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Vertical slot fishways: Mathematical modeling and optimal management

L.J. Alvarez-Vázquez[a,*], A. Martínez[a], M.E. Vázquez-Méndez[b], M.A. Vilar[b]

[a]Departamento de Matemática Aplicada II, ETSI Telecomunicación, Universidad de Vigo, 36310 Vigo, Spain
[b]Departamento de Matemática Aplicada, Escola Politécnica Superior, Universidad de Santiago de Compostela, 27002 Lugo, Spain

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Abstract

Fishways are the main type of hydraulic devices currently used to facilitate migration of fish past obstructions (dams, waterfalls, rapids, . . .) in rivers. In this paper we present a mathematical formulation of an optimal control problem related to the optimal management of a vertical slot fishway, where the state system is given by the shallow water equations, the control is the flux of inflow water, and the cost function reflects the need of rest areas for fish and of a water velocity suitable for fish leaping and swimming capabilities. We give a first-order optimality condition for characterizing the optimal solutions of this problem. From a numerical point of view, we use a characteristic-Galerkin method for solving the shallow water equations, and we use an optimization algorithm for the computation of the optimal control. Finally, we present numerical results obtained for the realistic case of a standard nine pools fishway.
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1. An introduction to the mathematical problem

Many countries with important fisheries dependent on populations of migratory fish have recognized for the past century the importance of preserving and enhancing natural stocks of diadromous and resident fish, by means of specific laws or regulations protecting fish that are affected by dam construction and other water-use projects. They provide that the owner of any obstruction to migration be responsible for providing facilities for fish passage. The structural design of a fishway is done along with the design of the dam by the owner, however he is not expected to be an expert in fishway design. And here is where the awareness of the importance of the subject appears.

Diadromous fish are fish that migrate between freshwater and saltwater. Their migration patterns differ for each species: some diadromous fish migrate great distances, while others migrate much shorter ones. In both cases, fish undergo physiological changes that allow them to survive as they migrate. There are three types of diadromous fish, depending on their specific migration patterns: anadromous, catadromous and amphidromous. Anadromous fish spend

* Corresponding author. Tel.: +34 986812166; fax: +34 986812116.
E-mail addresses: lino@dma.uvigo.es (L.J. Alvarez-Vázquez), aurea@dma.uvigo.es (A. Martínez), ernesto@lugo.usc.es (M.E. Vázquez-Méndez), miguel@lugo.usc.es (M.A. Vilar).

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most of their adult lives in saltwater, and migrate to freshwater rivers and lakes to reproduce. Anadromous fish species include lamprey (*lampetra fluviatilis*, *petromyzon marinus*), sturgeon (*acipenser sturio*), salmon (*salmo salar*), and trout (*salmo trutta*). More than half of all diadromous fish in the world are anadromous. Catadromous fish spend most of their adult lives in freshwater, and migrate to saltwater to spawn. Juvenile fish migrate back upstream where they stay until maturing into adults, at which time the cycle starts again. One of the main catadromous species is the eel (*anguilla anguilla*). About one quarter of all diadromous fish are catadromous. Finally, amphidromous species move between estuaries and coastal rivers and streams, usually associated with the search for food or refuge rather than the need to reproduce. Amphidromous fish can spawn in either freshwater or a marine environment. Less than one fifth of all diadromous fish are amphidromous.

A fishway (also known in the literature as fish-ladder or fish-pass) is an hydraulic structure that enable fish to overcome obstructions to their spawning and other river migrations, and is built whenever it is required, based on ecological, economical, or legal considerations. Fishways are generally divided into three groups: pool and weir type [9], Denil type [11], and vertical slot type [23]. Pool and weir fishways were the earliest type constructed and are still built with the addition of orifices in their walls. A pool and weir fishway consists of a number of pools formed by a series of weirs. The fish passes over a weir by swimming at burst speed (or in some cases—salmon, trout, etc.—by jumping over it). The fish then rests in the pool, then passes over the next weir, and so on, till it completes the ascent. The success of this type of fishway depends on the maintenance of water levels, which can be facilitated by the provision of a set of orifices in the weir walls close to the floor.

The Denil fishway is essentially a straight rectangular flume provided with closely spaced baffles or vanes on the bottom and sides. (The first of the classical works of G. Denil on the scientific design of fish-passes was already published in 1909 in *Annales de Travaux Publiques de Belgique*). Of the many types of Denil fishway studied in the scientific literature, the more commonly used are the simple Denil fishway and the more complex "Alaska Steep-pass".

