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Asymptotic Behaviour of Almost Nonexpansive Sequences in a Hilbert Space

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The results in [9, 11, 12] for nonexpansive sequences are generalized to almost nonexpansive sequences in a Hilbert space, and using this notion a direct proof of a result of H. Brézis and F. E. Browder [6] is given.

1. INTRODUCTION

Let H denote a real Hilbert space with inner product $(\ |\ )$ and norm $\|\ \|$. We denote weak convergence in H by $\rightharpoonup$ and strong convergence by $\to$.

Let $(x_n)_{n\geqslant 0}$ be a sequence in H and define: $\forall n \geqslant 1$, $s_n = (1/n)\sum_{i=0}^{n-1} x_i$. If $(x_n)_{n\geqslant 0}$ is nonexpansive, then the asymptotic behavior of s_n as well as x_n was studied in [9–12], generalizing the nonlinear ergodic theorems of J. B. Baillon [2, 3]. (Strong convergence of s_n was proved for odd nonexpansive sequences in H.) In [13] we defined the notion of an almost nonexpansive sequence, and proved ergodic theorems for such sequences in H by suitably modifying our previous proofs for nonexpansive sequences, and applied the results to the study of the asymptotic behaviour of quasi-autonomous dissipative systems in H.

In this paper, after recalling the basic definitions, we study first the weak convergence of s_n and x_n for general sequences in H, and then consider the special cases of nonexpansive and almost nonexpansive sequences. Then we study the strong convergence of s_n and give a direct proof of a result of H. Brézis and F. E. Browder [6].

2. PRELIMINARIES

DEFINITION 2.1. (a) the sequence $(x_n)_{n\geqslant 0}$ in H is nonexpansive if $\forall i, j \geqslant 0$ $\|x_{i+1} - x_{j+1}\| \leqslant \|x_i - x_j\|$.

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(b) $(x_n)_{n\geq 0}$ is an odd nonexpansive sequence (abbreviated as ONES) if $(x_n)_{n\geq 0}$ is nonexpansive and $\forall i, j\geq 0 \|x_{i+1}+x_{j+1}\| \leq \|x_i+x_j\|$.

(c) $(x_n)_{n\geq 0}$ is an almost nonexpansive sequence (abbreviated as ANES) if $\forall i, j, k \geq 0 \|x_{i+k} - x_{j+k}\|^2 \leq \|x_i - x_j\|^2 + \varepsilon(i, j)$, where $\lim_{i,j\to+\infty} \varepsilon(i, j)=0$.

(d) $(x_n)_{n\geq 0}$ is an almost odd nonexpansive sequence (abbreviated as AONES) if $(x_n)_{n\geq 0}$ is an ANES and $\forall i, j, k\geq 0$ $\|x_{i+k}+x_{j+k}\|^2 \leq \|x_i+x_j\|^2+\varepsilon_1(i, j)$, where $\lim_{i,j} \varepsilon_1(i, j)=0$.

(e) $(x_n)_{n\geq 0}$ is asymptotically regular (abbreviated as a.r.) if $x_{n+1}-x_n \to_{n\to+\infty} 0$.

(f) $(x_n)_{n\geq 0}$ is weakly asymptotically regular (abbreviated as w.a.r.) if $x_{n+1}-x_n \rightharpoonup_{n\to+\infty} 0$.

DEFINITION 2.2. Given a bounded sequence $(x_n)_{n\geq 0}$ in H, the asymptotic center c of $(x_n)_{n\geq 0}$ is defined as follows (cf. [5, 14]). For every $u\in H$, let $\varphi(u)=\limsup_{n\to+\infty} \|x_n-u\|^2$. Then φ is a continuous, strictly convex function on H, satisfying $\varphi(u)\to_{\|u\|\to+\infty}+\infty$. Thus φ achieves its minimum on H at a unique point c, called the asymptotic center of the sequence $(x_n)_{n\geq 0}$.

