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Advances in Mathematics 231 (2012) 3054–3078

ADVANCES IN Mathematics

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Maximal velocity of photons in non-relativistic QED

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Received 3 November 2011; accepted 20 July 2012
Available online 12 September 2012

Communicated by C. Fefferman

Abstract

We consider the problem of propagation of photons in the quantum theory of non-relativistic matter coupled to electromagnetic radiation, which is, presently, the only consistent quantum theory of matter and radiation. Assuming that the matter system is in a localized state (i.e. for energies below the ionization threshold), we show that the probability to find photons at time t at the distance greater than ct, where c is the speed of light, vanishes as $t \to \infty$ as an inverse power of t.

Keywords: Non-relativistic QED; Scattering theory; Propagation estimates; Schrödinger equation; Wave operators; Quantized electromagnetic field

1. Introduction

One of the key postulates in the theory of relativity is that the speed of light is constant and the same in all inertial reference frames. This postulate, verified to begin with experimentally, can also be easily checked theoretically for propagation of disturbances in the free Maxwell equations. However, one would like to show it for the physical model of matter interacting with electromagnetic radiation. To have a sensible model, one would have to consider both matter and radiation as quantum. This, in turn, requires reformulation of the problem in terms of quantum probabilities. The latter are given through localization observables for photons. We define it below. Now we proceed to the model of quantum matter interacting with (quantum) radiation. (By radiation we always mean the electromagnetic radiation.) In what follows we use the units in which the speed of light and the Planck constant divided by 2π are 1.

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.07.019

Presently, the only mathematically well-defined such a model, which is in a good agreement with experiments, is the one in which matter is treated non-relativistically. In this model, the state space of the total system is given by $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, where $\mathcal{H}_p$ is the state space of the particles, say $\mathcal{H}_p = \mathrm{L}^2(\mathbb{R}^{3n})$, and $\mathcal{H}_f$ is the state spaces of photons (i.e. of the quantized electromagnetic field), defined as the bosonic (symmetric) Fock space, $\mathcal{F}$, over the one-photon space $\mathfrak{h}$ (see Appendix B for the definition of $\mathcal{F}$). In the Coulomb gauge, which we assume from now on, $\mathfrak{h}$ is the L^2-space, $\mathrm{L}^2_{\mathrm{transv}}(\mathbb{R}^3; \mathbb{C}^3)$, of complex vector fields $f : \mathbb{R}^3 \to \mathbb{C}^3$ satisfying $k \cdot f = 0$, where $k = -\mathrm{i}\nabla_y$ in the coordinate representation. In what follows, we use the momentum representation. Then, by choosing orthonormal vector fields $\varepsilon_\lambda(k) : \mathbb{R}^3 \to \mathbb{R}^3$, $\lambda = 1, 2$, satisfying $k \cdot \varepsilon_\lambda(k) = 0$ and $\varepsilon_\lambda(-k) = \pm\varepsilon_\lambda(k)$ ($\varepsilon_\lambda(k)$, $\lambda = 1, 2$, are called the polarization vectors), we identify $\mathfrak{h}$ with the space $\mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$ of square integrable functions of photon momentum $k \in \mathbb{R}^3$ and polarization index $\lambda = 1, 2$.

The dynamics of the system is described by the Schrödinger equation,

$$\mathrm{i}\partial_t \psi_t = H\psi_t, \tag{1.1}$$

on the state space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, with the standard quantum Hamiltonian (see [17,46])

$$H = \sum_{j=1}^{n} \frac{1}{2m_j}\big(-\mathrm{i}\nabla_{x_j} - g_j A_\kappa(x_j)\big)^2 + V(x) + H_f.$$

Here, m_j and x_j, $j = 1, \dots, n$, are the ('bare') particle masses and the particle positions, $V(x)$, $x = (x_1, \dots, x_n)$, is the total potential affecting the particles and g_j are coupling constants related to the particle charges. Moreover, $A_\kappa := \check{\kappa} * A$, where $A(y)$ is the *quantized vector potential* in the Coulomb gauge ($\operatorname{div} A(y) = 0$), describing the quantized electromagnetic field, and given by

$$A_\kappa(y) = \sum_{\lambda=1,2} \int \varepsilon_\lambda(k)\big(e^{\mathrm{i}k\cdot y} a_\lambda(k) + e^{-\mathrm{i}k\cdot y} a^*_\lambda(k)\big)\kappa(k)\frac{\mathrm{d}k}{\sqrt{2|k|}}, \tag{1.2}$$

where $\kappa \in \mathrm{C}^\infty_0(\mathbb{R}^3)$ is a radial *ultraviolet cut-off*. The operator H_f is the quantum Hamiltonian of the quantized electromagnetic field, describing the dynamics of the latter,

$$H_f = \sum_{\lambda=1,2} \int \omega(k) a^*_\lambda(k) a_\lambda(k)\,\mathrm{d}k, \tag{1.3}$$

where $\omega(k) = |k|$ is the dispersion relation. The integrals without indication of the domain of integration are taken over entire $\mathbb{R}^3$. Above, λ is the polarization, $a_\lambda(k)$ and $a^*_\lambda(k)$ are annihilation and creation operators acting on the Fock space $\mathcal{H}_f = \mathcal{F}$ (see Appendix B for the definition of annihilation and creation operators).

Assuming for simplicity that our matter consists of electrons and nuclei and that the nuclei are infinitely heavy and therefore are manifested through the interactions only (put differently, the molecules are treated in the Born–Oppenheimer approximation), one arrives at the operator H with the coupling constants $g_j := \alpha^{1/2}$, where $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$ is the fine-structure constant. After that one can relax the conditions on the potentials $V(x)$ allowing say general many-body ones. For a general discussion of the Hamiltonian H see [12,28]. The spectral theory was developed in [1,4,7,10,21,23,25,26,31,32,41,47,50]. The beginnings of the scattering theory appeared in [6,11,18–20,24,27,33,48]. (For more extensive references see [5,51,54].) Since the structure of the particle system is immaterial for us, to keep notation as simple as possible, we consider a single particle in an external potential, $V(x)$, coupled to the quantized electromagnetic field. Furthermore, since our results hold for any fixed value of α, we absorb it into the ultraviolet cut-off κ. In this case, the state space of such a system is $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathcal{F} = \mathrm{L}^2(\mathbb{R}^3; \mathcal{F})$ and the standard Hamiltonian operator acting on $\mathrm{L}^2(\mathbb{R}^3; \mathcal{F})$ is given by (we omit the subindex κ in $A(x)$)

$$H := \big(p + A(x)\big)^2 + H_f + V(x), \tag{1.4}$$

with the notation $p := -\mathrm{i}\nabla_x$, the particle momentum operator. We assume that V is real valued and infinitesimally bounded with respect to p^2.

Our goal is to show that photons departing a bound particle system, say an atom or a molecule, move away from it with a speed not higher than the speed of light. Let $\mathrm{d}\Gamma(b)$ denote the lifting of a one-photon operator b to the photon Fock space (and then to the Hilbert space of the total system, see the precise definition in Appendix B), $y := \mathrm{i}\nabla_k$ be the operator on $\mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$, canonically conjugate to the photon momentum k and let $\mathbb{1}_\Omega(y)$ denote the characteristic function of a subset Ω of $\mathbb{R}^3$. To test the photon localization, we define the observables $\mathrm{d}\Gamma(\mathbb{1}_\Omega(y))$, which can be interpreted as giving the number of photons in Borel sets $\Omega \subset \mathbb{R}^3$. These observables are closely related to those used in [19,24,39] and are consistent with a theoretical description of detection of photons (usually via the photoelectric effect, see e.g. [43]).[1] The fact that they depend on the choice of polarization vector fields, $\varepsilon_\lambda(k)$, $\lambda = 1, 2$, is not an impediment here as our results imply analogous results for e.g. similarly constructed observables[2] based on the space $\mathrm{L}^2_{\mathrm{transv}}(\mathbb{R}^3; \mathbb{C}^3)$ instead of $\mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$, or localization observables constructed by Amrein [2]. (Both observables are also covariant under rigid motions, g, of $\mathbb{R}^3$,

$$T_g \mathrm{d}\Gamma(\mathbb{1}_\Omega(y)) T_g^{-1} = \mathrm{d}\Gamma(\mathbb{1}_{g^{-1}\Omega}(y)),$$

where $T_g = \Gamma(t_g)$ is generated by one particle transformations $t_g : f(y) \to f(g^{-1}y)$, as is usually required for localization observables.)

With the definition of localization observables given, we say that photons propagate with speed $\leq \mathrm{c}'$ if for any initial condition ψ_0 and for any $\mathrm{c} > \mathrm{c}'$, the state, ψ_t, of the system at time t, satisfies the estimate

$$\big\| \mathrm{d}\Gamma\big(F(|y| \geq \mathrm{c}t)\big)^{\frac{1}{2}} \psi_t \big\| \longrightarrow 0 \quad \text{as } t \to \infty,$$

for any bounded function $F(s \geq 1)$ supported in the domain $\{s \geq 1\}$. Similarly, one can define the propagation with speed $\geq \mathrm{c}'$. As with any other quantum models, this definition allows for a non-zero probability that photons propagate with arbitrary high speed. However, as estimates of such probabilities for massive free relativistic particles show (see [49]), these events (as with the problem of reversibility) have so low probabilities as to make them undetectable.

[1] The issue of localizability of photons is a tricky one and has been intensely discussed in the literature since the 1930 and 1932 papers by Landau and Peierls [38] and Pauli [45] (see also a review in [37]). A set of axioms for localization observables was proposed by Newton and Wigner [44] and Wightman [55] and further generalized by Jauch and Piron [36]. Localization observables for massless particles satisfying the Jauch–Piron version of the Wightman axioms were constructed by Amrein [2].

[2] These observables are similar to those introduced by Mandel [42]. Since polarization vector fields are not smooth, using them to reduce the results from one set of localization observables to another would limit the possible time decay. However, these vector fields can be avoided by using the approach of [40].

To formulate our result, we let Σ denote the ionization threshold defined by

$$\Sigma := \lim_{R\to\infty} \inf_{\substack{\varphi\in D_R \\ \|\varphi\|=1}} \langle \varphi, H\varphi\rangle,$$

where $D_R = \{\varphi \in \mathcal{D}(H);\ \varphi(x) = 0 \text{ if } |x| < R\}$ (see [25]). We also define the Hilbert space $X := \mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle)^{\frac{1}{2}})$, with the norm

$$|||u||| := \left\| \left(\mathrm{d}\Gamma(\langle y\rangle) + 1\right)^{\frac{1}{2}} u \right\|.$$

Let $f \in \mathrm{C}_0^\infty(\mathbb{R}; [0, 1])$ be such that $\operatorname{supp}(f) \subset [1, 2]$ and define $F(s) = \int_{-\infty}^s f(\tau)\,\mathrm{d}\tau$. We will localize the photon position using the following operator

$$F(|y| \geq \mathrm{c}t) = F(|v| \geq 1) := F(|v|), \tag{1.5}$$

where $v := y/\mathrm{c}t$. Throughout the paper, the notation $f \lesssim g$, for functions f and g, stands for $f \leq \mathrm{C}g$ where C is a positive constant. The norm in $\mathcal{H}$, as well as the operator norm, are denoted by $\|\cdot\|$, while the norms in $\mathcal{F}$ and $\mathfrak{h}$ are denoted respectively by $\|\cdot\|_{\mathcal{F}}$, $\|\cdot\|_{\mathfrak{h}}$.

Our main result is the following.

Theorem 1.1. *Let F be as above, $\chi \in \mathrm{C}_0^\infty((-\infty, \Sigma))$ and $\mathrm{c} > 1$. For all $u \in X$, the evolution $u_t := e^{-\mathrm{i}tH}\chi(H)u$ obeys the estimates*

$$\left\| \mathrm{d}\Gamma\big(F(|y| \geq \mathrm{c}t)\big)^{\frac{1}{2}} u_t \right\| \lesssim t^{-\gamma} |||u|||,$$

where

$$\gamma < \min\left(\frac{1}{2}\left(1 - \frac{1}{\mathrm{c}}\right), \frac{1}{10}\right). \tag{1.6}$$

Thus $e^{-\mathrm{i}tH}\chi(H)u$ is supported asymptotically in the set $|y| \leq \mathrm{c}t$. In other words, photons do not propagate faster than the speed of light.

The estimate of Theorem 1.1 is usually called a *strong propagation estimate* in the literature (see [13,52]). In order to prove it, we first need to 'improve' the infrared behavior of the electron–photon interaction given by (1.2), which can be done, as usual, by performing a *Pauli–Fierz transformation*. For technical convenience, we use a generalized Pauli–Fierz transformation as in [50]. Next, we employ the method of propagation observables by constructing a positive, unbounded observable, whose Heisenberg derivative is negative (up to integrable remainder terms). In our proof, the required estimates on the remainder terms are obtained thanks to Hardy's inequality in $\mathbb{R}^3$, together with a suitable control of the growth of $\mathrm{d}\Gamma(|k|^{-\delta})$ along the evolution, for some $0 \leq \delta \leq 1$.

For massive Pauli–Fierz Hamiltonians (that is with a dispersion relation of the form $\omega(k) = \sqrt{k^2 + m^2}$, $m > 0$), a weak version of the maximal velocity estimate is derived in [14] (see also [19] for a different weak maximal velocity estimate). Compared to [14], the main difficulty we encounter is that, in our case, the number of photons operator is *not* relatively bounded with respect to the Hamiltonian. It is presently not known whether or not the number of photons remains bounded along the evolution (see, however, the recent paper [15] for the case of massless spin-boson model). Another difficulty here is due to the lack of smoothness of the relativistic dispersion relation $\omega(k) = |k|$ at the origin. In Quantum Mechanics, maximal and minimal velocity estimates were proven in [35,53]. The result of Theorem 1.1 is used in the proof of asymptotic completeness for Rayleigh scattering in [16].