We deal here with the third type of fishway, that is the more generally adopted for upstream passage of fish in streams obstructions: the vertical slot fishway. It consists of a rectangular channel with a sloping floor that is divided into a number of pools. Water runs downstream in this channel, through a series of vertical slots from one pool to the next one below. The water flow forms a jet at the slot, and the energy is dissipated by mixing in the pool. The fish ascends, using its burst speed, to get past the slot, then it rests in the pool till the next slot is tried [6]. Thus, a fishway can be considered as a water passage around or through an obstruction, so designed as to dissipate the energy in the water in such a manner as to enable the fish to ascend without undue stress.

Our main aim consists of finding the optimal normal flux of incoming water for the vertical slot fishway so that the higher number of fish can ascend through the obstacle in the river in their best conditions. (The authors have recently studied a related problem where the subject was the optimal shape design of the vertical slot fishway [4]). In order to develop our mathematical study we make use of several tools related to the optimal control theory and the optimization techniques, which have been very useful in the mathematical resolution of other environmental problems previously addressed by the authors [18,3,2]. Other nice applications of optimal control theory to environmental problems can be found, among others, in [8,12,22].

Section 2 is devoted to present a mathematical formulation of the optimal control problem for a standard nine pools channel, where the state system is given by the shallow water equations determining the height of water and its velocity (averaged in height), the control is the normal flux of water on the inflow, and the objective function is related to the existence of rest areas for fish and a water velocity suitable for fish leaping and swimming capabilities. In Section 3 we derive a first order optimality condition for characterizing the optimal solutions, obtained by adjoint state techniques. From a numerical point of view, in last Sections we use a characteristic-Galerkin method for solving the shallow water (Saint Venant) equations, and a derivative-free algorithm for the computation of the optimal control. Finally, we present numerical results obtained for the nine pools fishway under study.

2. Mathematical setting of the ecological problem

In our study we consider a fishway $\omega \subset \mathbb{R}^2$ consisting of nine pools built in a rectangular channel. Each pool has a width of $W = 0.97$ m and a length of $L = 1.213$ m. We also consider two transition pools, one at the beginning and other at the end of the channel, of the same width and a length of 2.0155 m The baffles separating the pools have a width of $r = 0.061$ m and are made vertical to a flume bed slope that ranges from 2% to 20%. The standard fishway used in our numerical experiments is schematized in Fig. 1: the large baffle is 0.845 m long, the short one is 0.061 m long, the horizontal distance between both baffles is 0.060 m, and the bed slope is 5%. In the figure water enters by the left side and runs downstream to the right side, and fish ascend in the opposite direction.

Water flow in the channel along the time interval $(0, T)$ is governed by the shallow water (2D Saint Venant) equations:

$$\left.\begin{aligned} &\frac{\partial H}{\partial t} + \vec{\nabla}.\vec{Q} = 0 && \text{in } \omega \times (0, T), \\ &\frac{\partial \vec{Q}}{\partial t} + \vec{\nabla}.\left(\vec{Q} \otimes \frac{\vec{Q}}{H}\right) + gH\vec{\nabla}(H - \eta) = \vec{f} && \text{in } \omega \times (0, T), \end{aligned}\right\} \tag{1}$$

where $H(x, t)$ is the height of water at point $x = (x_1, x_2) \in \omega$ at time $t \in (0, T)$, $\vec{u}(x, t) = (u_1, u_2)$ is the averaged horizontal velocity of water, $\vec{Q} = (Q_1, Q_2) = H\vec{u}$ is the flux, g is the gravity acceleration, $\eta(x)$ represents the bottom geometry of the fishway, and $\vec{f}$ is the source term. These equations must be completed with a set of initial and boundary conditions. In order to do that, we need to define three different parts in the boundary of ω: the lateral boundary of the channel, denoted by γ_0, the inflow boundary, denoted by γ_1 and the outflow boundary, denoted by γ_2. We also consider $\vec{n} = (n_1, n_2)$ the unit outer normal vector to boundary $\partial\omega = \gamma_0 \cup \gamma_1 \cup \gamma_2$, and the unit tangent vector $\vec{\tau} = (-n_2, n_1)$. Thus, the boundary and initial conditions read in the following classical form (cf. [20] or [1]): we assume the normal flux and the vorticity to be null on the lateral walls of the fishway, we impose an inflow flux in the normal direction, and we fix the height of water on the outflow boundary, that is,