Notation 2.3. (a) $\omega_w((x_n)_{n\geq 0})$ denotes the weak-ω-limit set of $(x_n)_{n\geq 0}$ (i.e., $y\in\omega_w((x_n)_{n\geq 0})\Leftrightarrow \exists (n_k)_{k\geq 1}$ such that $x_{n_k}\rightharpoonup_{k\to\infty} y$).

(b) We denote by $F((x_n)_{n\geq 0})$, or for simplicity by F, the following subset (possibly empty) of H : $F=\{q\in H \mid \lim_{n\to+\infty}\|x_n-q\| \text{ exists}\}$. Note that if $F\neq\emptyset$ then the sequence $(x_n)_{n\geq 0}$ is bounded.

(c) We denote by $F_1((x_n)_{n\geq 0})$, or for simplicity by F_1, the following subset (possibly empty) of F: $F_1=\{q\in H/\text{the sequence } (\|x_n-q\|)_{n\geq 0} \text{ is nonincreasing}\}\subset F$.

(d) If K is a nonempty closed convex subset of H, we denote by P_K the nearest point projection map of H onto K.

3. WEAK CONVERGENCE

Let $(x_n)_{n\geq 0}$ be any sequence in H, and $\forall n\geq 1$, $s_n=(1/n)\sum_{i=0}^{n-1} x_i$.

LEMMA 3.1. *F and F_1 are closed convex (possibly empty) subsets of H.*

Proof. The proof for F was given in [13, Lemma 2.5] and we omit it here. For F_1 we do as follows. Let $\forall n\geq 0$, $F_{1,n}=\{q\in H \mid \|x_{n+1}-q\|\leq \|x_n-q\|\}$; then we have $\|x_{n+1}-q\|^2\leq\|x_n-q\|^2 \Leftrightarrow (x_n-x_{n+1}\mid q)\leq \frac{1}{2}(\|x_n\|^2-\|x_{n+1}\|^2)$. Hence $\forall n\geq 0$, $F_{1,n}=\{q\in H/(x_n-x_{n+1}\mid q)\leq\frac{1}{2}(\|x_n\|^2-$

$\|x_{n+1}\|^2$). Now by the continuity and linearity of the inner product, it is clear that for every $n \geqslant 0$, $F_{1,n}$ is a closed convex subset of H. Therefore, since we have $F_1 = \bigcap_{n=0}^{\infty} F_{1,n}$ the same is also true for F_1.

EXAMPLE 3.2. The following example shows that F_1 may be a proper subset of F (even for an ANES). Take $H = R$ (the real line) and $x_0 = 0$ and

$$\forall k \geqslant 0 \begin{cases} x_{4k+1} = \dfrac{1}{2k+1} = -x_{4k+3} \\ x_{4k+2} = \dfrac{1}{2k+2} = -x_{4k+4}. \end{cases}$$

Then $(x_n)_{n \geqslant 0}$ is an ANES, $x_n \to_{n \to +\infty} 0$, and we have $F = \{0\}$ but $F_1 = \emptyset$.

THEOREM 3.3. *Let $(x_n)_{n \geqslant 0}$ be any sequence in H for which $F \neq \emptyset$. Assume that the weak limit of* every *weakly convergent subsequence $(s_{m_l})_{l \geqslant 1}$ of s_n belongs to F. Then s_n converges weakly to some $p \in H$. Moreover under these conditions we have*:

(a) $\overline{\text{conv}}\, \omega_n((x_n)_{n \geqslant 0}) \cap F = \{p\}$.

(b) *p is the asymptotic center of the sequence $(x_n)_{n \geqslant 0}$.*

Proof. The proof is similar to that of Theorem 2.8 in [13] and we omit it here.