Our paper is organized as follows. In Section 2, we introduce a generalized Pauli–Fierz transformation and prove our main theorem. Various ingredients of the proof of Theorem 1.1 are deferred to the next sections. In Section 3, we estimate interaction terms. Section 4 is devoted to the estimate of the growth of $\mathrm{d}\Gamma(|k|^{-\delta})$ along the evolution. In Section 5, we control remainder terms by estimating some commutators. Domain questions are discussed in Appendix A. Finally, for the convenience of the reader, standard definitions of operators in Fock space and some standard bound and commutator formulas are recalled in Appendix B, and our main notations are listed in Appendix C.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we use the generalized Pauli–Fierz transformation (see [50]) defined as follows. For any $h \in \mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$, we define the operator-valued field

$$\Phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)). \tag{2.1}$$

Using it, we can write

$$A(x) = \Phi(g_x), \qquad g_x(k,\lambda) := \frac{\kappa(k)}{|k|^{\frac{1}{2}}}\varepsilon_\lambda(k)e^{\mathrm{i}k\cdot x}. \tag{2.2}$$

Let $\varphi \in \mathrm{C}^\infty(\mathbb{R};\mathbb{R})$ be a non-decreasing function such that $\varphi(r) = r$ if $|r| \le 1/2$ and $|\varphi(r)| = 1$ if $|r| \ge 1$. For $0 < \mu < 1/2$, we define the function

$$q_x(k,\lambda) := -\frac{\kappa(k)}{|k|^{\frac{1}{2}+\mu}}\varphi(|k|^\mu \varepsilon_\lambda(k)\cdot x),$$

and the unitary operator

$$\mathcal{U} := e^{-\mathrm{i}\Phi(q_x)},$$

on $\mathrm{L}^2(\mathbb{R}^3;\mathcal{F})$. We also introduce the Pauli–Fierz transformed Hamiltonian $\widetilde{H}$ by $\widetilde{H} := \mathcal{U}H\mathcal{U}^*$. Using the properties of $\varepsilon_\lambda(k)$ and the relations (B.4) and (B.6) of Appendix B, we compute

$$\widetilde{H} = \big(p + \widetilde{A}(x)\big)^2 + E(x) + H_f + \widetilde{V}(x),$$

where

$$\widetilde{A}(x) := \Phi(\widetilde{g}_x), \qquad \widetilde{g}_x(k,\lambda) := g_x(k,\lambda) + \nabla_x q_x(k,\lambda), \tag{2.3}$$

$$E(x) := \Phi(e_x), \qquad e_x(k,\lambda) := \mathrm{i}|k|q_x(k,\lambda), \tag{2.4}$$

$$\widetilde{V}(x) := V(x) + \frac{1}{2}\sum_{\lambda=1,2}\int_{\mathbb{R}^3}|k||q_x(k,\lambda)|^2\mathrm{d}k. \tag{2.5}$$

The operator $\widetilde{H}$ is self-adjoint with domain $\mathcal{D}(\widetilde{H}) = \mathcal{D}(H) = \mathcal{D}(p^2 + H_f)$ (see Theorem A.1 in Appendix A).

The generalized Pauli–Fierz transformation is technically convenient since the coupling functions $q_x(k,\lambda)$, $\widetilde{g}_x(k,\lambda)$ and $e_x(k,\lambda)$ satisfy the estimates

$$|\partial_k^m q_x(k,\lambda)| \lesssim \kappa_m(k)|k|^{-\frac{1}{2}-|m|}\langle x\rangle^{1+|m|}, \tag{2.6}$$

$$|\partial_k^m \widetilde{g}_x(k,\lambda)| \lesssim \kappa_m(k)|k|^{\frac{1}{2}-|m|}\langle x\rangle^{\frac{1}{\mu}+|m|}, \tag{2.7}$$

$$|\partial_k^m e_x(k,\lambda)| \lesssim \kappa_m(k)|k|^{\frac{1}{2}-|m|}\langle x\rangle^{1+|m|}, \tag{2.8}$$

where $\kappa_m(k) \geq 0$ is compactly supported and bounds $\kappa(k)$ and all its derivatives up to the order $|m|$. These estimates will play an important role in our analysis. Eqs. (2.6) and (2.8) follow directly from the definition of q_x and e_x. To obtain (2.7) for $m = 0$, we use

$$\begin{aligned} |\widetilde{g}_x(k,\lambda)| &= \frac{|\kappa(k)|}{|k|^{\frac{1}{2}}}\left|e^{\mathrm{i}k\cdot x} - \varphi'(|k|^{\mu}\,\varepsilon_\lambda(k)\cdot x)\right| \\ &\leq \frac{|\kappa(k)|}{|k|^{\frac{1}{2}}}\Big(\left|e^{\mathrm{i}k\cdot x} - 1\right| + \left|1 - \varphi'(|k|^{\mu}\,\varepsilon_\lambda(k)\cdot x)\right|\Big), \end{aligned}$$

and the estimates $|e^{\mathrm{i}k\cdot x} - 1| \lesssim |k||x|$ and $|1 - \varphi'(|k|^{\mu}\,\varepsilon_\lambda(k)\cdot x)| \lesssim (|k|^{\mu}|x|)^r$ for all $r > 0$. The latter is implied by the property that $1 - \varphi'(|k|^{\mu}\varepsilon_\lambda(k)\cdot x) = 0$ for $|k|^{\mu}\varepsilon_\lambda(k)\cdot x \leq \frac{1}{2}$. Choosing $r = 1/\mu$, we arrive at (2.7) for $m = 0$. The case of $|m| > 0$ is treated similarly.

We define the Hilbert spaces $X_\delta := \mathcal{D}\big(\mathrm{d}\Gamma(|k|^{-\delta})^{\frac{1}{2}}\big)$, with the norms

$$\|u\|_\delta := \left\|\big(\mathrm{d}\Gamma(|k|^{-\delta}) + 1\big)^{\frac{1}{2}}u\right\|,$$

and $X_{\delta,\beta} := \mathcal{D}\big(\mathrm{d}\Gamma(|k|^{-\delta})^{\frac{1}{2}}\big) \cap \mathcal{D}\big(\mathrm{d}\Gamma(|y|^{2\beta})^{\frac{1}{2}}\big)$, with the norms

$$\|u\|_{\delta,\beta} := \left\|\big(\mathrm{d}\Gamma\big(|k|^{-\delta} + |y|^{2\beta}\big) + 1\big)^{\frac{1}{2}}u\right\|.$$

We shall prove the following theorem.

Theorem 2.1. *Let F be as in (1.5), $\chi \in \mathrm{C}_0^\infty((-\infty,\Sigma))$ and $\mathrm{c} > 1$. For all parameters β, γ, δ such that*

$$0 \leq 2\beta < \delta < 1, \tag{2.9}$$

$$0 \leq \gamma < \min\Big(\Big(1 - \frac{1}{\mathrm{c}}\Big)\beta, \frac{3\delta - 2}{10}\Big), \tag{2.10}$$

and for $u \in X_{\delta,\beta}$, the evolution $\tilde{u}_t := e^{-\mathrm{i}t\widetilde{H}}\chi(\widetilde{H})u$ satisfies

$$\left\|\mathrm{d}\Gamma\big(F(|y| \geq \mathrm{c}t)\big)^{\frac{1}{2}}\tilde{u}_t\right\| \lesssim t^{-\gamma}\|u\|_{\delta,\beta}. \tag{2.11}$$

We first verify that Theorem 2.1 implies Theorem 1.1 and next proceed to the proof of Theorem 2.1.

Proof of Theorem 1.1. For γ as in (1.6), we fix β and δ satisfying (2.9) and (2.10). Let $\widehat{\chi} \in \mathrm{C}_0^\infty((-\infty,\Sigma))$ be such that $\chi\widehat{\chi} = \chi$. We set $\widehat{u} := \widehat{\chi}(H)u$ and $\hat{u}_t := e^{-\mathrm{i}t\widetilde{H}}\chi(\widetilde{H})\mathcal{U}\widehat{u}$. We also recall the definition $u_t := e^{-\mathrm{i}tH}\chi(H)u$. Using the Pauli–Fierz transformation $\mathcal{U}$, we write

$$\left\|\mathrm{d}\Gamma(F(|v|))^{\frac{1}{2}}u_t\right\|^2 = \left\langle\hat{u}_t, \mathcal{U}\mathrm{d}\Gamma(F(|v|))\mathcal{U}^*\hat{u}_t\right\rangle.$$

Using the relation (B.6) of Appendix B, we compute

$$\mathcal{U}\mathrm{d}\Gamma(F(|v|))\mathcal{U}^* = \mathrm{d}\Gamma(F(|v|)) - \Phi(\mathrm{i}F(|v|)q_x) + \frac{1}{2}\operatorname{Re}\left\langle F(|v|)q_x, q_x\right\rangle_{\mathfrak{h}}. \tag{2.12}$$

We can estimate the second term given by (2.12) as

$$\left|\left\langle \hat{u}_t, \Phi(\mathrm{i}F(|v|)q_x)\hat{u}_t\right\rangle\right| \lesssim \left\| \Phi(\mathrm{i}F(|v|)q_x)\langle x\rangle^{-\tau_1}(H_f+1)^{-\frac{1}{2}}\right\| \\ \times\left\|(H_f+1)^{\frac{1}{2}}\langle x\rangle^{\tau_1}\chi(\widetilde{H})\right\|\|\widehat{u}\|^2. \tag{2.13}$$

Corollary 3.2 implies

$$\left\| \Phi(\mathrm{i}F(|v|)q_x)\langle x\rangle^{-\tau_1}(H_f+1)^{-\frac{1}{2}}\right\| \lesssim t^{-d_1},$$

with $0 \le d_1 < 1/2$ and $\tau_1 = 3/2 + d_1$. Moreover, since $\langle x\rangle$ and H_f commute, we obtain

$$\begin{aligned}\left\|(H_f+1)^{\frac{1}{2}}\langle x\rangle^{\tau_1}\chi(\widetilde{H})\right\|^2 &= \left\|\chi(\widetilde{H})(H_f+1)\langle x\rangle^{2\tau_1}\chi(\widetilde{H})\right\| \\ &\le \left\|(H_f+1)\chi(\widetilde{H})\right\|\left\|\langle x\rangle^{2\tau_1}\chi(\widetilde{H})\right\| \\ &\lesssim 1,\end{aligned} \tag{2.14}$$

where we used Theorem A.2 of Appendix A. Thus, (2.13) becomes

$$\left|\left\langle \hat{u}_t, \Phi(\mathrm{i}F(|v|)q_x)\hat{u}_t\right\rangle\right| \lesssim t^{-d_1}\|u\|^2. \tag{2.15}$$

Similarly, using Lemma 3.1 and Theorem A.2, the last term given by (2.12) is estimated as

$$\begin{aligned}\left|\left\langle \hat{u}_t, \mathrm{Re}\left\langle F(|v|)q_x, q_x\right\rangle_{\mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2)}\hat{u}_t\right\rangle\right| &\lesssim \left\|F(|v|)q_x(k,\lambda)\langle x\rangle^{-\tau_2}\right\|\left\|\langle x\rangle^{\tau_2}\chi(\widetilde{H})\right\|\|\widehat{u}\|^2 \\ &\lesssim t^{-d_2}\|u\|^2,\end{aligned} \tag{2.16}$$

with $0 \le d_2 < 1$ and $\tau_2 = 1 + d_2$. Now, by Theorem 2.1, we have

$$\left\langle \hat{u}_t, \mathrm{d}\Gamma(F(|v|))\hat{u}_t\right\rangle \lesssim t^{-2\gamma}\|\mathcal{U}\widehat{u}\|^2_{\delta,\beta}. \tag{2.17}$$

Therefore it remains to show that

$$\|\mathcal{U}\widehat{u}\|^2_{\delta,\beta} \lesssim \left\langle u, \left(\mathrm{d}\Gamma(\langle y\rangle)+1\right)u\right\rangle.$$

Using the definition of the norm $\|\mathcal{U}\widehat{u}\|_{\delta,\beta}$ and the relation (B.6) of Appendix B, we can compute as above

$$\mathcal{U}^*\left(\mathrm{d}\Gamma(|k|^{-\delta}) + \mathrm{d}\Gamma(|y|^{2\beta})+1\right)\mathcal{U} = \left(\mathrm{d}\Gamma(|k|^{-\delta}) + \mathrm{d}\Gamma(|y|^{2\beta})+1\right) \\ + \Phi\left(\mathrm{i}(|k|^{-\delta}+|y|^{2\beta})q_x\right) + \frac{1}{2}\mathrm{Re}\left\langle(|k|^{-\delta}+|y|^{2\beta})q_x, q_x\right\rangle_{\mathfrak{h}}. \tag{2.18}$$

Next, we use the standard Hardy's inequality: $\mathcal{D}(|y|^s) \subset \mathcal{D}(|k|^{-s})$, for all $0 \le s < 3/2$, and, for all $u \in \mathcal{D}(|y|^s)$,

$$\left\||k|^{-s}u\right\| \lesssim \left\||y|^s u\right\|. \tag{2.19}$$

(For $s = 1$, this is the refined uncertainty principle, see [28], and for $0 < s < 1$, this can be obtained by interpolation. For the general case, $s < 3/2$, see [29,30,8].) Since $0 \le 2\beta \le \delta < 1$, Hardy's inequality, together with Lemma B.3 of Appendix B, implies

$$\left\|\left(\mathrm{d}\Gamma(\langle y\rangle^\delta)+1\right)^{-\frac{1}{2}}\left(\mathrm{d}\Gamma(|k|^{-\delta}) + \mathrm{d}\Gamma(|y|^{2\beta})+1\right)\left(\mathrm{d}\Gamma(\langle y\rangle^\delta)+1\right)^{-\frac{1}{2}}\right\| \lesssim 1.$$

Besides, Lemma B.1 of Appendix B gives

$$\begin{aligned}\Big\| \Phi\big(\mathrm{i}|k|^{-\delta}q_x\big)(N+1)^{-\frac{1}{2}}\langle x\rangle^{-1}\psi\Big\|^2 &= \int_{\mathbb{R}^3} \big\| \Phi\big(\mathrm{i}|k|^{-\delta}q_x\big)(N+1)^{-\frac{1}{2}}\langle x\rangle^{-1}\psi(x)\big\|^2_{\mathcal{F}}\,dx \\ &\lesssim \int_{\mathbb{R}^3} \big\||k|^{-\delta}q_x(k,\lambda)\langle x\rangle^{-1}\big\|^2_{\mathfrak{h}}\|\psi(x)\|^2_{\mathcal{F}}\,dx \\ &\lesssim \sup_{x\in\mathbb{R}^3} \big\||k|^{-\delta}q_x(k,\lambda)\langle x\rangle^{-1}\big\|_{\mathfrak{h}}\|\psi\|^2, \end{aligned} \tag{2.20}$$

where $N := \mathrm{d}\Gamma(\mathbb{1})$ is the number operator. Using now (2.6) and $\delta < 1$, this yields

$$\Big\| \Phi\big(\mathrm{i}|k|^{-\delta}q_x\big)(N+1)^{-\frac{1}{2}}\langle x\rangle^{-1}\Big\| \lesssim 1.$$