$$\left.\begin{aligned} &\vec{Q}.\vec{n} = 0 && \text{on } \gamma_0 \times (0, T), \\ &\operatorname{curl}\left(\frac{\vec{Q}}{H}\right) = 0 && \text{on } \gamma_0 \times (0, T), \\ &\vec{Q} = q\vec{n} && \text{on } \gamma_1 \times (0, T), \\ &H = \Psi && \text{on } \gamma_2 \times (0, T), \\ &H(0) = H_0 && \text{in } \omega, \\ &\vec{Q}(0) = \vec{Q}_0 && \text{in } \omega. \end{aligned}\right\} \tag{2}$$

We must remark that, since $\{\vec{n}, \vec{\tau}\}$ form an orthonormal basis of $\mathbb{R}^2$, the boundary condition $\vec{Q} = q\,\vec{n}$ on $\gamma_1 \times (0, T)$ is equivalent to the two conditions $\vec{Q}.\vec{n} = q$ and $\vec{Q}.\vec{\tau} = 0$ on $\gamma_1 \times (0, T)$.

There exist several works in the mathematical literature related to the study of solution of shallow water equations in particular cases (in a first attempt to deal with the well-posedness of the shallow water equations, Ton [25] proved local—in time—existence and uniqueness of strong solution to the Dirichlet problem using Hölder estimates and smooth initial data. Later, Kloeden [14] proved global existence and uniqueness of strong solution to the homogeneous Dirichlet problem using Sobolev estimates. Subsequently, Sundbye [24] proved global existence and uniqueness of strong solution to the Dirichlet problem for small initial data and small forcing. From another point of view, Orenga [21] obtained an existence result for the weak solution to the Dirichlet problem with non-smooth data. In the same spirit, Chatelon and Orenga [7] obtained smoothness and uniqueness results for the weak solution to the problem in the case that $\operatorname{curl}(\vec{u}) = 0$ and $\vec{u}.\vec{n}$ is prescribed on the whole boundary). However, the analysis of the general case is still an open (and hard) problem.

In our case, the control will be the normal flux $q(t)$ on the inflow boundary γ_1. Since we need to inject water through γ_1 (i.e., we need $q \leqslant 0$) and due to technological reasons, we are led to consider only the admissible fluxes in the set:

$$U_{\text{ad}} = \{l \in L^2(0, T) : -B \leqslant l \leqslant 0\} \tag{3}$$

with $B > 0$ a technological bound for water inflow.

Finally, we introduce the objective function which is intended for obtaining an optimal velocity of water in such a way that in the zone of the channel near the slots (say the lower third) the velocity be as close as possible to a desired horizontal velocity c suitable for fish leaping and swimming capabilities (it is usually known as threshold velocity, and corresponds to the minimum current velocity that leads to the appearance of fish orientation against the current, highly depending on the species of fish). In the remaining of the fishway, the velocity must be very small for making possible the rest of the fish. Moreover, in all the channel, we must minimize the existence of flow turbulence. Thus, if we define the target velocity $\vec{v}$ by

$$\vec{v}(x_1, x_2) = \begin{cases} (c, 0) & \text{if } x_2 \leqslant \frac{1}{3}W, \\ (0, 0) & \text{otherwise,} \end{cases} \tag{4}$$

the objective function is given by

$$J(q) = \frac{1}{2}\int_0^T \int_\omega \left\| \frac{\vec{Q}}{H} - \vec{v} \right\|^2 \mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{2}\int_0^T \int_\omega \left| \mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\right|^2 \mathrm{d}x\,\mathrm{d}t, \tag{5}$$

where $\alpha \geqslant 0$ is a weight parameter for the role of the vorticity in the whole cost function, and $(H, \vec{Q})$ is solution of the state system (1) with boundary and initial conditions (2).

Then the optimization problem, denoted by $(\mathscr{P})$, consists of finding the optimal flux $q \in U_{\mathrm{ad}}$ on the fishway inflow in such a way that, verifying the state system (1)–(2), minimizes the cost function J given by (5). Thus, the problem can be written as

$$(\mathscr{P}) \qquad \min_{q \in U_{\mathrm{ad}}} J(q).$$

3. The control problem

Since the theory regarding existence, uniqueness and regularity for solutions to shallow water equations is still incomplete, as remarked in previous section, the question about the existence of solution for problem $(\mathscr{P})$ will not be discussed here. Moreover, the problem will be non-convex because of the nonlinearity of the state system, so uniqueness of solution is not expected.

