_1, \ldots, t_k) \mapsto e^{t_1 X_1} \ldots e^{t_k X_k} m_1.$$

These X_k are vector fields on M_1. Write Y_k for some vector fields on M_0 which satisfy $\phi_* Y_k = X_k$. Clearly ϕ is a surjective submersion. Let $U_1 \subset M_1$ be the associated distinguished set; on U_1 these t_j are now coordinates. Let $U_0 = \phi^{-1} U_1 \subset M_0$. Let Z be the fiber of $\phi : M_0 \to M_1$ above the origin of the distinguished chart. Map

$$u_0 \in U_0 \mapsto (u_1, z) \in U_1 \times Y$$

by $u_1 = \phi(u_0)$ and

$$z = e^{-t_k Y_k} \ldots e^{-t_1 Y_1} u_0.$$

Clearly this gives M_0 the local structure of a product. The transition maps have a similar form, composing various flows, so $M_0 \to M_1$ is a fiber bundle. $\quad\square$

Keep in mind that all vector fields on compact manifolds are complete. Even though the orbits might not be compact, our theorem says that the orbits upstairs will fiber over the orbits downstairs.

Example 9. Take $M_1 = \mathbb{R}^2_{x,y}$, and $M_0 \subset M_1$ a pair of disjoint disks, say those of unit radius around two points of the x axis which are distantly separated. As the family $\mathfrak{F}_0$ up in the disks, take the translation vector fields ∂_x, ∂_y along coordinate axes in the right disk, and in the left, the pair of vector fields ∂_x, 0. Obviously these are not complete. As the family $\mathfrak{F}_1$, take the translation vector field ∂_x, and a vector field $f(x, y)\partial_y$ which vanishes in the left disk, and nowhere outside of closure of the left disk, and equals ∂_y in the right disk. The orbits downstairs are all two dimensional, while those upstairs are one dimensional in the left disk, and two dimensional in the right.

Example 10. Take $E \to M$ any fiber bundle, and pick a plane field on E transverse to the fibers. Every vector field on M lifts to a unique vector field on E tangent to the 2-plane field. Suppose that the fibers of $E \to M$ are compact. Lifting all complete vector fields, we get a family of complete vector fields on E. Their orbits must be connected and fiber over M.

Example 11. Take any 2-plane field on SO(3) transverse to the leaves of the Hopf fibration SO(3) $\to S^2$, and lift vector fields as in the last example. A two dimensional orbit would have to be diffeomorphic to S^2, since S^2 is simply connected. The Hopf fibration admits no section, so therefore all orbits must be three dimensional, hence open and disjoint, and cover SO(3), which is connected. Hence every 2-plane field transverse to the Hopf fibration has all of SO(3) as orbit, even though the 2-plane field may be holonomic on an open set. The same result works for any circle bundle on any compact manifold: either every plane field transverse to the circle fibers has a single orbit, or the circle bundle trivializes on a covering space.

Example 12. Consider the Hopf fibration $S^3 \to S^7 \to S^4$. Take any 4-plane field on S^7 transverse to the fibers. The orbits must be bundles over S^4. The fibers of such a bundle cannot be zero dimensional, since S^4 is simply connected and the Hopf fibration is not a trivial bundle. Suppose that $F \to B \to S^4$ is a fiber bundle, and that $B \subset S^7$ is a subbundle. The bundle B cannot be trivial, since that would give rise to a section of the Hopf fibration. The bundle B is determined completely by slicing S^4 along the equatorial S^3, and mapping S^3 to the diffeomorphism group of the fiber F. The fiber F cannot be the real line, the circle, or a closed surface other than the sphere, since the diffeomorphism groups of these manifolds retract to finite dimensional groups which are aspherical. Therefore F must be a sphere or noncompact surface, or a component of the complement in S^3 of a set of disjoint spheres and noncompact surfaces. Our theorem does not suffice to give a complete analysis of the possible orbits, but clearly it makes a substantial contribution to this question.

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