The same way, (2.6) and $\delta < 1$ imply

$$\big\|\big\langle|k|^{-\delta}q_x, q_x\big\rangle_{\mathfrak{h}}\langle x\rangle^{-2}\big\| \lesssim \sup_{x\in\mathbb{R}^3} \big\||k|^{-\frac{\delta}{2}}q_x(k,\lambda)\langle x\rangle^{-1}\big\|^2_{\mathfrak{h}} \lesssim 1.$$

Similarly, by Lemma 3.1 (with $t = 1$) and Lemma B.1 of Appendix B, we have

$$\Big\| \Phi\big(\mathrm{i}|y|^{2\beta}q_x\big)(N+1)^{-\frac{1}{2}}\langle x\rangle^{-2}\Big\| \lesssim \sup_{x\in\mathbb{R}^3} \big\||y|^{2\beta}q_x(k,\lambda)\langle x\rangle^{-2}\big\|_{\mathfrak{h}} \lesssim 1,$$

$$\big\|\big\langle|y|^{2\beta}q_x, q_x\big\rangle_{\mathfrak{h}}\langle x\rangle^{-3}\big\| \lesssim \sup_{x\in\mathbb{R}^3} \big\||y|^{\beta}q_x(k,\lambda)\langle x\rangle^{-\frac{3}{2}}\big\|^2_{\mathfrak{h}} \lesssim 1,$$

since $0 < \beta < 1/2$. Combining (2.18), the previous estimates and an interpolation argument, we obtain

$$\begin{aligned}\Big\langle\widehat{u}, \mathcal{U}^*\big(\mathrm{d}\Gamma(|k|^{-\delta}) + \mathrm{d}\Gamma(|y|^{2\beta}) + 1\big)\mathcal{U}\widehat{u}\Big\rangle &\lesssim \big\langle\widehat{u}, \big(\mathrm{d}\Gamma(\langle y\rangle^{\delta}) + N + \langle x\rangle^4 + 1\big)\widehat{u}\big\rangle \\ &\lesssim \big\langle\widehat{u}, \big(\mathrm{d}\Gamma(\langle y\rangle^{\delta}) + \langle x\rangle^4 + 1\big)\widehat{u}\big\rangle.\end{aligned}$$

To conclude, it suffices to use that

$$\Big\|\mathrm{d}\Gamma(\langle y\rangle^{\delta})^{\frac{1}{2}}\widehat{\chi}(H)\big(\mathrm{d}\Gamma(\langle y\rangle) + 1\big)^{-\frac{1}{2}}\Big\| \lesssim 1,$$

by Proposition A.4 of Appendix A, and

$$\big\|\langle x\rangle^4\widehat{u}\big\| \lesssim \|u\|,$$

by Theorem A.2 of Appendix A. Then, (2.17) becomes

$$\Big\langle\hat{u}_t, \mathrm{d}\Gamma(F(|v|))\hat{u}_t\Big\rangle \lesssim t^{-2\gamma}\|u\|^2. \tag{2.21}$$

Eventually, Theorem 1.1 follows from (2.12) together with the estimates (2.15), (2.16) and (2.21). □

Proof of Theorem 2.1. We use the method of propagation observables. We construct a family of operators Φ_t (called a *propagation observable*) such that, on one hand, $\Phi_t \geq t^{2\gamma}\mathrm{d}\Gamma(F(|v|))$, and, on the other hand, the Heisenberg derivative

$$D\Phi_t := \partial_t\Phi_t - \mathrm{i}\big[\Phi_t, \widetilde{H}\big],$$

can be decomposed into a non-positive part and an integrable remainder term (plus possibly a term which can be treated by another observable). Recall the notation $\widetilde{u}_t = e^{-\mathrm{i}t\widetilde{H}}\chi(\widetilde{H})u$. Fix

β, γ, δ satisfying (2.9)–(2.10). We set

$$J_\beta(s) := s^\beta F(s^{\frac{1}{2}}) \in \mathrm{C}^\infty(\mathbb{R}). \tag{2.22}$$

The family Φ_t is defined, as a quadratic form on $\chi(\widetilde{H})\mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle^\beta))$, by

$$\Phi_t := t^{2\gamma}\mathrm{d}\Gamma\big(J_\beta(v^2)\big).$$

The fact that Φ_t is well-defined follows from $\beta < 1$, the bound

$$\Big\|\big(\mathrm{d}\Gamma(\langle y\rangle^\beta)+1\big)^{-1}\mathrm{d}\Gamma\big(J_\beta(v^2)\big)\big(\mathrm{d}\Gamma(\langle y\rangle^\beta)+1\big)^{-1}\Big\| \lesssim 1,$$

and Proposition A.4. We show below the following lemma.

Lemma 2.2. *Assume* $0 \le \beta < \delta < 1$, $0 \le \gamma < \min((1-1/\mathrm{c})\beta, 1/4)$ *and* $0 < \varepsilon < 1/2 - 2\gamma$. *In the sense of quadratic forms on* $\chi(\widetilde{H})\mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle^\beta))$,

$$\Phi_t \ge t^{2\gamma}\mathrm{d}\Gamma(F(|v|)), \tag{2.23}$$

and there exists $\mathrm{C} > 0$ *such that*

$$D\Phi_t \le -\frac{\theta}{t}\Phi_t + \mathrm{C}t^{-1-\delta+2\gamma}\mathrm{d}\Gamma(|k|^{-\delta}) + \mathrm{C}t^{-1-\varepsilon}, \tag{2.24}$$

where $\theta := 2((1-1/\mathrm{c})\beta - \gamma) > 0$.

Rewriting inequality (2.24) in terms of quadratic forms on the vectors $\widetilde{u}_t = e^{-\mathrm{i}t\widetilde{H}}\chi(\widetilde{H})u$ and using $\Phi_t \ge 0$ and $\langle \widetilde{u}_t, D\Phi_t\widetilde{u}_t\rangle = \partial_t\langle \widetilde{u}_t, \Phi_t\widetilde{u}_t\rangle$, we obtain

$$\partial_t\langle \widetilde{u}_t, \Phi_t\widetilde{u}_t\rangle \lesssim t^{-1-\delta+2\gamma}\big\langle \widetilde{u}_t, \mathrm{d}\Gamma(|k|^{-\delta})\widetilde{u}_t\big\rangle + t^{-1-\varepsilon}\|u\|^2.$$

It then follows from Lemma 4.1 that

$$\partial_t\langle \widetilde{u}_t, \Phi_t\widetilde{u}_t\rangle \lesssim t^{-\frac{3}{5}(1+\delta)+2\gamma}\|u\|_\delta^2 + t^{-1-\varepsilon}\|u\|^2.$$

Assuming $3\delta > 10\gamma + 2$, this yields

$$\partial_t\langle \widetilde{u}_t, \Phi_t\widetilde{u}_t\rangle \lesssim t^{-1-\widetilde{\varepsilon}}\|u\|_\delta^2,$$

for some $\widetilde{\varepsilon} > 0$. Integrating this inequality from 1 to t, this implies

$$\langle \widetilde{u}_t, \Phi_t\widetilde{u}_t\rangle \le \big\langle \widetilde{u}_{t=1}, \Phi_{t=1}\widetilde{u}_{t=1}\big\rangle + \mathrm{C}\|u\|_\delta^2.$$

Combined with (2.23) and the fact that

$$\Phi_{t=1} := \mathrm{d}\Gamma\Big(\Big(\frac{|y|}{\mathrm{c}}\Big)^{2\beta}F\Big(\frac{|y|}{\mathrm{c}}\Big)\Big) \lesssim \mathrm{d}\Gamma\big(|y|^{2\beta}\big),$$

which follows from the definition of Φ_t and Lemma B.3, this gives the desired inequality (2.11). This completes the proof of Theorem 2.1. □

Proof of Lemma 2.2. Estimate (2.23) is straightforward. To prove (2.24), we start with computing $D\Phi_t$. The relations below are understood in the sense of quadratic forms on $\chi(\widetilde{H})\mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle^\beta))$. With $J_\beta = J_\beta(v^2)$ defined in (2.22), and the notation $p_{\widetilde{A}} := p + \widetilde{A}(x)$, we compute

$$D\Phi_t = 2t^{2\gamma-1}\,\mathrm{d}\Gamma\big(\gamma J_\beta - v^2 J_\beta'\big) \tag{2.25}$$

$$-t^{2\gamma}\big[\mathrm{d}\Gamma\big(J_\beta\big), \mathrm{i}\mathrm{d}\Gamma(|k|)\big] \tag{2.26}$$

$$-t^{2\gamma}\Big[\mathrm{d}\Gamma\big(J_\beta\big), \mathrm{i}p_{\widetilde{A}}^2 + \mathrm{i}E(x)\Big]. \tag{2.27}$$

Consider the term given by (2.26). We have (see (B.2) of Appendix B)

$$\big[\mathrm{d}\Gamma\big(J_\beta\big), \mathrm{i}\mathrm{d}\Gamma(|k|)\big] = \mathrm{d}\Gamma\big(\big[J_\beta, \mathrm{i}|k|\big]\big),$$

and it follows from Lemma 5.2 that

$$\big[J_\beta, \mathrm{i}|k|\big] = \frac{1}{(\mathrm{c}t)^2}(J'_\beta)^{\frac{1}{2}} b (J'_\beta)^{\frac{1}{2}} + \mathcal{R}, \tag{2.28}$$

where $b := y \cdot \widehat{k} + \widehat{k} \cdot y$, $\widehat{k} := k/|k|$ and

$$\big\| |k|^{\frac{\delta}{2}} \mathcal{R} |k|^{\frac{\delta}{2}} \big\| \lesssim t^{-1-\delta}, \tag{2.29}$$

for all $\beta < \delta \le 1$. Observe that for all $w \in \mathcal{D}(|v|^\beta) = \mathcal{D}(|y|^\beta)$,

$$\begin{aligned} -\Big\langle w, (J'_\beta)^{\frac{1}{2}} b (J'_\beta)^{\frac{1}{2}} w \Big\rangle &\le 2 \Big\| |y| (J'_\beta)^{\frac{1}{2}} w \Big\| \Big\| (J'_\beta)^{\frac{1}{2}} w \Big\| \\ &\le 2\mathrm{c}t \Big\| |v| (J'_\beta)^{\frac{1}{2}} w \Big\|^2, \end{aligned}$$

since $\mathrm{supp}(J'_\beta) \subset [1, \infty)$. This gives

$$-\,\mathrm{d}\Gamma\Big((J'_\beta)^{\frac{1}{2}} b (J'_\beta)^{\frac{1}{2}}\Big) \le 2\mathrm{c}t\,\mathrm{d}\Gamma\big(v^2 J'_\beta\big). \tag{2.30}$$

Combining (2.28) with (2.29) and (2.30), we obtain

$$-\,[\mathrm{d}\Gamma(J_\beta), \mathrm{i}\mathrm{d}\Gamma(|k|)] \le \frac{2}{\mathrm{c}t}\mathrm{d}\Gamma\big(v^2 J'_\beta\big) - \mathrm{C}t^{-1-\delta}\mathrm{d}\Gamma(|k|^{-\delta}). \tag{2.31}$$

It remains to estimate the term (2.27). Using the relation $\mathrm{i}[\mathrm{d}\Gamma(b), \Phi(g)] = \Phi(\mathrm{i}bg)$ (see (B.5) of Appendix B), we compute

$$(2.27) = t^{2\gamma} p_{\widetilde{A}} \cdot \Phi\big(\mathrm{i}J_\beta \widetilde{g}_x\big) + t^{2\gamma} \Phi\big(\mathrm{i}J_\beta \widetilde{g}_x\big) \cdot p_{\widetilde{A}} + \mathrm{i}t^{2\gamma} \Phi\big(\mathrm{i}J_\beta e_x\big).$$

Since $\mathcal{D}(\widetilde{H}) \subset \mathcal{D}(p) \cap \mathcal{D}(\widetilde{A})$, we have

$$\big\| p_{\widetilde{A}} \mathbb{1}_{\mathrm{supp}(\chi)}(\widetilde{H}) \big\| \lesssim 1.$$

Moreover, it follows from Corollary 3.2 and an estimate similar to (2.14) that

$$\begin{aligned} \big\| \Phi\big(\mathrm{i}J_\beta \widetilde{g}_x\big) \mathbb{1}_{\mathrm{supp}(\chi)}(\widetilde{H}) \big\| &\lesssim \Big\| \Phi\big(\mathrm{i}J_\beta \widetilde{g}_x\big) \langle x \rangle^{-\tau_2} (H_f + 1)^{-\frac{1}{2}} \Big\| \\ &\quad \times \Big\| (H_f + 1)^{\frac{1}{2}} \langle x \rangle^{\tau_2} \mathbb{1}_{\mathrm{supp}(\chi)}(\widetilde{H}) \Big\| \\ &\lesssim t^{-d}, \end{aligned}$$

for $0 \le d < 3/2$ and $\tau_2 = 1/2 + \mu^{-1} + 2\beta + d$. Likewise,

$$\big\| \Phi\big(\mathrm{i}J_\beta e_x\big) \mathbb{1}_{\mathrm{supp}(\chi)}(\widetilde{H}) \big\| \lesssim t^{-d}.$$

Then, the previous estimates imply

$$\| \mathbb{1}_{\mathrm{supp}(\chi)}(\widetilde{H}) (2.27) \mathbb{1}_{\mathrm{supp}(\chi)}(\widetilde{H}) \| \lesssim t^{-1-\varepsilon}, \tag{2.32}$$

for all $0 < \varepsilon < 1/2 - 2\gamma$.