We will center our attention in obtaining a formal first-order optimality condition satisfied by the solutions of problem $(\mathscr{P})$. A complete review of the mathematical tools related to optimal control theory, optimality conditions and adjoint method can be found, for instance, in the classical books of Lions [16] and Marchuk [17].

In order to express this necessary optimality condition in a simpler way we introduce the functions $(p, \vec{r})$ solutions of the adjoint system:

$$\left.\begin{aligned} & -\frac{\partial p}{\partial t} + \frac{1}{H^2}(\vec{Q}.\vec{\nabla})\vec{r}.\vec{Q} - gH(\vec{\nabla}.\vec{r}) - g\vec{\nabla}\eta.\vec{r} \\ & \quad = -\left(\frac{\vec{Q}}{H} - \vec{v}\right).\frac{\vec{Q}}{H^2} - \alpha\,\vec{\mathrm{curl}}\left(\mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\right).\frac{\vec{Q}}{H^2} && \text{in } \omega \times (0, T), \\ & -\frac{\partial \vec{r}}{\partial t} - \vec{\nabla} p - \frac{1}{H}(\vec{\nabla}\vec{r})^T\vec{Q} - \frac{1}{H}(\vec{Q}.\vec{\nabla})\vec{r} \\ & \quad = \frac{1}{H}\left(\frac{\vec{Q}}{H} - \vec{v}\right) + \frac{\alpha}{H}\vec{\mathrm{curl}}\left(\mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\right) && \text{in } \omega \times (0, T) \end{aligned}\right\} \tag{6}$$

with boundary and final conditions:

$$\left.\begin{aligned} & H\vec{r}.\vec{n} = 0 && \text{on } \gamma_0 \times (0, T), \\ & \left(gH - \frac{q^2}{H^2}\right)\vec{r}.\vec{n} = 0 && \text{on } \gamma_1 \times (0, T), \\ & \left\{p + \frac{1}{\Psi}(\vec{Q}.\vec{r})\right\}\vec{n} + \frac{1}{\Psi}(\vec{Q}.\vec{n})\vec{r} - \frac{\alpha}{\Psi}\mathrm{curl}\left(\frac{\vec{Q}}{\Psi}\right)\vec{\tau} = 0 && \text{on } \gamma_2 \times (0, T), \\ & p(T) = 0 && \text{in } \omega, \\ & \vec{r}(T) = \vec{0} && \text{in } \omega. \end{aligned}\right\} \tag{7}$$

We must recall that, for a given vector field $\vec{w} = (w_1, w_2)$, we denote $\mathrm{curl}(\vec{w}) = \partial w_2/\partial x - \partial w_1/\partial y$; and, for a given scalar field s, we denote $\vec{\mathrm{curl}}(s) = (\partial s/\partial y, -\partial s/\partial x)$.

Thus, assuming the solvability of the control problem, and of the state and the adjoint systems, we can state the following necessary optimality condition:

Theorem 1. *Let $q \in U_{\mathrm{ad}}$ be a solution of the control problem $(\mathscr{P})$. Let $(H, \vec{Q})$ and $(p, \vec{r})$ be, respectively, the corresponding solutions of the state system* (1)–(2) *and the adjoint system* (6)–(7). *Then, the following relation is verified*:

$$\int_0^T (l-q)\int_{\gamma_1}\left\{p+2\frac{q}{H}\vec{r}.\vec{n}\right\}\mathrm{d}\gamma\,\mathrm{d}t \leqslant 0 \quad \forall l \in U_{\mathrm{ad}}. \tag{8}$$