THEOREM 3.4. *Let $(x_n)_{n \geqslant 0}$ be any sequence in H for which $F_1 \neq \emptyset$. Assume that the weak limit of* every *weakly convergent subsequence $(s_{m_l})_{l \geqslant 1}$ of s_n belongs to F_1. Then in addition to the conclusions of Theorem* 3.3 *we have*

(c) $p = \lim_{n \to +\infty} P_{F_1} x_n$.

Proof. Let $y_n = P_{F_1} x_n$ and let us first prove that $(y_n)_{n \geqslant 0}$ is a Cauchy sequence in H.

First note that we have $\|x_{n+1} - y_{n+1}\| \leqslant \|x_{n+1} - y_n\| \leqslant \|x_n - y_n\|$ (the first inequality holds since $y_{n+1} = P_{F_1} x_{n+1}$ and the second because $y_n \in F_1$). Therefore $(\|x_n - y_n\|)_{n \geqslant 0}$ is a nonincreasing sequence and hence converges. Next by the property of the projection map P_{F_1} we have $\forall n,\ k \geqslant 0$, $\|y_{n+k} - y_n\|^2 + \|x_{n+k} - y_{n+k}\|^2 - \|x_{n+k} - y_n\|^2 = 2(x_{n+k} - y_{n+k} \mid y_n - y_{n+k}) \leqslant 0$. Hence, $\|y_{n+k} - y_n\|^2 \leqslant \|x_{n+k} - y_n\|^2 - \|x_{n+k} - y_{n+k}\|^2 \leqslant \|x_n - y_n\|^2 - \|x_{n+k} - y_{n+k}\|^2 \to_{n \to +\infty} 0$ (uniformly in $k \geqslant 0$, since $\|x_n - y_n\|$ converges as $n \to +\infty$). The second inequality above holds since $y_n \in F_1$. This proves that $(y_n)_{n \geqslant 0}$ is a Cauchy sequence in H, thus converges $y_n \to_{n \to +\infty} q$. Now let us prove that $q = p$. We have $\forall k \geqslant 0$, $(x_k - y_k \mid p - y_k) \leqslant 0$ (since $p \in F_1$). Hence $(x_k \mid p - y_k) \leqslant (y_k \mid p - y_k)$. But we have

$(y_k | p-y_k) \to_{k\to+\infty} (q|p-q)$ and $(1/n)\sum_{k=0}^{n-1}(x_k|p-y_k)=(s_n|p-q)+(1/n)\sum_{k=0}^{n-1}(x_k|q-y_k) \to_{n\to+\infty} (p|p-q)$ since $s_n \rightharpoonup_{n\to+\infty} p$ and $F_1 \neq \varnothing$; therefore $\|x_n\|$ is bounded ($\forall n \geqslant 0$, $\|x_n\| \leqslant M$). Hence $|(1/n)\sum_{k=0}^{n-1}(x_k|q-y_k)| \leqslant (M/n)\sum_{k=0}^{n-1}\|y_k-q\| \to_{n\to+\infty} 0$. Therefore we proved that $(p|p-q) \leqslant (q|p-q)$; hence $\|p-q\|^2 \leqslant 0$, which implies that $q=p$, and completes the proof of the Theorem.

EXAMPLE 3.5. The following example shows that in Theorem 3.4 (hence also in Theorem 3.3), the sequence $z_n = P_F x_n$ may not be convergent in H (even when $F_1 \neq \varnothing$ and $(x_n)_{n\geqslant 0}$ is a nonexpansive (even isometric) sequence).