The estimates (2.31) and (2.32), together with (2.25)–(2.27), imply

$$D\Phi_t \le 2t^{2\gamma-1}\mathrm{d}\Gamma\Big(\gamma J_\beta - v^2 J'_\beta + \frac{1}{\mathrm{c}} v^2 J'_\beta\Big) - \mathrm{C}t^{2\gamma-1-\delta}\mathrm{d}\Gamma(|k|^{-\delta}) - \mathrm{C}t^{-1-\varepsilon},$$

as a quadratic form on $\chi(\widetilde{H})\mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle^{\frac{\beta}{2}}))$. Using

$$v^2 J_\beta' = \beta J_\beta + \frac{1}{2}|v|^{2\beta+1} F'(|v|) \geq \beta J_\beta,$$

this becomes

$$D\Phi_t \leq -\theta t^{-1}\Phi_t - \mathrm{C}t^{2\gamma-1-\delta}\mathrm{d}\Gamma(|k|^{-\delta}) - \mathrm{C}t^{-1-\varepsilon}, \tag{2.33}$$

which concludes the proof of the lemma. □

3. Estimates on interaction

In this section we prove estimates on the interaction used, in particular, to prove (2.32). Recall that $\kappa \in \mathrm{C}_0^\infty(\mathbb{R}^3)$ is the ultraviolet cut-off entering (1.2) and the cut-off operator $F(|v|)$ is defined in (1.5).

Lemma 3.1. *Let $a \in [0, 3/2)$, $b \in \mathbb{R}$, $c \geq 0$, $\kappa \in \mathrm{C}_0^\infty(\mathbb{R}^3)$ and $\rho_x^b(k)$ be such that, for all $m \in \mathbb{N}^3$, $|\partial_k^m \rho_x^b(k)| \lesssim |k|^{b-|m|}\langle x\rangle^{|m|}$. Assume that $b > a + c - 3/2$. Then, for all $d \in [0, b - a - c + 3/2)$,*

$$\forall x \in \mathbb{R}^3, \quad \big\| |k|^{-a}|y|^c F(|v|)\kappa(k)\rho_x^b(k)\big\|_{L^2(\mathbb{R}_k^3)} \lesssim t^{-d}\langle x\rangle^{a+c+d}.$$

Proof. Let $\ell_x(k) = \kappa(k)\rho_x^b(k)$. Using Hardy's inequality, (2.19), we can write

$$\begin{aligned}\big\| |k|^{-a}|y|^c F(|v|)\ell_x(k)\big\| &\lesssim \big\| |y|^{a+c} F(|v|)\ell_x(k)\big\| \\ &\lesssim \big\| F(|v|)|y|^{-d}\big\|_{L^\infty} \big\| |y|^{a+c+d}\ell_x(k)\big\| \\ &\lesssim t^{-d}\big\| |y|^{a+c+d}\ell_x(k)\big\|.\end{aligned} \tag{3.1}$$

Next, to handle fractional derivatives $|y|^s$, we use a dyadic decomposition of κ. More precisely, let $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^3 \setminus \{0\})$ be such that

$$\forall k \in \operatorname{supp}(\kappa), \quad \sum_{\nu \in \mathcal{V}} \varphi(\nu k) = 1, \tag{3.2}$$

where $\mathcal{V} = \{2^{-j};\ j \in \mathbb{N}\}$ is the set of dyadic numbers. For $n \in \mathbb{N}$, we have

$$\big\| |y|^n \varphi(\nu k)\ell_x(k)\big\| \lesssim \sum_{i_1,\dots,i_n \in \{1,2,3\}} \big\| y_{i_1}\cdots y_{i_n}\varphi(\nu k)\ell_x(k)\big\|,$$

and $y_{i_1}\cdots y_{i_n}\varphi(\nu k)\ell_x(k)$ can be written as a finite sum of terms of the form

$$w = \nu^\alpha \widetilde{\varphi}(\nu k)\widetilde{\kappa}(k)\widetilde{\rho}_x^{b-\beta}(k)\langle x\rangle^\beta,$$

where $\alpha, \beta \in \mathbb{N}$ with $\alpha + \beta \leq n$, $\widetilde{\varphi} \in \mathrm{C}_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $\widetilde{\kappa} \in \mathrm{C}_0^\infty(\mathbb{R}^3)$ and $\widetilde{\rho}_x^{b-\beta}$ is such that $|\widetilde{\rho}_x^{b-\beta}(k)| \lesssim |k|^{b-\beta}$. Then,

$$\|w\| \lesssim \nu^{\alpha+\beta-b}\langle x\rangle^\beta \|\widetilde{\varphi}(\nu k)\| \lesssim \nu^{\alpha+\beta-b-\frac{3}{2}}\langle x\rangle^\beta \leq \nu^{n-b-\frac{3}{2}}\langle x\rangle^n.$$

This gives

$$\big\| |y|^n \varphi(\nu k)\ell_x(k)\big\| \lesssim \nu^{n-b-\frac{3}{2}}\langle x\rangle^n.$$

Now, an interpolation argument implies that, for all $s \geq 0$,

$$\big\| |y|^s \varphi(\nu k)\ell_x(k)\big\| \lesssim \nu^{s-b-\frac{3}{2}}\langle x\rangle^s. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\begin{aligned}\big\| |y|^s \ell_x(k) \big\| &\le \sum_{\nu \in \mathcal{V}} \big\| |y|^s \varphi(\nu k) \ell_x(k) \big\| \\ &\lesssim \sum_{\nu \in \mathcal{V}} \nu^{s-b-\frac{3}{2}} \langle x \rangle^s \lesssim \langle x \rangle^s, \end{aligned} \tag{3.4}$$

provided that $b + 3/2 - s > 0$. Taking $s = a + c + d$ and $d \in [0, b - a - c + 3/2)$ and recalling (3.1), we arrive at the statement of the lemma. □

Recall that $v = y/ct$ and that the coupling functions $q_x, \widetilde{g}_x$ and e_x are defined at the beginning of Section 2 and satisfy (2.6)–(2.8).

Corollary 3.2. *For all $0 < \mu < 1/2$, $0 \le \beta \le 1/2$ and $\varepsilon > 0$,*

$$\Big\| \Phi\big(\mathrm{i}F(|v|)q_x\big) \langle x \rangle^{-\tau_1} (H_f + 1)^{-\frac{1}{2}} \Big\| \lesssim t^{-d}, \quad 0 \le d < \frac{1}{2}, \tag{3.5}$$

$$\Big\| \Phi\big(\mathrm{i}|v|^{2\beta} F(|v|)\widetilde{g}_x\big) \langle x \rangle^{-\tau_2} (H_f + 1)^{-\frac{1}{2}} \Big\| \lesssim t^{-d}, \quad 0 \le d < \frac{3}{2}, \tag{3.6}$$

$$\Big\| \Phi\big(\mathrm{i}|v|^{2\beta} F(|v|)e_x\big) \langle x \rangle^{-\tau_3} (H_f + 1)^{-\frac{1}{2}} \Big\| \lesssim t^{-d}, \quad 0 \le d < \frac{3}{2}, \tag{3.7}$$

where $\tau_1 = 3/2 + d$, $\tau_2 = 1/2 + \mu^{-1} + 2\beta + d$ and $\tau_3 = 3/2 + 2\beta + d$.

Proof. It follows from Lemma B.2 of Appendix B that, for all $u \in \mathcal{H} = \mathrm{L}^2(\mathbb{R}^3; \mathcal{F})$,

$$\begin{aligned}\Big\| \Phi\big(\mathrm{i}F(|v|)q_x\big) \langle x \rangle^{-\tau_1} (H_f + 1)^{-\frac{1}{2}} u \Big\|^2 \lesssim \int_{\mathbb{R}^3} \langle x \rangle^{-2\tau_1} \Big(\big\| |k|^{-\frac{1}{2}} F(|v|) q_x(k,\lambda) \big\|_{\mathfrak{h}}^2 \\ + \big\| F(|v|) q_x(k,\lambda) \big\|_{\mathfrak{h}}^2 \Big) \|u(x)\|_{\mathcal{F}}^2 \, \mathrm{d}x. \end{aligned} \tag{3.8}$$

Using (2.6) and applying Lemma 3.1 with $a = 1/2$, $b = -1/2$, $c = 0$ to the first term on the right hand side, and with $a = 0$, $b = -1/2$, $c = 0$ to the second term, we obtain

$$\Big\| \Phi\big(\mathrm{i}F(|v|)q_x\big) \langle x \rangle^{-\tau_1} (H_f + 1)^{-\frac{1}{2}} u \Big\|^2 \lesssim t^{-2d} \int_{\mathbb{R}^3} \|u(x)\|_{\mathcal{F}}^2 \, \mathrm{d}x = t^{-2d} \|u\|^2, \tag{3.9}$$

which gives (3.5). To prove (3.6) or (3.7), we proceed as above, applying Lemma 3.1 with $a = 1/2$, $b = 1/2$, $c = 2\beta$ and with $a = 0$, $b = 1/2$, $c = 2\beta$. □

4. Control of small momenta

In this section we estimate the growth of $\mathrm{d}\Gamma(|k|^{-\delta})$ (for $-1 < \delta < 3/2$) along the evolution, which was used in the proof of Theorem 2.1. The proof of the following lemma is similar to [24, (4.8)].

Lemma 4.1. *Let $-1 < \delta < 3/2$ and $\chi \in \mathrm{C}_0^\infty((-\infty, \Sigma))$. Then, for all $u \in X_\delta$, the evolution $\tilde{u}_t := e^{-\mathrm{i}t\widetilde{H}} \chi(\widetilde{H}) u$ satisfies the estimates*

$$\Big\langle \tilde{u}_t, \mathrm{d}\Gamma(|k|^{-\delta}) \tilde{u}_t \Big\rangle \lesssim t^{\frac{2(1+\delta)}{5}} \|u\|_\delta^2.$$

Proof. Let $h \in \mathrm{C}^\infty([0,\infty); \mathbb{R})$ be a decreasing function such that $h(s) = 1$ on $[0, 1]$ and $h(s) = 0$ on $[2, +\infty)$, and let $\bar{h} = \mathbb{1} - h$. For $\nu > 0$, we decompose

$$\mathrm{d}\Gamma(|k|^{-\delta}) = \mathrm{d}\Gamma\big(|k|^{-\delta}h(t^\nu|k|)\big) + \mathrm{d}\Gamma\big(|k|^{-\delta}\bar{h}(t^\nu|k|)\big). \tag{4.1}$$

The contribution of the second term of (4.1) is estimated as

$$\begin{aligned}\big\langle\widetilde{u}_t, \mathrm{d}\Gamma\big(|k|^{-\delta}\bar{h}(t^\nu|k|)\big)\widetilde{u}_t\big\rangle &\le t^{(1+\delta)\nu}\big\langle\widetilde{u}_t, \mathrm{d}\Gamma\big(|k|\bar{h}(t^\nu|k|)\big)\widetilde{u}_t\big\rangle \\ &\lesssim t^{(1+\delta)\nu}\|u\|^2,\end{aligned} \tag{4.2}$$

since $\mathrm{d}\Gamma(|k|\bar{h}(t^\nu|k|))\chi(\widetilde{H})$ is bounded. To estimate the first term, we use the propagation observable

$$\Psi_t := \mathrm{d}\Gamma\big(|k|^{-\delta}h(t^\nu|k|)\big),$$

and compute

$$\begin{aligned}D\Psi_t = \partial_t\Psi_t - \mathrm{i}\big[\Psi_t, \widetilde{H}\big] &= \nu t^{\nu-1}\mathrm{d}\Gamma\big(|k|^{1-\delta}h'(t^\nu|k|)\big) - \big[\Psi_t, \mathrm{i}\widetilde{H}\big] \\ &\le -\big[\Psi_t, \mathrm{i}\widetilde{H}\big],\end{aligned}$$

since $h' \le 0$. Using (B.2), (B.5) of Appendix B and the notation $p_{\widetilde{A}} = p + \widetilde{A}(x)$, the commutator above can be expressed as follows

$$\big[\Psi_t, \mathrm{i}\widetilde{H}\big] = p_{\widetilde{A}}\cdot\Phi\big(|k|^{-\delta}h(t^\nu|k|)\widetilde{g}_x\big) + \Phi\big(\mathrm{i}|k|^{-\delta}h(t^\nu|k|)\widetilde{g}_x\big)\cdot p_{\widetilde{A}} + \mathrm{i}\Phi\big(\mathrm{i}|k|^{-\delta}h(t^\nu|k|)e_x\big).$$

As in (2.20), using (2.7) and Lemma B.2 of Appendix B, we find that

$$\begin{aligned}\Big\|\Phi\big(\mathrm{i}|k|^{-\delta}h(t^\nu|k|)\widetilde{g}_x\big)(H_f+1)^{-\frac12}\langle x\rangle^{-\frac1\mu}\Big\| &\le \sup_{x\in\mathbb{R}^3}\big\||k|^{-\delta}h(t^\nu|k|)\widetilde{g}_x(k,\lambda)\langle x\rangle^{-\frac1\mu}\big\|_{\mathfrak{h}} \\ &\quad + \sup_{x\in\mathbb{R}^3}\big\||k|^{-\delta-\frac12}h(t^\nu|k|)\widetilde{g}_x(k,\lambda)\langle x\rangle^{-\frac1\mu}\big\|_{\mathfrak{h}} \\ &\lesssim \big\||k|^{-\delta}h(t^\nu|k|)\kappa(k)\big\|_{\mathfrak{h}} \\ &\lesssim t^{-(\frac32-\delta)\nu}.\end{aligned}$$

Likewise, using (2.8) and Lemma B.2, we obtain

$$\begin{aligned}\Big\|\Phi\big(\mathrm{i}|k|^{-\delta}h(t^\nu|k|)e_x\big)(H_f+1)^{-\frac12}\langle x\rangle^{-1}\Big\| &\le \sup_{x\in\mathbb{R}^3}\big\||k|^{-\delta}h(t^\nu|k|)e_x(k,\lambda)\langle x\rangle^{-1}\big\|_{\mathfrak{h}} \\ &\quad + \sup_{x\in\mathbb{R}^3}\big\||k|^{-\delta-\frac12}h(t^\nu|k|)e_x(k,\lambda)\langle x\rangle^{-1}\big\|_{\mathfrak{h}} \\ &\lesssim t^{-(\frac32-\delta)\nu}.\end{aligned}$$

The last two inequalities, an estimate similar to (2.14), $\|p_{\widetilde{A}}\chi(\widetilde{H})\| \lesssim 1$ and $\partial_t\langle\widetilde{u}_t, \Psi_t\widetilde{u}_t\rangle = \langle\widetilde{u}_t, D\Psi_t\widetilde{u}_t\rangle$ imply

$$\partial_t\langle\widetilde{u}_t, \Psi_t\widetilde{u}_t\rangle \lesssim t^{-(\frac32-\delta)\nu}\|u\|^2.$$

Hence, assuming $(\frac32-\delta)\nu < 1$, we obtain

$$\langle\widetilde{u}_t, \Psi_t\widetilde{u}_t\rangle \lesssim t^{-(\frac32-\delta)\nu+1}\|u\|^2 + \big\|\mathrm{d}\Gamma(|k|^{-\delta})^{\frac12}u\big\|^2. \tag{4.3}$$

The statement of the lemma follows from (4.1)–(4.3), after choosing $\nu = 2/5$. $\square$

5. Some commutator estimates

In this part, we estimate some commutators appearing in Section 2. As usual, for $\rho \in \mathbb{R}$, we define the set of functions

$$S^{\rho}(\mathbb{R}) := \{ f \in C^{\infty}(\mathbb{R});\ \left|\partial_s^n f(s)\right| \leq C_n \langle s\rangle^{\rho-n} \text{ for } n \geq 0\}. \tag{5.1}$$

Recall the notations $v = y/ct$ and $b = y \cdot \widehat{k} + \widehat{k} \cdot y$.