Proof. Since q is solution of the minimization problem $(\mathscr{P})$, the following inequality holds:

$$DJ(q)\cdot(l-q) \geqslant 0 \quad \forall l \in U_{\mathrm{ad}}. \tag{9}$$

Let $(H, \vec{Q})$ be a state corresponding to the optimal control q, then we have

$$\begin{aligned} DJ(q)\cdot(l-q) = {} & \int_0^T\int_\omega \left(\frac{\vec{Q}}{H}-\vec{v}\right).\frac{\bar{\vec{Q}}}{H}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\omega \left(\frac{\vec{Q}}{H}-\vec{v}\right).\frac{\vec{Q}\bar{H}}{H^2}\,\mathrm{d}x\,\mathrm{d}t \\ & + \alpha\int_0^T\int_\omega \mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\mathrm{curl}\left(\frac{\bar{\vec{Q}}}{H}\right)\mathrm{d}x\,\mathrm{d}t - \alpha\int_0^T\int_\omega \mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\mathrm{curl}\left(\frac{\vec{Q}\bar{H}}{H^2}\right)\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

where $(\bar{H}, \bar{\vec{Q}}) = (D/Dq)(H, \vec{Q})(q)\cdot(l-q)$ is given by the linearized system:

$$\left.\begin{aligned} & \frac{\partial \bar{H}}{\partial t} + \vec{\nabla}.\bar{\vec{Q}} = 0 && \text{in } \omega\times(0,T), \\ & \frac{\partial \bar{\vec{Q}}}{\partial t} + \vec{\nabla}.\left(\bar{\vec{Q}}\otimes\frac{\vec{Q}}{H}\right) + \vec{\nabla}.\left(\vec{Q}\otimes\frac{\bar{\vec{Q}}}{H}\right) + \vec{\nabla}.\left(\vec{Q}\otimes\frac{\vec{Q}\bar{H}}{H^2}\right) && \\ & \quad + g\bar{H}\vec{\nabla}(H-\eta) + gH\vec{\nabla}\bar{H} = \vec{0} && \text{in } \omega\times(0,T) \end{aligned}\right\} \tag{10}$$

with boundary and initial conditions:

$$\left.\begin{aligned} & \bar{\vec{Q}}.\vec{n} = 0 && \text{on } \gamma_0\times(0,T), \\ & \mathrm{curl}\left(\frac{\bar{\vec{Q}}}{H}\right) = \mathrm{curl}\left(\frac{\vec{Q}\bar{H}}{H^2}\right) && \text{on } \gamma_0\times(0,T), \\ & \bar{\vec{Q}} = (l-q)\vec{n} && \text{on } \gamma_1\times(0,T), \\ & \bar{H} = 0 && \text{on } \gamma_2\times(0,T), \\ & \bar{H}(0) = 0 && \text{in } \omega, \\ & \bar{\vec{Q}}(0) = \vec{0} && \text{in } \omega. \end{aligned}\right\} \tag{11}$$

Taking into account the boundary conditions (2) and (11), we have

$$\begin{aligned} DJ(q)\cdot(l-q) = {} & \int_0^T\int_\omega \left\{\frac{1}{H}\left(\frac{\vec{Q}}{H}-\vec{v}\right) + \frac{\alpha}{H}\vec{\mathrm{curl}}\left(\mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\right)\right\}.\bar{\vec{Q}}\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_0^T\int_\omega \left\{\left(\frac{\vec{Q}}{H}-\vec{v}\right).\frac{\vec{Q}}{H^2} + \alpha\,\vec{\mathrm{curl}}\left(\mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\right)\frac{\vec{Q}}{H^2}\right\}\bar{H}\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_0^T\int_{\gamma_2}\frac{\alpha}{H}\mathrm{curl}\left(\frac{\vec{Q}}{H}\right)\tau.\bar{\vec{Q}}\,\mathrm{d}\gamma\,\mathrm{d}t. \end{aligned}$$

Then, by using the adjoint and linearized systems:

$$
\begin{aligned}
DJ(q)\cdot(l-q) &= \int_0^T \int_\omega \left\{ -\frac{\partial \vec{r}}{\partial t} - \vec{\nabla} p - \frac{1}{H}(\vec{\nabla}\vec{r})^T \vec{Q} - \frac{1}{H}(\vec{Q}.\vec{\nabla})\vec{r} \right\} .\bar{\vec{Q}}\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_\omega \left\{ -\frac{\partial p}{\partial t} + \frac{1}{H^2}(\vec{q}.\vec{\nabla})\vec{r}.\vec{Q} + g\vec{\nabla}(H-\eta).\vec{r} - g\vec{\nabla}.(H\vec{r}) \right\} \bar{H}\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_{\gamma_2} \left\{ p + \frac{1}{\Psi}(\vec{Q}.\vec{r}) \right\} \vec{n} + \left\{ \frac{1}{\Psi}(\vec{Q}.\vec{n})\vec{r} \right\} .\bar{\vec{Q}}\, \mathrm{d}\gamma\, \mathrm{d}t \\
&= \int_0^T \int_\omega \left\{ \frac{\partial \bar{H}}{\partial t} + \vec{\nabla}.\bar{\vec{Q}} \right\} p\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\omega \left\{ \frac{\partial \bar{\vec{Q}}}{\partial t} + \vec{\nabla}.\left(\bar{\vec{Q}} \otimes \frac{\vec{Q}}{H} \right) \right\} .\vec{r}\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_\omega \left\{ \vec{\nabla}.\left(\vec{Q} \otimes \frac{\bar{\vec{Q}}}{H} \right) + \vec{\nabla}.\left(\vec{Q} \otimes \frac{\vec{Q}\bar{H}}{H^2} \right) + g\bar{H}\vec{\nabla}(H-\eta) + gH\vec{\nabla}\bar{H} \right\} .\vec{r}\, \mathrm{d}x\, \mathrm{d}t \\
&\quad - \int_0^T \int_{\gamma_0} gH\bar{H}\vec{r}.\vec{n}\, \mathrm{d}\gamma\, \mathrm{d}t - \int_0^T \int_{\gamma_1} \left\{ gH\bar{H}\vec{r}.\vec{n} - \frac{\bar{H}}{H^2} q^2 \vec{r}.\vec{n} \right\} \mathrm{d}\gamma\, \mathrm{d}t \\
&\quad - \int_0^T \int_{\gamma_1} \left\{ p\,\bar{\vec{Q}}.\vec{n} + \frac{1}{H}(\vec{Q}.\vec{r})(\bar{\vec{Q}}.\vec{n}) + \frac{1}{H}(\vec{Q}.\vec{n})(\bar{\vec{Q}}.\vec{r}) \right\} \mathrm{d}\gamma\, \mathrm{d}t \\
&= -\int_0^T \int_{\gamma_1} \left\{ p(l-q) + \frac{1}{H}(l-q)\vec{Q}.\vec{r} + \frac{1}{H} q\,\bar{\vec{Q}}.\vec{r} \right\} \mathrm{d}\gamma\, \mathrm{d}t.
\end{aligned}
$$

Finally, from the boundary conditions for the state and the linearized systems, we can deduce that $\vec{Q}.\vec{r} = q\,\vec{r}.\vec{n}$, and $\bar{\vec{Q}}.\vec{r} = (l-q)\vec{r}.\vec{n}$ on $\gamma_1 \times (0, T)$, and, consequently

$$
DJ(q)\cdot(l-q) = -\int_0^T (l-q) \int_{\gamma_1} \left\{ p + 2\frac{q}{H}\vec{r}.\vec{n} \right\} \mathrm{d}\gamma\, \mathrm{d}t. \tag{12}
$$

Taking now this expression to (9) we obtain the optimality condition (8). □

4. The numerical problem

In order to minimize the objective function J the standard proposal consists of using a gradient-type algorithm, where the gradient $\vec{\nabla} J(q)$ can be directly obtained from expression (12) *via* the computation of the adjoint system (6)–(7). However, due to the high computational cost arisen from the numerical resolution of this adjoint system, in this paper we alternatively propose a gradient-free algorithm for solving the discretized optimization problem, where the adjoint system is not needed to be solved. (This algorithm has already given very good results in the case of the optimal shape design [4]). In order to do this, we will change our problem into an unconstrained optimization problem by introducing a penalty function involving the constraints $q \leqslant 0$ and $-B - q \leqslant 0$.

Due to technological reasons (flow control mechanisms cannot act upon water flow in a continuous way, but discontinuously at short time periods) we choose to seek the control between the piecewise-constant $L^2(0, T)$ functions. So, for the time interval $[0, T]$ we choose a number $M \in \mathbb{N}$, we consider the time step $\Delta\tau = T/M > 0$ and we define the discrete times $\tau_m = m\Delta\tau$ for $m = 0, 1, \ldots, M$. Thus, a function $q \in L^2(0, T)$ which is constant at each subinterval determined by the grid $\{\tau_0, \tau_1, \ldots, \tau_M\}$ is completely fixed by the set of values $q^{\Delta\tau} = (q^0, q^1, \ldots, q^{M-1}) \in \mathbb{R}^M$, where $q^m = q(\tau_m),\ m = 0, \ldots, M-1$. For this $q^{\Delta\tau}$, the shallow water equations are solved by using an implicit discretization in time, upwinding the convective term by the method of characteristics, and Raviart–Thomas finite elements for the space discretization (the whole details of the numerical scheme can be seen in [5]). So, for the time interval $[0, T]$ we choose a number $N \in \mathbb{N}$, consider the time step $\Delta t = T/N > 0$ and define the discrete times $t_n = n\Delta t$