Take $H = l^2(N)$ and let $(e_n)_{n\geqslant 0}$ be the standard basis for H. Then we have $e_n \rightharpoonup_{n\to+\infty} 0$ and $\forall i,\ j \geqslant 0\ \|e_i - e_j\| = \sqrt{2}$; hence $(e_n)_{n\geqslant 0}$ is an isometric (thus nonexpansive) sequence in H. We have $F = H$ since $\forall q \in H$, $\|e_n - q\|^2 = 1 + \|q\|^2 - 2(e_n|q) \to_{n\to+\infty} 1 + \|q\|^2$ and $q \in F_1 \Leftrightarrow \forall n \geqslant 0$, $\|e_{n+1} - q\|^2 \leqslant \|e_n - q\|^2 \Leftrightarrow \forall n \geqslant 0$, $(e_n - e_{n+1}|q) \leqslant 0 \Leftrightarrow \forall n \geqslant 0$, $\alpha_n \leqslant \alpha_{n+1}$ (where $q = \sum_{i=0}^{\infty} \alpha_i e_i$). Thus $F_1 = \{q = (\alpha_i)_{i\geqslant 0} | \sum_{i=0}^{\infty} \alpha_i^2 < +\infty$ and $\forall i \geqslant 0$, $\alpha_i \leqslant \alpha_{i+1}\}$ (of course this implies that $\forall i \geqslant 0$, $\alpha_i \leqslant 0$). Now we have $z_n = P_F e_n = P_H e_n = e_n$ which does not converge (strongly) in H. Of course, we have $\forall n \geqslant 0$, $y_n = P_{F_1} e_n = 0$ which is in accord with the conclusion of Theorem 3.4. To see this, we have $\forall q = \sum_{i=0}^{\infty} \alpha_i e_i \in F_1$, $\|e_n - \sum_{i=0}^{\infty} \alpha_i e_i\|^2 = 1 + \sum_{i=0}^{\infty} \alpha_i^2 - 2\alpha_n \geqslant 1 = \|e_n\|^2$ (since $\alpha_n \leqslant 0$). Thus $P_{F_1} e_n = 0$.

COROLLARY 3.6. *Let $(x_n)_{n\geqslant 0}$ be an ANES in H satisfying $\liminf_{n\to+\infty} \|s_n\| < +\infty$. Then the conclusions of Theorem* 3.3 *hold.*

Proof. In fact, in this case, if $s_{m_l} \rightharpoonup_{l\to+\infty} p$, then $p \in F$ (cf. Proposition 2.6 and Theorem 2.8 in [13]).

COROLLARY 3.7. *Let $(x_n)_{n\geqslant 0}$ be a nonexpansive sequence in H satisfying $\liminf_{n\to+\infty} \|s_n\| < +\infty$. Then the conclusions of Theorems* 3.3 *and* 3.4 *hold.*

Proof. In fact, in this case if $s_{m_l} \rightharpoonup_{l\to+\infty} p$, then $p \in F_1$ (cf. Theorem 1 in [11]).

EXAMPLE 3.8. The following example shows that in Theorem 3.4 (and Theorem 3.3) the word *every* cannot be replaced by *some*. Of course, by Corollary 3.6 the sequence $(x_n)_{n\geqslant 0}$ is not an ANES. Let $H = R$ (the real line) and take $x_0 = 1$, $x_1 = -1$, and

$$x_k = \begin{cases} 1 & \text{if} \quad 2\cdot 3^n \leqslant k \leqslant 4\cdot 3^n - 1 \\ -1 & \text{if} \quad 4\cdot 3^n \leqslant k \leqslant 2\cdot 3^{n+1} - 1 \end{cases} \qquad \text{for} \quad n = 0.1, 2 \ldots .$$

Then we have $F_1 = \{0\} \neq \emptyset$ and

$$\forall n \geqslant 0 \begin{cases} s_{2.3^n} = 0 \in F_1 \\ s_{4.3^n} = 1/2. \end{cases}$$

Thus the sequence $(s_n)_{n \geqslant 1}$ is divergent.

THEOREM 3.9. *Let $(x_n)_{n \geqslant 0}$ be any sequence in H satisfying $\omega_w((x_n)_{n \geqslant 0}) \subset F \neq \emptyset$. Then $(x_n)_{n \geqslant 0}$ is weakly convergent in H to its asymptotic center.*

Proof. The proof is similar to that of Theorem 2.11 in [13] and we omit it here.