Lemma 5.1. *Let $G \in S^{\rho}(\mathbb{R})$ with $\rho < 0$ and $\max(1+2\rho, 0) < \delta \leq 1$. We have*

$$\left\| |k|^{\frac{\delta}{2}} \big[G(v^2), b\big] |k|^{\frac{\delta}{2}} \right\| \lesssim t^{1-\delta}.$$

Proof. Let $\widetilde{G}$ denote an almost analytic extension of G. This means that $\widetilde{G}$ is a C^{∞} function on $\mathbb{C}$ such that $\widetilde{G}|_{\mathbb{R}} = G$,

$$\operatorname{supp} \widetilde{G} \subset \{ z \in \mathbb{C};\ |\operatorname{Im} z| \leq C\langle \operatorname{Re} z\rangle \}, \tag{5.2}$$

$|\widetilde{G}(z)| \leq C\langle \operatorname{Re} z\rangle^{\rho}$ and, for all $n \in \mathbb{N}$,

$$\left|\frac{\partial \widetilde{G}}{\partial \bar{z}}(z)\right| \leq C_n \langle \operatorname{Re} z\rangle^{\rho-1-n} |\operatorname{Im} z|^n. \tag{5.3}$$

Moreover, if G is compactly supported, we can assume that this is also the case for $\widetilde{G}$. Using the Helffer–Sjöstrand formula (see e.g. [13,34])

$$G(v^2) = \frac{1}{\pi} \int \frac{\partial \widetilde{G}(z)}{\partial \bar{z}} (v^2 - z)^{-1} \, d \operatorname{Re} z \, d \operatorname{Im} z,$$

we can write

$$\begin{aligned} \big[G(v^2), b\big] &= \frac{1}{\pi} \int \frac{\partial \widetilde{G}(z)}{\partial \bar{z}} \big[(v^2 - z)^{-1}, b\big] \, d \operatorname{Re} z \, d \operatorname{Im} z \\ &= -\frac{1}{\pi c^2 t^2} \int \frac{\partial \widetilde{G}(z)}{\partial \bar{z}} (v^2 - z)^{-1} \big[y^2, b\big] (v^2 - z)^{-1} \, d \operatorname{Re} z \, d \operatorname{Im} z. \end{aligned} \tag{5.4}$$

Let us first prove that

$$(v^2 - z)^{-1} \big[y^2, b\big] (v^2 - z)^{-1} |k| = \mathcal{O}\big(t^2 |z|^2 |\operatorname{Im} z|^{-3}\big). \tag{5.5}$$

A direct calculation gives

$$\begin{aligned} \frac{1}{i}\big[y^2, b\big] &= y^2 |k|^{-1} + |k|^{-1} y^2 + 2 \sum_i y_i |k|^{-1} y_i \\ &\quad - \sum_{i,j} y_i y_j \big(k_i k_j |k|^{-3}\big) + 2 y_i \big(k_i k_j |k|^{-3}\big) y_j + \big(k_i k_j |k|^{-3}\big) y_i y_j. \end{aligned} \tag{5.6}$$

Using Hardy's inequality (see (2.19)) and the functional calculus, we get

$$\begin{aligned} (v^2 - z)^{-1} |k| &= |k| (v^2 - z)^{-1} - \frac{i}{t^2} (v^2 - z)^{-1} \big(2\widehat{k} \cdot y + 2i|k|^{-1}\big) (v^2 - z)^{-1} \\ &= |k| \mathcal{O}\big(|z| |\operatorname{Im} z|^{-2}\big), \end{aligned} \tag{5.7}$$

$$\begin{aligned}
y_i(v^2-z)^{-1}|k| &= |k|y_i(v^2-z)^{-1} + \mathrm{i}\widehat{k}_i(v^2-z)^{-1} \\
&\quad - \frac{\mathrm{i}}{t^2} y_i(v^2-z)^{-1}\big(2\widehat{k}\cdot y + 2\mathrm{i}|k|^{-1}\big)(v^2-z)^{-1} \\
&= |k|\mathcal{O}\big(t|z|^{\frac{3}{2}}|\operatorname{Im} z|^{-2}\big), \\
y_i y_j(v^2-z)^{-1}|k| &= |k|y_i y_j(v^2-z)^{-1} \\
&\quad + \big(\mathrm{i}\widehat{k}_i y_j + \mathrm{i}\widehat{k}_j y_i + k_i k_j|k|^{-3} - \delta_{i,j}|k|^{-1}\big)(v^2-z)^{-1} \\
&\quad - \frac{\mathrm{i}}{t^2} y_i y_j(v^2-z)^{-1}\big(2\widehat{k}\cdot y + 2\mathrm{i}|k|^{-1}\big)(v^2-z)^{-1} \\
&= |k|\mathcal{O}\big(t^2|z||\operatorname{Im} z|^{-1}\big) + \mathcal{O}\big(t|z|^{\frac{3}{2}}|\operatorname{Im} z|^{-2}\big),
\end{aligned}$$

and

$$\begin{aligned}
&(v^2-z)^{-1}y_i y_j = \mathcal{O}\big(t^2|z||\operatorname{Im} z|^{-1}\big), \qquad (v^2-z)^{-1}y_i = \mathcal{O}\big(t|z|^{\frac{1}{2}}|\operatorname{Im} z|^{-1}\big), \\
&(v^2-z)^{-1}|k|^{-1} = \mathcal{O}\big(t|z|^{\frac{1}{2}}|\operatorname{Im} z|^{-1}\big), \qquad (v^2-z)^{-1} = \mathcal{O}\big(|\operatorname{Im} z|^{-1}\big).
\end{aligned}$$

Combining (5.6) with the previous estimates, we obtain (5.5).

Now, using again (5.6) and the previous estimates, one easily verifies that

$$(v^2-z)^{-1}\big[y^2, b\big](v^2-z)^{-1} = \mathcal{O}\big(t^3|z|^{\frac{3}{2}}|\operatorname{Im} z|^{-2}\big). \tag{5.8}$$

By an interpolation argument, we then obtain from (5.5) (and its adjoint) and (5.8) that

$$|k|^{\frac{\delta}{2}}(v^2-z)^{-1}\big[y^2, b\big](v^2-z)^{-1}|k|^{\frac{\delta}{2}} = \mathcal{O}\big(t^{3-\delta}|z|^{\frac{3}{2}+\frac{\delta}{2}}|\operatorname{Im} z|^{-2-\delta}\big), \tag{5.9}$$

for all $0 \le \delta \le 1$.

Introducing (5.9) into (5.4) gives

$$\begin{aligned}
\big\||k|^{\frac{\delta}{2}}\big[G(v^2), b\big]|k|^{\frac{\delta}{2}}u\big\| &\lesssim t^{1-\delta}\int \left|\frac{\partial \widetilde{G}(z)}{\partial \bar{z}}\right| |z|^{\frac{3}{2}+\frac{\delta}{2}}|\operatorname{Im} z|^{-2-\delta}\|u\|\, \mathrm{d}\operatorname{Re} z\, \mathrm{d}\operatorname{Im} z \\
&\lesssim t^{1-\delta}\int \langle \operatorname{Re} z\rangle^{-\frac{1}{2}+\rho-\frac{\delta}{2}}\|u\|\, \mathrm{d}\operatorname{Re} z \lesssim t^{1-\delta}\|u\|,
\end{aligned} \tag{5.10}$$

provided that $\delta > 1 + 2\rho$, which concludes the proof of the lemma. □

Lemma 5.2. *Let $G \in \mathrm{S}^{\rho}(\mathbb{R})$ with $\rho < 1$ and $\max(2\rho - 1, 0) < \delta \le 1$. We have*

$$\big[G(v^2), \mathrm{i}|k|\big] = \frac{1}{\mathrm{c}t}G'(v^2)b + \mathcal{R},$$

as a quadratic form on $\mathcal{D}(|y|^{2\rho}) \cap D(|k|)$, with

$$\big\||k|^{\frac{\delta}{2}}\mathcal{R}|k|^{\frac{\delta}{2}}\big\| \lesssim t^{-1-\delta}.$$

Proof. Since ρ may be non-negative, we cannot directly express $G(v^2)$ with the Helffer–Sjöstrand formula. Therefore, we use an artificial cut-off. Consider $\varphi \in \mathrm{C}_0^{\infty}(\mathbb{R}; [0,1])$ equal to 1 near 0 and $\varphi_R(\cdot) = \varphi(\cdot/R)$ for $R > 0$. Let $\widetilde{G}$ (resp. $\widetilde{\varphi} \in \mathrm{C}_0^{\infty}(\mathbb{C})$) be an almost analytic extension of G (resp. φ) as in (5.2)–(5.3). Then, as a quadratic form on $\mathcal{D}(|y|^{2\rho}) \cap D(|k|)$,

$$\big[G(v^2), \mathrm{i}|k|\big] = \operatorname*{s-lim}_{R\to\infty}\big[(\varphi_R G)(v^2), \mathrm{i}|k|\big], \tag{5.11}$$

where

$$\begin{aligned}\big[(\varphi_R G)(v^2), \mathrm{i}|k|\big] &= \frac{1}{\pi}\int \frac{\partial(\widetilde{\varphi}_R\widetilde{G})(z)}{\partial\bar{z}}\big[(v^2-z)^{-1}, \mathrm{i}|k|\big]\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z\\ &= -\frac{1}{\pi}\int \frac{\partial(\widetilde{\varphi}_R\widetilde{G})(z)}{\partial\bar{z}}(v^2-z)^{-1}\big[v^2, \mathrm{i}|k|\big](v^2-z)^{-1}\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z\\ &= \frac{1}{\pi(\mathrm{c}t)^2}\int \frac{\partial(\widetilde{\varphi}_R\widetilde{G})(z)}{\partial\bar{z}}(v^2-z)^{-1}b(v^2-z)^{-1}\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z\\ &= \frac{1}{(\mathrm{c}t)^2}(\varphi_R G)'(v^2)b + \mathcal{R}_R, \end{aligned}\tag{5.12}$$

and

$$\begin{aligned}\mathcal{R}_R &= \frac{1}{\pi(\mathrm{c}t)^2}\int \frac{\partial(\widetilde{\varphi}_R\widetilde{G})(z)}{\partial\bar{z}}(v^2-z)^{-1}\big[b, (v^2-z)^{-1}\big]\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z\\ &= \frac{1}{\pi(\mathrm{c}t)^5}\int \frac{\partial(\widetilde{\varphi}_R\widetilde{G})(z)}{\partial\bar{z}}(v^2-z)^{-2}\big[y^2, b\big](v^2-z)^{-1}\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z. \end{aligned}\tag{5.13}$$

From (5.5), (5.7) and (5.8), we obtain

$$\begin{aligned}&(v^2-z)^{-2}\big[y^2, b\big](v^2-z)^{-1}|k| = \mathcal{O}\big(t^2|z|^2|\,\mathrm{Im}\,z|^{-4}\big),\\ &|k|(v^2-z)^{-2}\big[y^2, b\big](v^2-z)^{-1} = \mathcal{O}\big(t^2|z|^3|\,\mathrm{Im}\,z|^{-5}\big),\\ &(v^2-z)^{-2}\big[y^2, b\big](v^2-z)^{-1} = \mathcal{O}\big(t^3|z|^{\frac{3}{2}}|\,\mathrm{Im}\,z|^{-3}\big).\end{aligned}$$

Then, an interpolation argument gives

$$|k|^{\frac{\delta}{2}}(v^2-z)^{-2}\big[y^2, b\big](v^2-z)^{-1}|k|^{\frac{\delta}{2}} = \mathcal{O}\big(t^{3-\delta}|z|^{\frac{3}{2}(1+\delta)}|\,\mathrm{Im}\,z|^{-3-2\delta}\big).\tag{5.14}$$

On the other hand, for all $n \in \mathbb{N}$,

$$\Big|\frac{\partial(\widetilde{\varphi}_R\widetilde{G})}{\partial\bar{z}}(z)\Big| \leq \mathrm{C}_n\langle \mathrm{Re}\,z\rangle^{\rho-1-n}|\,\mathrm{Im}\,z|^n,\tag{5.15}$$

where $\mathrm{C}_n > 0$ does not depend on $R \geq 1$. Using (5.13) together with (5.14) and (5.15), there exists $\mathrm{C} > 0$ such that

$$\big\||k|^{\frac{\delta}{2}}\mathcal{R}_R|k|^{\frac{\delta}{2}}\big\| \leq \mathrm{C}t^{-1-\delta},$$

for all $R \geq 1$. Eventually, since $(\varphi_R G)'(v^2)$ converges strongly to $G'(v^2)$ on $\mathcal{D}(|v|^{2\rho})$, the lemma follows from (5.12) and the previous estimate. □

Acknowledgments

The authors are grateful to the anonymous referee for useful remarks and to Volker Bach, Jürg Fröhlich, Marcel Griesemer and Avy Soffer for many discussions and collaborations. The research of the last author was supported by NSERC under Grant No. NA790127.