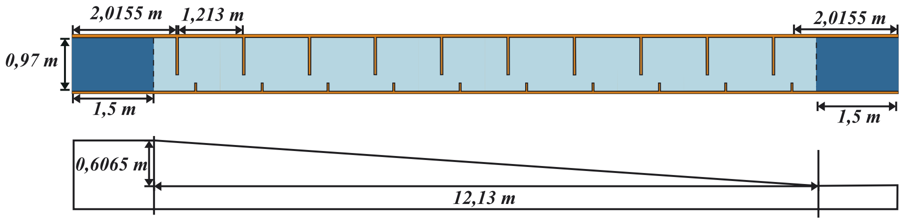

Fig. 1. Ground plant and elevation of the standard nine pools fishway ω under study.

for $n = 0, \ldots, N$. We also consider a Lagrange–Galerkin finite element triangulation τ_h of the domain ω. Thus, the numerical scheme provides us, for each discrete time t_n, with an approximated flux $\vec{Q}_h^n$ and an approximated height H_h^n, which are piecewise-linear polynomials and discontinuous piecewise-constant functions, respectively. With these approximated fields we can compute the approximated velocity $\vec{u}_h^n = \vec{Q}_h^n / H_h^n$, and approach the objective function value $J(q)$ by the expression:

$$\tilde{\Phi}(q^{\Delta\tau}) = \frac{1}{2}\Delta t \sum_{n=1}^{N} \sum_{T \in \tau_h} \int_T \|\vec{u}_h^n - \vec{v}\|^2 \, \mathrm{d}x + \frac{\alpha}{2}\Delta t \sum_{n=1}^{N} \sum_{T \in \tau_h} \int_T |\mathrm{curl}(\vec{u}_h^n)|^2 \, \mathrm{d}x. \tag{13}$$

Thus, we define the penalty function Φ in the following way:

$$\Phi(q^{\Delta\tau}) = \tilde{\Phi}(q^{\Delta\tau}) + \beta \sum_{m=0}^{M} \max\{q^m, -B - q^m, 0\}, \tag{14}$$

where the parameter $\beta > 0$ determines the relative contribution of the objective function and the penalty terms. Function Φ is an exact penalty function in the sense that, for sufficiently large β, the solutions of our original constrained problem $(\mathscr{P})$ are equivalent to the minimizers of function Φ (cf. [10]).

For computing a minimum of this non-differentiable function Φ we use a direct search algorithm: the Nelder–Mead simplex method [19]. This is a gradient-free method, which merely compares function values; the values of the objective function being taken from a set of sample points (simplex) are used to continue the sampling. A short description of the above algorithm can be found, for instance, in the nice paper of Kelley [13]. Although the Nelder–Mead algorithm is not guaranteed to converge in the general case, it has good convergence properties in low dimensions (cf. [15] for a detailed analysis of its convergence under convexity requirements). Moreover, to prevent stagnation at a non-optimal point, we use a modification proposed by Kelley: when stagnation is detected, we modify the simplex by an oriented restart, replacing it by a new smaller simplex.

In the final part of this section we present the numerical results obtained by using above method to determine the optimal inflow flux for the nine pools channel introduced in Fig. 1, with a slope of 5%. Both initial and boundary conditions were taken as constant, particularly, $\vec{Q}_0 = (0, 0)\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $H_0 = 0.5\,\mathrm{m}$, $\Psi = 0.5\,\mathrm{m}$. The time interval for the simulation was $T = 300\,\mathrm{s}$. Moreover, for the sake of simplicity, for the second member $\vec{f}$ we have only considered the bottom friction stress for a Chezy coefficient of $57.36\,\mathrm{m}^{0.5}\,\mathrm{s}^{-1}$. For the objective function we have taken a target velocity value $c = 0.8\,\mathrm{m\,s}^{-1}$ and a technological bound $B = 0.12\,\mathrm{m}^2\,\mathrm{s}^{-1}$, and we have chosen the parameters $\alpha = 0$, $\beta = 10^4$. For the time discretization we have taken $N = 3000$ (that is, a time step of $\Delta t = 0.1\,\mathrm{s}$), and for the space discretization we have tried a regular triangulation of 10492 elements.