COROLLARY 3.10. *Let $(x_n)_{n \geqslant 0}$ be a w.a.r. ANES in H satisfying $\liminf_{n \to +\infty} \|x_n\| < +\infty$. Then $(x_n)_{n \geqslant 0}$ converges weakly in H to its asymptotic center.*

Proof. In fact in this case we have $\emptyset \neq \omega_w((x_n)_{n \geqslant 0}) \subset F$ (cf. Proposition 2.10 and Theorem 2.11 in [13]).

EXAMPLE 3.11. The following example shows that the weak asymptotic regularity of the sequence $(x_n)_{n \geqslant 0}$ does not imply $\liminf_{n \to +\infty} \|x_n\| < +\infty$, even for a nonexpansive (even isometric) sequence. Take $H = l^2(N)$ and let $(e_n)_{n \geqslant 0}$ be the standard basis for H. Let $\forall n \geqslant 0$, $x_n = \sum_{i=0}^{n-1} e_i$; then $x_{n+1} - x_n = e_n \rightharpoonup_{n \to +\infty} 0$ and $\|x_{i+1} - x_{j+1}\|^2 = |i-j| = \|x_i - x_j\|^2$. However, $\|x_n\|^2 = n \to_{n \to +\infty} +\infty$.

4. STRONG CONVERGENCE

PROPOSITION 4.1. *Let $(x_n)_{n \geqslant 0}$ be any sequence in H. Then the following are equivalent:*

(i) *$(x_i \mid x_{i+m}) \to_{i \to +\infty} \alpha_m$ uniformly in $m \geqslant 0$ (we call this the property (*)).*

(ii) *$(x_n)_{n \geqslant 0}$ is an AONES.*

Proof. (i)$\Rightarrow$(ii): We have $\|x_n\|^2 \to_{n \to +\infty} \alpha_0$; hence $(x_n)_{n \geqslant 0}$ is a bounded sequence in H and $\forall \varepsilon > 0 \; \exists N_0 \ni \forall i, j \geqslant N_0 \forall k \geqslant 0 \, |(x_{i+k} \mid x_{j+k}) - (x_i \mid x_j)| < \varepsilon$. Therefore by taking $\varepsilon(i, j) = | \, \|x_{i+k} - x_{j+k}\|^2 - \|x_i - x_j\|^2 | \leqslant | \, \|x_{i+k}\|^2 + \|x_{j+k}\|^2 - \|x_i\|^2 - \|x_j\|^2 | + 2 |(x_{i+k} \mid x_{j+k}) - (x_i \mid x_j)|$ we get $\limsup_{i,j \to +\infty} \varepsilon(i, j) \leqslant 2\varepsilon$, uniformly in $k \geqslant 0$, which implies that $(x_n)_{n \geqslant 0}$ is an ANES in H since $\varepsilon > 0$ was arbitrary. Similarly from the equality $\|x_{i+k} + x_{j+k}\|^2 - \|x_i + x_j\|^2 = \|x_{i+k}\|^2 + \|x_{j+k}\|^2 - \|x_i\|^2 - \|x_j\|^2 +$

$2((x_{i+k}\mid x_{j+k})-(x_i\mid x_j))$, it follows that $\|x_{i+k}+x_{j+k}\|^2-\|x_i+x_j\|^2 \to_{i,j\to+\infty} 0$ (uniformly in $k\geqslant 0$). Therefore $(x_n)_{n\geqslant 0}$ is an AONES in H.