Appendix A. Properties of the Hamiltonians H and $\widetilde{H}$

In this appendix, we collect a few properties of the Hamiltonians H and $\widetilde{H}$. We begin with the following two important results.

Theorem A.1 (*Self-adjointness [31,32]*). *The Hamiltonians H and $\widetilde{H}$ are self-adjoint operators on the domain*

$$\mathcal{D}(H) = \mathcal{D}(\widetilde{H}) = \mathcal{D}\big(p^2 + H_f\big).$$

The fact that H is self-adjoint on $\mathcal{D}(p^2 + H_f)$ is proved in [32] by functional integral methods. Another proof is given in [31] using abstract results based on commutator arguments. Self-adjointness of $\widetilde{H}$ on $\mathcal{D}(p^2 + H_f)$ is another application of [31], using that $|\widetilde{g}_x(k, \lambda)| \lesssim \kappa(k)|k|^{-\frac{1}{2}-\mu}$ with $0 < \mu < 1/2$.

Theorem A.2 (*Exponential decay below the ionization threshold [25]*). *For all real numbers δ and ξ such that $\xi + \delta^2 < \Sigma$,*

$$\big\| e^{\delta|x|} \mathbb{1}_{(-\infty,\xi]}(H) \big\| = \big\| e^{\delta|x|} \mathbb{1}_{(-\infty,\xi]}(\widetilde{H}) \big\| \lesssim 1.$$

We now establish a property used in the proof of Theorem 1.1. It shows in particular that the propagation observable Φ_t of Theorem 1.1 is well-defined. Recall that the notion of regularity with respect to an operator is defined by

Definition A.3. Let $(A, \mathcal{D}(A))$ and $(H, \mathcal{D}(H))$ be self-adjoint operators on a separable Hilbert space $\mathcal{H}$. The operator H is of class $\mathrm{C}^k(A)$ for $k > 0$, if there is $z \in \mathbb{C} \setminus \sigma(H)$ such that

$$\mathbb{R} \ni t \longrightarrow e^{\mathrm{i}tA}(H - z)^{-1} e^{-\mathrm{i}tA},$$

is C^k for the strong topology of $\mathcal{L}(\mathcal{H})$.

We refer to [3] for properties of $\mathrm{C}^k(\cdot)$. Since H and $\widetilde{H}$ are *not* of class $\mathrm{C}^1(\mathrm{d}\Gamma(\langle y \rangle))$, the proof of the next proposition is not straightforward.

Proposition A.4. *Let $H^{\#}$ denote either H or $\widetilde{H}$. For all $\chi \in \mathrm{C}_0^\infty((-\infty, \Sigma))$ and $0 \le \beta < 1$, we have*

$$\chi(H^{\#}) \mathcal{D}\big(\mathrm{d}\Gamma(\langle y \rangle^\beta)\big) \subset \mathcal{D}\big(\mathrm{d}\Gamma(\langle y \rangle^\beta)\big).$$

Remark A.5. The allowed power of $\langle y \rangle$ in Proposition A.4 is related to the infrared singularity of the interaction. More precisely, the requirement that $\beta < 1$ is due to the fact that the infrared behavior of the interaction in H is of order $|k|^{-1/2}$. On the other hand, since the infrared behavior of the interaction in $\widetilde{H}$ is of order $|k|^{1/2}$, one could in fact show that

$$\chi(\widetilde{H}) \mathcal{D}\big(\mathrm{d}\Gamma(\langle y \rangle^\beta)\big) \subset \mathcal{D}\big(\mathrm{d}\Gamma(\langle y \rangle^\beta)\big),$$

for any $0 \le \beta < 2$. For our purpose, however, the stated result is sufficient.

We shall need the following two lemmas to prove Proposition A.4.

Lemma A.6. *Let $H^{\#}$ denote either H or $\widetilde{H}$. Then*

$$H^{\#} \in \mathrm{C}^1(N).$$

In particular, for all $\chi \in C_0^\infty(\mathbb{R})$,

$$\chi(H^\#)\mathcal{D}(N) \subset \mathcal{D}(N).$$

Proof. Let us prove that $\widetilde{H} \in C^1(N)$. Since $\mathcal{D}(\widetilde{H}) = \mathcal{D}(p^2 + H_f)$ and since N commutes with $p^2 + H_f$, we obviously have that

$$\forall s \in \mathbb{R}, \qquad e^{\mathrm{i}sN}\mathcal{D}(\widetilde{H}) \subset \mathcal{D}(\widetilde{H}).$$

Therefore, by [3, Theorem 6.3.4] (see also [22]), it suffices to prove that

$$\big|\langle \widetilde{H}u, Nu\rangle - \langle Nu, \widetilde{H}u\rangle\big| \lesssim \big(\|\widetilde{H}u\|^2 + \|u\|^2\big), \tag{A.1}$$

for all $u \in \mathcal{D}(\widetilde{H}) \cap \mathcal{D}(N)$. In the sense of quadratic forms on $\mathcal{D}(\widetilde{H}) \cap \mathcal{D}(N)$, using (B.2) of Appendix B and the notation $p_{\widetilde{A}} = p + \widetilde{A}(x)$, we can compute

$$[\widetilde{H}, N] = \mathrm{i}p_{\widetilde{A}} \cdot \Phi(\mathrm{i}\widetilde{g}_x) + \mathrm{i}\Phi(\mathrm{i}\widetilde{g}_x) \cdot p_{\widetilde{A}} + \mathrm{i}\Phi(\mathrm{i}e_x).$$

Using Lemma B.2 of Appendix B, estimate (A.1) easily follows. In the case of H, the proof is similar. The fact that $\chi(H^\#)\mathcal{D}(N) \subset \mathcal{D}(N)$ is then a consequence of [3, Theorem 6.2.10]. □

Lemma A.7. *Let* $H^\#$ *denote either* H *or* $\widetilde{H}$*. For all* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}$, $0 < \pm \operatorname{Im} z \leq 1$*, the operator* $\langle x\rangle^{-n}(H^\# - z)^{-1}\langle x\rangle^n$ *defined on* $\mathcal{D}(\langle x\rangle^n)$ *extends by continuity to a bounded operator on* $\mathcal{H}$ *satisfying*

$$\big\|\langle x\rangle^{-n}(H^\# - z)^{-1}\langle x\rangle^n\big\| \leq \frac{C}{|\operatorname{Im} z|^{n+1}}. \tag{A.2}$$

Moreover, $\langle x\rangle^{-n}(H^\# - z)^{-1}\langle x\rangle^n(H^\# - z)$ *defined on* $\mathcal{D}(H^\#)$ *extends by continuity to a bounded operator on* $\mathcal{H}$ *satisfying*

$$\big\|\langle x\rangle^{-n}(H^\# - z)^{-1}\langle x\rangle^n(H^\# - z)\big\| \leq \frac{C}{|\operatorname{Im} z|^{n}}. \tag{A.3}$$

Estimates (A.2)–(A.3) are established in [9, Lemma A.5] in the case of H. Since the proof is the same in the case of $\widetilde{H}$, we do not reproduce it.

Proof of Proposition A.4. We show the proposition for $\widetilde{H}$, the case of H being similar. Let $\eta \in C_0^\infty((-\infty, \Sigma))$ be such that $\chi\eta = \chi$. Consider $\varphi \in C_0^\infty(\mathbb{R}; [0, 1])$ equal to 1 near 0 and $\varphi_R(\cdot) = \varphi(\cdot/R)$ for $R > 0$. Let $u \in \mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle^\beta))$. We want to prove that for all $v \in \mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle^\beta))$,

$$\big|\langle \mathrm{d}\Gamma(\langle y\rangle^\beta)v, \chi(\widetilde{H})u\rangle\big| \leq C_u\|v\|.$$

We write

$$\begin{aligned}\big|\langle \mathrm{d}\Gamma(\langle y\rangle^\beta)v, \chi(\widetilde{H})u\rangle\big| &= \lim_{R\to\infty}\big|\langle \mathrm{d}\Gamma\big(\langle y\rangle^\beta\varphi_R(y^2)\big)v, \chi(\widetilde{H})\eta(\widetilde{H})u\rangle\big| \\ &\leq \limsup_{R\to\infty}\big|\langle v, \chi(\widetilde{H})\eta(\widetilde{H})\mathrm{d}\Gamma\big(\langle y\rangle^\beta\varphi_R(y^2)\big)u\rangle\big| \\ &\quad + \limsup_{R\to\infty}\big|\langle v, \big[\mathrm{d}\Gamma\big(\langle y\rangle^\beta\varphi_R(y^2)\big), \chi(\widetilde{H})\big]\eta(\widetilde{H})u\rangle\big| \\ &\quad + \limsup_{R\to\infty}\big|\langle v, \chi(\widetilde{H})\big[\mathrm{d}\Gamma\big(\langle y\rangle^\beta\varphi_R(y^2)\big), \eta(\widetilde{H})\big]u\rangle\big|,\end{aligned} \tag{A.4}$$

where the commutators should be understood in the sense of quadratic forms on $\mathcal{D}(N)$. By Lemma A.6, the previous expressions are justified since $\chi(\widetilde{H})$ and $\eta(\widetilde{H})$ preserve $\mathcal{D}(N)$. The first term is easily estimated as

$$\left|\langle v, \chi(\widetilde{H})\eta(\widetilde{H})\mathrm{d}\Gamma\big(\langle y\rangle^{\beta}\varphi_R(y^2)\big)u\rangle\right| \le \mathrm{C}\|v\|\,\big\|\mathrm{d}\Gamma(\langle y\rangle^{\beta})u\big\|. \tag{A.5}$$

Let $\widetilde{\chi} \in \mathrm{C}_0^{\infty}(\mathbb{C})$ denote an almost analytic extension of χ (see the beginning of the proof of Lemma 5.1). To estimate the second term of (A.4), we write

$$\begin{aligned}
&\big\|\big[\mathrm{d}\Gamma\big(\langle y\rangle^{\beta}\varphi_R(y^2)\big), \chi(\widetilde{H})\big]\eta(\widetilde{H})u\big\| \\
&\quad \le \frac{1}{\pi}\int \left|\frac{\partial\widetilde{\chi}(z)}{\partial\bar{z}}\right| \big\|\big[\mathrm{d}\Gamma\big(\langle y\rangle^{\beta}\varphi_R(y^2)\big), (\widetilde{H}-z)^{-1}\big]\eta(\widetilde{H})u\big\|\,\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z \\
&\quad \le \frac{1}{\pi}\int \left|\frac{\partial\widetilde{\chi}(z)}{\partial\bar{z}}\right| \big\|(\widetilde{H}-z)^{-1}B_R(\widetilde{H}-z)^{-1}\eta(\widetilde{H})u\big\|\,\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z \\
&\quad \le \frac{1}{\pi}\int \left|\frac{\partial\widetilde{\chi}(z)}{\partial\bar{z}}\right| \big\|(\widetilde{H}-z)^{-1}\langle\widetilde{H}\rangle^{\frac{1}{2}}\big\|\,\big\|\langle\widetilde{H}\rangle^{-\frac{1}{2}}B_R\big(N+\langle x\rangle^{\frac{4}{\mu}+2\beta}\big)^{-1}\big\| \\
&\qquad \times\big\|\big(N+\langle x\rangle^{\frac{4}{\mu}+2\beta}\big)(\widetilde{H}-z)^{-1}\eta(\widetilde{H})(N+1)^{-1}\big\|\,\|(N+1)u\|\,\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z,
\end{aligned} \tag{A.6}$$

where B_R is the quadratic form on $\mathcal{D}(\widetilde{H}) \cap \mathcal{D}(N)$ defined by

$$B_R := \big[\widetilde{H}, \mathrm{d}\Gamma\big(\langle y\rangle^{\beta}\varphi_R(y^2)\big)\big].$$

Using Lemma A.6, one verifies that

$$\big\|N(\widetilde{H}-z)^{-1}\eta(\widetilde{H})(N+1)^{-1}\big\| \lesssim |\mathrm{Im}\,z|^{-2},$$

and by Theorem A.2,

$$\big\|\langle x\rangle^{\frac{4}{\mu}+2\beta}(\widetilde{H}-z)^{-1}\eta(\widetilde{H})\big\| \lesssim |\mathrm{Im}\,z|^{-1}\big\|\langle x\rangle^{\frac{4}{\mu}+2\beta}\eta(\widetilde{H})\big\| \lesssim |\mathrm{Im}\,z|^{-1}.$$

We claim that

$$\left\|\langle\widetilde{H}\rangle^{-\frac{1}{2}}B_R\big(N+\langle x\rangle^{\frac{4}{\mu}+2\beta}\big)^{-1}\right\| \lesssim 1. \tag{A.7}$$

Then (A.6)–(A.7) together with the properties of $\widetilde{\chi}$ imply that

$$\big\|\big[\mathrm{d}\Gamma\big(\langle y\rangle^{\beta}\varphi_R(y^2)\big), \chi(\widetilde{H})\big]\eta(\widetilde{H})u\big\| \lesssim \big\|\big(\mathrm{d}\Gamma(\langle y\rangle^{\beta})+1\big)u\big\|. \tag{A.8}$$