Although we have developed many numerical experiences, we present here only one example corresponding to the case of $M = 4$ time subintervals. Thus, applying the Nelder–Mead algorithm, we have passed, after 66 function evaluations, from an initial random cost $\Phi = 612.37$ to the minimum cost $\Phi = 431.27$, corresponding to the optimal flux $q^0 = -0.114,\ q^1 = -0.085,\ q^2 = -0.066,\ q^3 = -0.116$.

Fig. 2 shows water velocities at times $t = 100, 200, 300\,\mathrm{s}$ in the fifth pool, corresponding to the initial random flux (left), and to the optimal flux (right). It can be seen how, in the controlled case, the optimal velocity is close to the target velocity $\vec{v}$ (a horizontal velocity in the lower third and a rest area in the remaining) at all times. Moreover, the

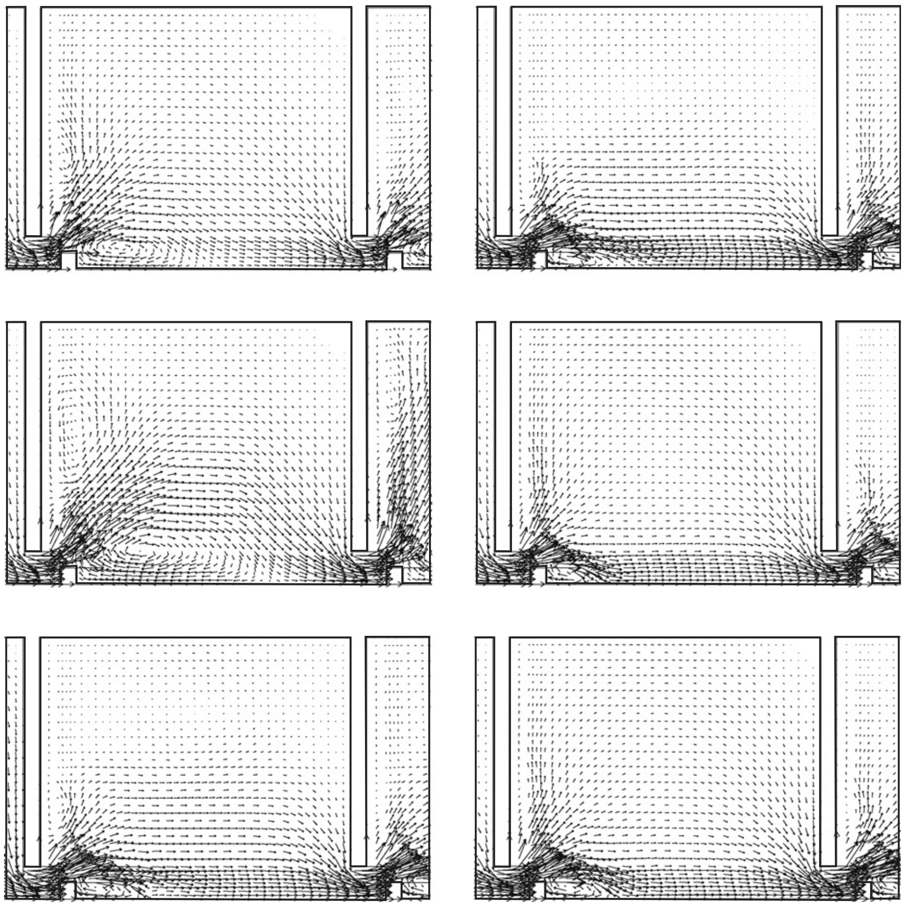

Fig. 2. Uncontrolled (left) and controlled (right) velocities for the central pool at times $t = 100, 200, 300$ s.

two large recirculation regions at both sides of the slot (which can be clearly noticed in the figure corresponding to time $t = 200$ s for the uncontrolled case) are also highly reduced.

5. Conclusions

In this paper the authors have formulated and solved an optimal control problem related to the management of vertical slot fishways in rivers. Once the ecological problem is mathematically well posed in terms of water height and flux, a numerical discretization method is presented for solving the shallow water equations involved in the modeling. Also a direct search method (the Nelder–Mead algorithm) is proposed for solving the discrete optimization problem. Finally, the efficiency of the algorithm is confirmed by the numerical experiments developed by the authors for a realistic case.

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