(ii) $\Rightarrow$ (i): First we prove that from (ii) it follows that $\|x_n\|$ converges. Taking $i=j$ and using the oddness, we get $4\|x_{i+m}\|^2 \leqslant 4\|x_i\|^2+\varepsilon_1(i)$. Hence $\forall i\geqslant 0$, $\limsup_{j\to+\infty}\|x_j\|^2\leqslant\|x_i\|^2+\varepsilon_1(i)/4$. Therefore $\limsup_{j\to+\infty}\|x_j\|^2\leqslant\liminf_{i\to+\infty}\|x_i\|^2$, which implies that $\|x_n\|$ converges. Now let us prove that $(x_i\mid x_{i+m})$ is Cauchy, uniformly in $m\geqslant 0$, and this completes the proof. For $l\leqslant k$, we have $\|x_{m+k}-x_k\|^2\leqslant(x_{m+l}-x_l\|^2+\varepsilon(m+l,l)$. Hence $2[(x_{m+l}\mid x_l)-(x_{m+k}\mid x_k)]\leqslant\|x_{m+l}\|^2+\|x_l\|^2-\|x_{m+k}\|^2-\|x_k\|^2+\varepsilon(m+l,l)$. Similarly from the inequality $\|x_{m+k}+x_k\|^2\leqslant\|x_{m+l}+x_l\|^2+\varepsilon_1(m+l,l)$ it follows that $2[(x_{m+k}\mid x_k)-(x_{m+l}\mid x_l)]\leqslant\|x_{m+l}\|^2+\|x_l\|^2-\|x_{m+k}\|^2-\|x_k\|^2+\varepsilon_1(m+l,l)$. Therefore we have

$$\begin{aligned}
&2\,|(x_{m+l}\mid x_l)-(x_{m+k}\mid x_k)|\\
&\quad\leqslant\|x_{m+l}\|^2+\|x_l\|^2-\|x_{m+k}\|^2-\|x_k\|^2\\
&\qquad+\operatorname{Max}(\varepsilon(m+l,l),\varepsilon_1(m+l,l))\xrightarrow[\substack{l,k\to\infty\\(l\leqslant k)}]{}0
\end{aligned}$$

uniformly in $m\geqslant 0$, since $\|x_n\|$ converges.

As an immediate corollary we have the following:

COROLLARY 4.2. *Let $(x_n)_{n\geqslant 0}$ be any sequence in H satisfying the property* (*). *Then,*

(1) *s_n converges weakly to some $p\in H$. Moreover we have:*

  (a) $\overline{\operatorname{conv}}\,\omega_w((x_n)_{n\geqslant 0})\cap F=\{p\}$

  (b) *p is the asymptotic center of the sequence $(x_n)_{n\geqslant 0}$*

  (c) $(x_n\mid p)\to_{n\to+\infty}\|p\|^2$

(2) *$(x_n)_{n\geqslant 0}$ converges weakly if and only if it is w.a.r.*

(3) *$(x_n)_{n\geqslant 0}$ converges strongly if and if it is asymptotically regular.*

Proof. (1) By Proposition 4.1, $(x_n)_{n\geqslant 0}$ is an AONES; hence it is an ANES, and since $\|x_n\|$ is convergent, therefore bounded, the weak convergence of s_n, (a), and (b) all follow from Corollary 3.6. To prove (c), we know that $p\in F$; hence $\lim_{n\to+\infty}\|x_n-p\|^2=d^2$ exists. But we have

$$(x_n\mid p)=\tfrac{1}{2}(\|x_n\|^2+\|p\|^2-\|x_n-p\|^2)\xrightarrow[n\to+\infty]{}\tfrac{1}{2}(\alpha_0+\|p\|^2-d^2).$$

Hence $\lim_{n\to+\infty}(x_n\mid p)$ exists, and since $(s_n\mid p)\to_{n\to+\infty}\|p\|^2$ it follows that $\lim_{n\to+\infty}(x_n\mid p)=\|p\|^2$.

(2) This follows from Corollary 3.10 since $(x_n)_{n\geqslant 0}$ is an ANES.