Let us now prove (A.7). In the sense of quadratic forms on $\mathcal{D}(\widetilde{H}) \cap \mathcal{D}(N)$, we have

$$\begin{aligned}
B_R &= \mathrm{d}\Gamma\big(\big[|k|, \langle y\rangle^{\beta}\varphi_R(y^2)\big]\big) + \mathrm{i}p_{\widetilde{A}}\cdot\Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x\big) \\
&\quad + \mathrm{i}\Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x\big)\cdot p_{\widetilde{A}} + \mathrm{i}\Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)e_x\big) \\
&= \mathrm{d}\Gamma\big(\big[|k|, \langle y\rangle^{\beta}\varphi_R(y^2)\big]\big) - \Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\nabla_x\widetilde{g}_x\big) + \mathrm{Im}\,\big\langle\widetilde{g}_x, \mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x\big\rangle_{\mathfrak{h}} \\
&\quad + 2\mathrm{i}p_{\widetilde{A}}\cdot\Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x\big) + \mathrm{i}\Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)e_x\big),
\end{aligned} \tag{A.9}$$

where we used again the notation $p_{\widetilde{A}} = p + \widetilde{A}(x)$. Using (2.7) and applying Lemma 3.1 (with $t = 1$), we obtain that, for all $x \in \mathbb{R}^3$,

$$\big\|\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x(k,\lambda)\big\|_{\mathfrak{h}} \le \big\|\langle y\rangle^{\beta}\widetilde{g}_x(k,\lambda)\big\|_{\mathfrak{h}} \lesssim \langle x\rangle^{\frac{1}{\mu}+\beta},$$

and likewise with $\nabla_x \widetilde{g}_x(k,\lambda)$ or $e_x(k,\lambda)$ in place of $\widetilde{g}_x(k,\lambda)$. Therefore, by Lemma B.1,

$$\left\| \langle \widetilde{g}_x, \mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x\rangle_{\mathfrak{h}}\langle x\rangle^{-\frac{2}{\mu}-\beta}\right\| \lesssim 1, \tag{A.10}$$

$$\left\| \Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\nabla_x\widetilde{g}_x\big)\langle x\rangle^{-\frac{1}{\mu}-\beta}(N+1)^{-\frac{1}{2}}\right\| \lesssim 1, \tag{A.11}$$

$$\left\| \Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)e_x\big)\langle x\rangle^{-1-\beta}(N+1)^{-\frac{1}{2}}\right\| \lesssim 1, \tag{A.12}$$

and, since $\|\langle\widetilde{H}\rangle^{-\frac{1}{2}}p_{\widetilde{A}}\| \lesssim 1$,

$$\left\| \langle\widetilde{H}\rangle^{-\frac{1}{2}}p_{\widetilde{A}}\cdot\Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x\big)\langle x\rangle^{-\frac{1}{\mu}-\beta}(N+1)^{-\frac{1}{2}}\right\| \lesssim 1. \tag{A.13}$$

Finally, using the representation formula

$$\langle y\rangle^{\beta}\varphi_R(y^2) = \frac{1}{\pi}\int \frac{\partial(\widetilde{\psi}\widetilde{\varphi}_R)(z)}{\partial\bar{z}}(y^2-z)^{-1}\,\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z,$$

where $\widetilde{\psi}$ (resp. $\widetilde{\varphi}$) is an almost analytic extension of $(\cdot+1)^{\frac{\beta}{2}} \in S^{\frac{\beta}{2}}(\mathbb{R})$ (resp. $\varphi \in \mathrm{C}_0^{\infty}(\mathbb{R})$), one can verify that

$$\left\| \big[|k|, \langle y\rangle^{\beta}\varphi_R(y^2)\big]\right\| \lesssim 1,$$

and hence, by Lemma B.3,

$$\left\| \mathrm{d}\Gamma\big(\big[|k|, \langle y\rangle^{\beta}\varphi_R(y^2)\big]\big)(N+1)^{-1}\right\| \lesssim 1. \tag{A.14}$$

Estimates (A.10)–(A.14) together with the fact that $\|\langle x\rangle^{\frac{2}{\mu}+\beta}(N+1)^{1/2}u\| \lesssim \|(N+\langle x\rangle^{\frac{4}{\mu}+2\beta})u\|$ imply (A.7).

It remains to estimate the third term in the right hand side of (A.4). To this end, let $\widetilde{\eta}$ denote an almost analytic extension of η and write similarly

$$\begin{aligned}
&\left\|\chi(\widetilde{H})\big[\mathrm{d}\Gamma\big(\langle y\rangle^{\beta}\varphi_R(y^2)\big), \eta(\widetilde{H})\big]u\right\| \\
&\quad\le \frac{1}{\pi}\int\left|\frac{\partial\widetilde{\eta}(z)}{\partial\bar{z}}\right|\left\|\chi(\widetilde{H})(\widetilde{H}-z)^{-1}B_R(\widetilde{H}-z)^{-1}u\right\|\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z \\
&\quad\le \frac{1}{\pi}\int\left|\frac{\partial\widetilde{\eta}(z)}{\partial\bar{z}}\right|\left\|\chi(\widetilde{H})\langle x\rangle^{\frac{2}{\mu}+\beta}\right\|\left\|\langle x\rangle^{-\frac{2}{\mu}-\beta}(\widetilde{H}-z)^{-1}\langle x\rangle^{\frac{2}{\mu}+\beta}\langle\widetilde{H}\rangle^{\frac{1}{2}}\right\| \\
&\qquad\times\left\|\langle\widetilde{H}\rangle^{-\frac{1}{2}}\langle x\rangle^{-\frac{2}{\mu}-\beta}B_R(N+1)^{-1}\right\|\left\|(N+1)(\widetilde{H}-z)^{-1}(N+1)^{-1}\right\| \\
&\qquad\times\|(N+1)u\|\,\mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z.
\end{aligned} \tag{A.15}$$

Theorem A.2 gives $\|\chi(\widetilde{H})\langle x\rangle^{\frac{2}{\mu}+\beta}\| \lesssim 1$, Lemma A.7 yields $\|\langle x\rangle^{-\frac{2}{\mu}-\beta}(\widetilde{H}-z)^{-1}\langle x\rangle^{\frac{2}{\mu}+\beta}\langle\widetilde{H}\rangle^{\frac{1}{2}}\| \lesssim |\mathrm{Im}\,z|^{-\frac{2}{\mu}-\beta-1}$, and Lemma A.6 implies $\|(N+1)(\widetilde{H}-z)^{-1}(N+1)^{-1}\| \lesssim |\mathrm{Im}\,z|^{-2}$. Moreover we claim that

$$\left\|\langle\widetilde{H}\rangle^{-\frac{1}{2}}\langle x\rangle^{-\frac{2}{\mu}-\beta}B_R(N+1)^{-1}\right\| \lesssim 1. \tag{A.16}$$

To prove (A.16), it suffices to proceed in the same way as for (A.7). The only difference is (A.13), which is replaced by

$$\left\|\langle\widetilde{H}\rangle^{-\frac{1}{2}}\langle x\rangle^{-\frac{1}{\mu}-\beta}p_{\widetilde{A}}\cdot\Phi\big(\mathrm{i}\langle y\rangle^{\beta}\varphi_R(y^2)\widetilde{g}_x\big)(N+1)^{-\frac{1}{2}}\right\|$$

$$\leq \left\| \langle \widetilde{H} \rangle^{-\frac{1}{2}} \langle x \rangle^{-\frac{1}{\mu}-\beta} p_{\widetilde{A}} \langle x \rangle^{\frac{1}{\mu}+\beta} \right\| \left\| \langle x \rangle^{-\frac{1}{\mu}-\beta} \Phi\big(\mathrm{i}\langle y \rangle^{\beta} \varphi_R(y^2) \widetilde{g}_x\big)(N+1)^{-\frac{1}{2}} \right\|$$
$$\lesssim \big\| \langle \widetilde{H} \rangle^{-\frac{1}{2}} p_{\widetilde{A}} \big\| + \left\| \langle \widetilde{H} \rangle^{-\frac{1}{2}} \langle x \rangle^{-1} \frac{\mathrm{i}x}{\langle x \rangle} \right\| \lesssim 1.$$

Therefore

$$\big\| \chi(\widetilde{H}) \big[\mathrm{d}\Gamma\big(\langle y \rangle^{\beta} \varphi_R(y^2)\big), \eta(\widetilde{H}) \big] u \big\| \lesssim \big\| \big(\mathrm{d}\Gamma(\langle y \rangle^{\beta}) + 1\big) u \big\|. \tag{A.17}$$

Eq. (A.4) together with the estimates (A.5), (A.8) and (A.17) conclude the proof of the proposition. □

Appendix B. Creation and annihilation operators

Let $\mathfrak{h} := \mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$ be the momentum representation Hilbert space of a photon. The variable $k \in \mathbb{R}^3$ is the wave vector or momentum of the photon. Recall that the propagation speed of the light and the Planck constant divided by 2π are set equal to 1. The Bosonic Fock space, $\mathcal{F}$, over $\mathfrak{h}$ is defined by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} S_n \mathfrak{h}^{\otimes n},$$

where S_n is the orthogonal projection onto the subspace of totally symmetric n-particle wave functions contained in the n-fold tensor product $\mathfrak{h}^{\otimes n}$ of $\mathfrak{h}$ and $S_0 \mathfrak{h}^{\otimes 0} := \mathbb{C}$. The vector $\Omega := (1, 0, \dots)$ is called the *vacuum vector* in $\mathcal{F}$. Vectors $\Psi \in \mathcal{F}$ can be identified with sequences $(\psi_n)_{n=0}^{\infty}$ of n-particle wave functions $\psi_n(k_1, \lambda_1, \dots, k_n, \lambda_n)$, where $\lambda_j \in \{1, 2\}$ are the polarization variables, which are totally symmetric in their n arguments, and $\psi_0 \in \mathbb{C}$.

The scalar product of two vectors Ψ and Φ is given by

$$\langle \Psi, \Phi \rangle := \sum_{n=0}^{\infty} \sum_{\lambda_1, \dots, \lambda_n} \int \prod_{j=1}^{n} \mathrm{d}k_j \overline{\psi_n(k_1, \lambda_1, \dots, k_n, \lambda_n)} \varphi_n(k_1, \lambda_1, \dots, k_n, \lambda_n). \tag{B.1}$$

Given a one particle dispersion relation $\omega(k)$, the energy of a configuration of n *non-interacting* field particles with wave vectors $k_1, \dots, k_n$ is given by $\sum_{j=1}^{n} \omega(k_j)$. We define the *free-field Hamiltonian*, H_f, giving the field dynamics, by

$$(H_f \Psi)_n(k_1, \lambda_1, \dots, k_n, \lambda_n) = \sum_{j=1}^{n} \omega(k_j) \psi_n(k_1, \lambda_1, \dots, k_n, \lambda_n),$$

for $n \geq 1$ and $(H_f \Psi)_n = 0$ for $n = 0$. Here $\Psi = (\psi_n)_{n=0}^{\infty}$ (to be sure that the right hand side makes sense, we can assume that $\psi_n = 0$, except for finitely many n, for which $\psi_n(k_1, \lambda_1, \dots, k_n, \lambda_n)$ decrease rapidly at infinity). Clearly, if $\omega(k) = |k|$, the operator H_f has the single eigenvalue 0 with the eigenvector Ω and the rest of the spectrum absolutely continuous.

With each function $f \in \mathfrak{h}$ one associates an *annihilation operator* $a(f)$ defined as follows. For $\Psi = (\psi_n)_{n=0}^{\infty} \in \mathcal{F}$ with the property that $\psi_n = 0$, for all but finitely many n, the vector $a(f)\Psi$ is defined by

$$(a(f)\Psi)_n(k_1, \lambda_1, \dots, k_n, \lambda_n) := \sqrt{n+1} \times \sum_{\lambda=1,2} \int \mathrm{d}k \overline{f(k, \lambda)} \psi_{n+1}(k, \lambda, k_1, \lambda_1, \dots, k_n, \lambda_n),$$

for $n \geq 1$ and $(a(f)\Psi)_n = 0$ for $n = 0$. These equations define a closable operator $a(f)$ whose closure is also denoted by $a(f)$. The creation operator $a^*(f)$ is defined to be the adjoint of $a(f)$ with respect to the scalar product defined in (B.1). Since $a(f)$ is anti-linear and $a^*(f)$ is linear in f, we write formally

$$a(f) = \sum_{\lambda=1,2} \int \mathrm{d}k \overline{f(k,\lambda)} a_\lambda(k), \qquad a^*(f) = \sum_{\lambda=1,2} \int \mathrm{d}k f(k,\lambda) a^*_\lambda(k),$$

where $a_\lambda(k)$ and $a^*_\lambda(k)$ are unbounded, operator-valued distributions. The latter are well-known to obey the *canonical commutation relations* (CCR):

$$\left[a^\#_\lambda(k), a^\#_{\lambda'}(k')\right] = 0, \qquad \left[a_\lambda(k), a^*_{\lambda'}(k')\right] = \delta_{\lambda,\lambda'}\delta(k-k'),$$

where $a^\#_\lambda = a_\lambda$ or a^*_λ. We have the following standard estimates for annihilation and creation operators $a(f)$ and $a^*(f)$, whose proof can be found, for instance, in [23, Section 3], [28]:

Lemma B.1. *For any $f \in \mathfrak{h}$, the operators $a(f)(N+1)^{-1/2}$ and $a^*(f)(N+1)^{-1/2}$ extend to bounded operators on $\mathcal{H}$ satisfying*

$$\left\|a(f)(N+1)^{-\frac{1}{2}}\right\| \leq \|f\|_{\mathfrak{h}},$$
$$\left\|a^*(f)(N+1)^{-\frac{1}{2}}\right\| \leq \sqrt{2}\|f\|_{\mathfrak{h}}.$$

Lemma B.2. *Let $f \in \mathfrak{h}$ be such that $(k,\lambda) \mapsto \omega(k)^{-1/2} f(k,\lambda) \in \mathfrak{h}$. Then, the operators $a(f)(H_f+1)^{-1/2}$ and $a^*(f)(H_f+1)^{-1/2}$ extend to bounded operators on $\mathcal{H}$ satisfying*

$$\left\|a(f)(H_f+1)^{-\frac{1}{2}}\right\| \leq \left\|\omega(k)^{-\frac{1}{2}} f\right\|_{\mathfrak{h}},$$
$$\left\|a^*(f)(H_f+1)^{-\frac{1}{2}}\right\| \leq \left\|\omega(k)^{-\frac{1}{2}} f\right\|_{\mathfrak{h}} + \|f\|_{\mathfrak{h}}.$$

Now, using the definitions, one can rewrite the quantum Hamiltonian H_f in terms of the creation and annihilation operators, a and a^*, as

$$H_f = \sum_{\lambda=1,2} \int \mathrm{d}k a^*_\lambda(k)\omega(k)a_\lambda(k),$$

acting on the Fock space $\mathcal{F}$. More generally, for any operator, τ, on the one-particle space $\mathfrak{h} = \mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2)$ we define the operator $\mathrm{d}\Gamma(\tau)$ on the Fock space $\mathcal{F}$ by the following formal expression

$$\mathrm{d}\Gamma(\tau) := \sum_{\lambda=1,2} \int \mathrm{d}k a^*_\lambda(k)\tau a_\lambda(k),$$

where the operator τ acts on the k-variable ($\mathrm{d}\Gamma(\tau)$ is the second quantization of τ). The precise meaning of the latter expression is

$$\mathrm{d}\Gamma(\tau)_{|S_n \mathfrak{h}^{\otimes n}} = \sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes \tau \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}.$$

Commutators of two such operators reduce to commutators of the one-photon operators:

$$[\mathrm{d}\Gamma(\tau), \mathrm{d}\Gamma(\tau')] = \mathrm{d}\Gamma([\tau,\tau']). \tag{B.2}$$

A proof of the following lemma can be found in [23, Section 3].