(3) Necessity is obvious. Now to prove the sufficiency, assume that $x_{n+1} - x_n \to_{n \to +\infty} 0$ then, $\forall m \geq 0$ fixed, $\|x_n - x_{n+m}\| \leq \sum_{i=0}^{m-1} \|x_{n+i} - x_{n+i+1}\| \to_{n \to +\infty} 0$. Therefore, $\forall m \geq 0$ fixed, we have $(x_i \mid x_{i+m}) = \frac{1}{2}(\|x_i\|^2 + \|x_{i+m}\|^2 - \|x_i - x_{i+m}\|^2) \to_{i \to +\infty} \alpha_0$. Hence, $\forall m \geq 0$, $\alpha_m = \alpha_0$. Thus, $\|x_{n+k} - x_n\|^2 = \|x_n\|^2 + \|x_{n+k}\|^2 - 2(x_n \mid x_{n+k}) \to_{n \to +\infty} \alpha_0 + \alpha_0 - 2\alpha_k = 0$ (uniformly in $k \geq 0$). Hence $(x_n)_{n \geq 0}$ is a Cauchy sequence in H and thus convergent.

EXAMPLE 4.3. The following example shows that $(x_n)_{n \geq 0}$ may not converge if it is not asymptotically regular. Take $H = R$ (the real line) and $\forall n \geq 0$, $x_n = (-1)^n$. Then $(x_n)_{n \geq 0}$ is an ONES ($s_n \to_{n \to +\infty} 0$), but $(x_n)_{n \geq 0}$ does not converge.

In the sequel, we prove that if a sequence $(x_n)_{n \geq 0}$ in H satisfies the property (*) (or equivalently if it is an AONES) then s_n is strongly convergent. This theorem is proved by H. Brézis and F. E. Browder [6]. Short proofs are given also in [9] and [12].

Here we give a direct proof using our results on ANES.

First we recall the following classical lemmas on numerical sequences. Let $(\beta_n)_{n \geq 1}$ be a sequence of real (complex) numbers $a_n = (1/n) \sum_{i=1}^n \beta_i$ and $b_n = (1/n^2) \sum_{i=1}^n i\beta_i$.

LEMMA 4.4. *If $a_n \to_{n \to +\infty} \beta$ then $(1/n^2) \sum_{i=1}^n ia_i \to_{n \to +\infty} (\beta/2)$.*

Proof. For $\varepsilon > 0$ given, assume $|a_n - \beta| \leq \varepsilon$ for $n \geq n_0$ and let $M = \operatorname{Max}_{1 \leq i \leq n_0} |a_i| + |\beta|$. Then for $n > n_0$ we have

$$\begin{aligned}
&\left| \frac{1}{n(n+1)} \sum_{i=1}^{n} ia_i - \frac{\beta}{2} \right| \\
&\quad = \frac{1}{n(n+1)} \left| \sum_{i=1}^{n} i(a_i - \beta) \right| \\
&\quad \leq \frac{1}{n(n+1)} \left[\sum_{i=1}^{n_0} i(|a_i| + |\beta|) + \sum_{i=n_0+1}^{n} |a_i - \beta| \right] \\
&\quad \leq \frac{1}{n(n+1)} \left[\frac{Mn_0(n_0+1)}{2} + \frac{\varepsilon n(n+1)}{2} \right] \\
&\quad = \frac{Mn_0(n_0+1)}{2n(n+1)} + \frac{\varepsilon}{2}.
\end{aligned}$$

Hence, $\limsup_{n \to +\infty} |(1/n(n+1)) \sum_{i=1}^n ia_i - (\beta/2)| \leq \varepsilon/2$ which completes the proof since $\varepsilon > 0$ was arbitrary.

LEMMA 4.5. *If $a_n \to_{n \to +\infty} \beta$ then $b_n \to_{n \to +\infty} \beta/2$.*

Proof. We have $\forall n \geqslant 1$, $\beta_n = na_n - (n-1)\, a_{n-1}$; hence

$$\begin{aligned} b_n &= \frac{1}{n^2} \sum_{i=1}^{n} i\beta_i = \frac{1}{n^2} \sum_{i=1}^{n} i(ia_i - (i-1)\, a_{i-1}) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} [(i^2 a_i - (i-1)^2\, a_{i-1} - (i-1)\, a_{i-1})] \\ &= a_n - \frac{1}{n^2} \sum_{i=1}^{n} (i-1)\, a_{i-1} = a_n - \frac{1}{n^2} \sum_{i=1}^{n-1} ia_i \\ &= a_n - \frac{1}{n^2} \sum_{i=1}^{n} ia_i + \frac{a_n}{n} \xrightarrow[n \to +\infty]{} \beta - \frac{\beta}{2} = \frac{\beta}{2} \end{aligned}$$

(by Lemma 4.4).

THEOREM 4.6. *If $(x_n)_{n \geqslant 0}$ is any sequence in H satisfying the property* (*) (*or equivalently if it is an AONES*) *then s_n converges strongly to the asymptotic center p of the sequence $(x_n)_{n \geqslant 0}$.*

Proof. We already know by Corollary 4.2 that $s_n \rightharpoonup_{n \to +\infty} p$. Therefore to prove strong convergence it is enough to show that $\limsup_{n \to +\infty} \|s_n\|^2 \leqslant \|p\|^2$. Assume $\forall n \geqslant 0$, $\|x_n\| \leqslant M$, and for $\varepsilon > 0$ given, $|(x_i \mid x_{i+r}) - (x_k \mid x_{k+r})| \leqslant \varepsilon$, $\forall k \geqslant N_0$, $\forall i \geqslant k$, $\forall r \geqslant 0$. Then for $k \geqslant N_0$ fixed we have

$$\begin{aligned} \|s_n\|^2 &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (x_i \mid x_j) = \frac{1}{n^2} \sum_{i=0}^{n-1} \|x_i\|^2 \\ &\quad + \frac{2}{n^2} \sum_{r=1}^{n-1} \sum_{i=0}^{n-1-r} (x_i \mid x_{i+r}) \\ &\leqslant \frac{M^2}{n} + \frac{2}{n^2} \sum_{r=1}^{n-1} \sum_{i=0}^{n-1-r} ((x_i \mid x_{i+r}) - (x_k \mid x_{k+r})) \\ &\quad + \frac{2}{n^2} \sum_{r=1}^{n-1} (n-r)(x_k \mid x_{k+r}) \\ &\leqslant \frac{M^2}{n} + \frac{2}{n^2} \sum_{r=1}^{n-1} \sum_{i=0}^{k-1} 2M^2 + \frac{2\varepsilon}{n^2} \sum_{r=1}^{n-1} \sum_{i=k}^{n-1-r} 1 \\ &\quad + 2\left(x_k \,\middle|\, \frac{1}{n} \sum_{r=1}^{n-1} \left(1 - \frac{r}{n}\right) x_{k+r}\right) \\ &\leqslant \frac{M^2}{n} + \frac{4M^2 k}{n} + 2\varepsilon + 2\left(x_k \,\middle|\, \frac{1}{n} \sum_{r=1}^{n-1} x_{k+r}\right) \\ &\quad - 2\left(x_k \,\middle|\, \frac{1}{n^2} \sum_{r=1}^{n-1} r x_{k+r}\right). \end{aligned}$$

Therefore for $k \geqslant N_0$ we get (by Lemma 4.5) $\limsup_{n \to +\infty} \|s_n\|^2 \leqslant 2\varepsilon + 2(x_k | p) - (x_k | p) = 2\varepsilon + (x_k | p)$. Now letting $k \to +\infty$, by Corollary 4.2.1(c) we get $\limsup_{n \to +\infty} \|s_n\|^2 \leqslant \|p\|^2 + 2\varepsilon$, from which the result follows since $\varepsilon > 0$ was arbitrary.

Remark 4.7. Theorem 2 in [6] extends Theorem 2.1(i) in [7] and Theorem 1(b) in [24], where it was first shown, using the concept of almost convergence, that the "properness" condition of Theorem 2 in [26] is not needed.

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