Lemma B.3. *Let τ, τ' be two self-adjoint operators on $\mathfrak{h}$ with $\tau' \geq 0$, $\mathcal{D}(\tau') \subset \mathcal{D}(\tau)$ and $\|\tau\varphi\|_{\mathfrak{h}} \leq \|\tau'\varphi\|_{\mathfrak{h}}$ for all $\varphi \in \mathcal{D}(\tau')$. Then $\mathcal{D}(\mathrm{d}\Gamma(\tau')) \subset \mathcal{D}(\mathrm{d}\Gamma(\tau))$ and $\|\mathrm{d}\Gamma(\tau)\Phi\| \leq \|\mathrm{d}\Gamma(\tau')\Phi\|$ for all $\Phi \in \mathcal{D}(\mathrm{d}\Gamma(\tau'))$.*

Finally, let ω be a one-photon self-adjoint operator. The following commutation relations involving the field operator $\Phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f))$ can be readily derived from the definitions of the operators involved:

$$[\Phi(f), \Phi(g)] = \mathrm{i}\,\mathrm{Im}\langle f, g\rangle_{\mathfrak{h}}, \tag{B.3}$$

$$e^{\mathrm{i}\Phi(f)}\Phi(g)e^{-\mathrm{i}\Phi(f)} = \Phi(g) - \mathrm{Im}\langle f, g\rangle_{\mathfrak{h}}, \tag{B.4}$$

$$[\Phi(f), \mathrm{d}\Gamma(\omega)] = \mathrm{i}\Phi(\mathrm{i}\omega f), \tag{B.5}$$

$$e^{\mathrm{i}\Phi(f)}\mathrm{d}\Gamma(\omega)e^{-\mathrm{i}\Phi(f)} = \mathrm{d}\Gamma(\omega) - \Phi(\mathrm{i}\omega f) + \frac{1}{2}\mathrm{Re}\langle \omega f, f\rangle_{\mathfrak{h}}. \tag{B.6}$$

Appendix C. Notations

Notation	*Definition/description of notation*	*Reference*
$A(x)$	$\Phi(g_x)$, vector potential of the quantized electromagnetic field	(2.2)
$\widetilde{A}(x)$	$\Phi(\widetilde{g}_x)$	(2.3)
b	$\mathrm{i}(\widehat{k}\cdot y + y\cdot\widehat{k})$	Section 2
$E(x)$	$\Phi(e_x)$	(2.4)
$\mathcal{F}$	Fock space over $\mathfrak{h}$	Section 1
$\mathfrak{h}$	$\mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2)$, one-photon space	Section 1
$\widehat{k}$	$k/\lvert k\rvert$	Section 2
$\mathcal{H}$	$\mathcal{H}_{\mathrm{el}}\otimes\mathcal{F}$, total Hilbert space	Section 1
$\mathcal{H}_{\mathrm{el}}$	Hilbert space for the electron	Section 1
H_f	$\mathrm{d}\Gamma(\lvert k\rvert)$	(1.3)
H	Hamiltonian of the standard model of non-relativistic QED	(1.4)
$\widetilde{H}$	$\mathcal{U}H\mathcal{U}^*$, Pauli–Fierz transformed Hamiltonian	Section 2
N	$\mathrm{d}\Gamma(\mathbb{1})$, number operator	Section 2
p	$-\mathrm{i}\nabla_x$, momentum of the electron	Section 1
$p_{\widetilde{A}}$	$p+\widetilde{A}(x)$	Section 2
Σ	Ionization threshold	Section 1
$\mathcal{U}$	$e^{-\mathrm{i}\Phi(q_x)}$, generalized Pauli–Fierz transformation	Section 2
v	y/ct	Section 2
X	$\mathcal{D}(\mathrm{d}\Gamma(\langle y\rangle)^{\frac{1}{2}})$	Section 1
X_δ	$\mathcal{D}\Big(\mathrm{d}\Gamma(\lvert k\rvert^{-\delta})^{\frac{1}{2}}\Big)$	Section 2
$X_{\delta,\beta}$	$\mathcal{D}\Big(\mathrm{d}\Gamma(\lvert k\rvert^{-\delta})^{\frac{1}{2}}\Big)\cap\mathcal{D}\Big(\mathrm{d}\Gamma(\lvert y\rvert^{2\beta})^{\frac{1}{2}}\Big)$	Section 2
y	$\mathrm{i}\nabla_k$	Section 1
$\lVert\!\lvert\cdot\rvert\!\rVert$	$\Big\lVert\Big(\mathrm{d}\Gamma(\langle y\rangle)+1\Big)^{\frac{1}{2}}\cdot\Big\rVert$	Section 1
$\lVert\cdot\rVert_\delta$	$\Big\lVert\Big(\mathrm{d}\Gamma(\lvert k\rvert^{-\delta})+1\Big)^{\frac{1}{2}}\cdot\Big\rVert$	Section 2
$\lVert\cdot\rVert_{\delta,\beta}$	$\Big\lVert\Big(\mathrm{d}\Gamma\Big(\lvert k\rvert^{-\delta}+\lvert y\rvert^{2\beta}\Big)+1\Big)^{\frac{1}{2}}\cdot\Big\rVert$	Section 2

References

[1] L. Amour, B. Grébert, J.-C. Guillot, The dressed mobile atoms and ions, J. Math. Pures Appl. 86 (2006) 177–200.
[2] W. Amrein, Localizability for particles of mass zero, Helv. Phys. Acta 42 (1969) 149–190.

[3] W. Amrein, A. Boutet de Monvel, V. Georgescu, C_0-groups, commutator methods and spectral theory of N-body Hamiltonians, in: Progress in Mathematics, vol. 135, Birkhäuser Verlag, 1996.

[4] A. Arai, M. Hirokawa, Ground states of a general class of quantum field Hamiltonians, Rev. Math. Phys. 12 (2000) 1085–1135.

[5] V. Bach, Mass renormalization in nonrelativisitic quantum electrodynamics, in: Quantum Theory from Small to Large Scales: Lecture Notes of the Les Houches Summer Schools, vol. 95, Oxford University Press, 2011.

[6] V. Bach, J. Fröhlich, A. Pizzo, An infrared-finite algorithm for Rayleigh scattering amplitudes, and Bohr's frequency condition, Comm. Math. Phys. 274 (2) (2007) 457–486.

[7] V. Bach, J. Fröhlich, I.M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, Comm. Math. Phys. 207 (2) (1999) 249–290.

[8] H. Bahouri, J.-Y. Chemin, I. Gallagher, Inégalitées de Hardy précisées, C.R. Math. Acad. Sci. Paris 345 (2005) 89–92.

[9] J.-F. Bony, J. Faupin, Resolvent smoothness and local decay at low energies for the standard model of non-relativistic QED, J. Funct. Anal. 262 (2012) 850–888.

[10] T. Chen, J. Faupin, J. Fröhlich, I.M. Sigal, Local decay in non-relativistic QED, Comm. Math. Phys. 309 (2012) 543–583.

[11] T. Chen, J. Fröhlich, A. Pizzo, Infraparticle scattering states in non-relativistic QED - I. The Bloch–Nordsieck paradigm, Comm. Math. Phys. 294 (2010) 761–825.

[12] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, Photons and atoms: introduction to quantum electrodynamics, in: Wiley Professional Paperback Series, Wiley, 1997.

[13] J. Dereziński, C. Gérard, Scattering theory of classical and quantum N-particle systems, in: Texts and Monographs in Physics, Springer-Verlag, 1997.

[14] J. Dereziński, C. Gérard, Asymptotic completeness in quantum field theory. Massive Pauli–Fierz Hamiltonians, Rev. Math. Phys. 11 (4) (1999) 383–450.

[15] W. De Roeck, A. Kupiainen, Approach to ground state and time-independent photon bound for massless spin-boson models, 2011, preprint arXiv:1109.5582.

[16] J. Faupin, I.M. Sigal, On quantum Huygens principle and Rayleigh scattering, preprint arXiv.

[17] E. Fermi, Quantum theory of radiation, Rev. Modern Phys. 4 (1) (1932) 87–132.

[18] J. Fröhlich, M. Griesemer, B. Schlein, Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field, Adv. Math. 164 (2) (2001) 349–398.

[19] J. Fröhlich, M. Griesemer, B. Schlein, Asymptotic completeness for Rayleigh scattering, Ann. Henri Poincaré 3 (1) (2002) 107–170.

[20] J. Fröhlich, M. Griesemer, B. Schlein, Asymptotic completeness for Compton scattering, Comm. Math. Phys. 252 (2004) 415–476.

[21] J. Fröhlich, M. Griesemer, I.M. Sigal, Spectral theory for the standard model of non-relativistic QED, Comm. Math. Phys. 283 (3) (2008) 613–646.

[22] V. Georgescu, C. Gérard, On the virial theorem in quantum mechanics, Comm. Math. Phys. 208 (2) (1999) 275–281.

[23] V. Georgescu, C. Gérard, J.S. Møller, Spectral theory of massless Pauli–Fierz models, Comm. Math. Phys. 249 (1) (2004) 29–78.

[24] C. Gérard, On the scattering theory of massless Nelson models, Rev. Math. Phys. 14 (11) (2002) 1165–1280.

[25] M. Griesemer, Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics, J. Funct. Anal. 210 (2) (2004) 321–340.

[26] M. Griesemer, E.H. Lieb, M. Loss, Ground states in non-relativistic quantum electrodynamics, Invent. Math. 145 (2001) 557–595.

[27] M. Griesemer, H. Zenk, On the atomic photoeffect in non-relativistic QED, Comm. Math. Phys. 300 (2010) 615–639.

[28] S. Gustafson, I.M. Sigal, Mathematical concepts of quantum mechanics, in: Universitext, 2nd edition, Springer-Verlag, 2011.

[29] G.H. Hardy, Note on a theorem of Hilbert, Math. Z. 6 (1920) 314–317.

[30] G.H. Hardy, An inequality between integrals, Messenger Math. 54 (1925) 150–156.

[31] D. Hasler, I. Herbst, On the self-adjointness and domain of Pauli–Fierz type Hamiltonians, Rev. Math. Phys. 20 (7) (2008) 787–800.

[32] F. Hiroshima, Self-adjointness of the Pauli–Fierz Hamiltonian for arbitrary values of coupling constants, Ann. Henri Poincaré 3 (1) (2002) 171–201.

[33] M. Hübner, H. Spohn, Radiative decay: nonperturbative approaches, Rev. Math. Phys. 7 (3) (1995) 363–387.

[34] W. Hunziker, I.M. Sigal, The quantum N-body problem, J. Math. Phys. 41 (6) (2000) 3448–3510.

[35] W. Hunziker, I.M. Sigal, A. Soffer, Minimal escape velocities, Comm. Partial Differential Equations 24 (11-12) (1999) 2279–2295.
[36] J.M. Jauch, C. Piron, Generalized localizability, Helv. Phys. Acta 40 (1967) 559–570.
[37] O. Keller, On the theory of spatial localization of photons, Phys. Rep. 411 (1–3) (2005) 1–232.
[38] L. Landau, R. Peierls, Quantenelektrodynamik im Konfigurationsraum, Z. Phys. 62 (1930) 188–200.
[39] E. Lieb, M. Loss, Existence of atoms and molecules in non-relativistic quantum electrodynamics, Adv. Theor. Math. Phys. 7 (4) (2003) 667–710.
[40] E. Lieb, M. Loss, A note on polarization vectors in quantum electrodynamics, Comm. Math. Phys. 252 (1-3) (2004) 477–483.
[41] M. Loss, T. Miyao, H. Spohn, Lowest energy states in nonrelativistic QED: Atoms and ions in motion, J. Funct. Anal. 243 (2007) 353–393.
[42] L. Mandel, Configuration-space photon number operators in quantum optics, Phys. Rev. 144 (1966) 1071–1077.
[43] L. Mandel, E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, 1995.
[44] T.D. Newton, E. Wigner, Localized states for elementary systems, Rev. Modern Phys. 21 (1949) 400–406.
[45] W. Pauli, Collected scientific papers, vol. 2, Interscience Publishers, 1964.
[46] W. Pauli, M. Fierz, Zur Theorie der Emission langwelliger Lichtquanten, Il Nuovo Cimento 15 (3) (1938) 167–188.
[47] A. Pizzo, One-particle (improper) States in Nelson's massless model, Ann. Henri Poincaré 4 (2003) 439–486.
[48] A. Pizzo, Scattering of an infraparticle: The one particle sector in Nelson's massless model, Ann. Henri Poincaré 6 (2005) 553–606.
[49] S. Ruijsenaars, On Newton–Wigner localization and superluminal propagation speeds, Ann. Physics 137 (1) (1981) 33–43.
[50] I.M. Sigal, Ground state and resonances in the standard model of the non-relativistic QED, J. Stat. Phys. 134 (5-6) (2009) 899–939.
[51] I.M. Sigal, Renormalization group and problem of radiation, in: Lecture Notes of Les Houches Summer School on Quantum Theory From Small to Large Scales, vol. 95, 2012, arXiv:1110.3841.
[52] I.M. Sigal, A. Soffer, Local decay and propagation estimates for time dependent and time independent hamiltonians, preprint, Princeton University 1988.
[53] A. Soffer, The maximal velocity of a photon, 2011, preprint arXiv:1103.3031.
[54] H. Spohn, Dynamics of Charged Particles and their Radiation Field, Cambridge University Press, Cambridge, 2004.
[55] A. Wightman, On the localizibility of quantum mechanical systems, Rev. Modern Phys. 34 (1962) 845–872.