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Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II

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This article is a continuation of a previous article by the author [Harmonic analysis on the quotient spaces of Heisenberg groups, *Nagoya Math. J.* **123** (1991), 103–117]. In this article, we construct an orthonormal basis of the irreducible invariant component $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ of the Hilbert space $L^2(H_R^{(g,h)}\backslash H_R^{(g,h)})$ in the previous article and also construct a nonholomorphic modular form of half integral weight using the Hermite functions.

1. INTRODUCTION

This article is a continuation of a previous article by the author. In [Y], we showed that the vector space $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ is an irreducible invariant subspace of the Hilbert space $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ with respect to the right regular representation of the Heisenberg group $H_R^{(g,h)}$ (see Section 3 for the precise definition). In this article, we construct an orthonormal basis for the vector space $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ using the *Hermite polynomials*. Hermite polynomials arise from the problem of a quantum harmonic oscillator in one dimension. They are solutions of the *confluent hypergeometric equation*. Thus Hermite polynomials can be expressed in terms of the hypergeometric functions (see (2.25a) and (2.25b)). Using the *Hermite functions*, we construct a nonholomorphic modular form of half integral weight. This implies that the hypergeometric functions are related to the theory of automorphic forms.

In Section 2, we review Hermite polynomials and Hermite functions. We collect their properties to be used in the following sections. In Section 3, we construct an orthonormal basis for the vector space $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ using Hermite polynomials. In Section 4, we prove that the theta series $\vartheta_J(\Omega)$ (see (4.14)) obtained by using the Hermite function is a nonholomorphic modular form of half integral weight. In fact, this result is a generalization of that of Vigneras [V].

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Notations. We denote Z, R, and C the ring of integers, the field of real numbers, and the field of complex numbers, respectively. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. E_g denotes the identity matrix of degree g. $\sigma(A)$ denotes the trace of a square matrix A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a real number α, $[\alpha]$ denotes the greatest integer not exceeding α. We denote by H_g the Siegel upper half plane of degree g.

$$Z_{\geqslant 0}^{(h,g)} = \{J = (J_{kl}) \in Z^{(h,g)} \mid J_{kl} \geqslant 0 \text{ for all } k, l\},$$

$$|J| = \sum_{k,l} J_{kl},$$

$$(\lambda + N + A)^J = (\lambda_{11} + N_{11} + A_{11})^{J_{11}} \cdots (\lambda_{hg} + N_{hg} + A_{hg})^{J_{hg}}.$$

2. The Hermite Functions

In this section, we collect some properties of the Hermite polynomials and the Hermite functions to be used in the following sections.

The *Hermite polynomials* $H_n(x)$ $(n = 0, 1, 2, \ldots)$ in one variable x are defined by the generating functions

$$e^{-(t^2 - 2xt)} := \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \tag{2.1}$$

The Hermite polynomial $H_n(x)$ is a solution of the differential equation, the so-called *Hermite equation*

$$y'' - 2xy' + 2ny = 0. \tag{2.2}$$

There are several ways to represent the Hermite polynomial (cf. [S]). Indeed,

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}, \qquad n = 0, 1, 2, \ldots. \tag{2.3}$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \qquad \text{(the Rodrigues formula)} \tag{2.4}$$

$$H_n(x) = \frac{i^n}{2\pi^{1/2}} \int_{-\infty}^{\infty} t^n e^{-(1/4)(t+2ix)^2} \, dt \tag{2.5}$$

$$H_n(x) = \frac{n!}{2\pi i} \oint_C \frac{e^{-(t^2 - 2xt)}}{t^{n+1}} \, dt, \tag{2.6}$$

where the contour C encloses the origin.

LEMMA 2.1. *We have the recursion formulas*

$$H_n'(x) = 2nH_{n-1}(x), \qquad n \geqslant 1. \tag{2.7}$$

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \qquad n \geqslant 1. \tag{2.8}$$

Proof. These formulas follows immediately from (2.6). Q.E.D.

LEMMA 2.2.

$$x^n = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{n!}{k!\,(n-k)!} H_{n-2k}(x). \tag{2.9}$$

Proof. (2.9) follows from (2.3). Q.E.D.

The Hermite polynomials satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx = 2^n n!\, \pi^{1/2} \delta_{mn}, \tag{2.10}$$

where δ_{mn} denotes the Kronecker delta symbol. We set

$$u_n(x) := 2^{-n/2}(n!)^{-1/2}\, \pi^{-1/4} e^{-x^2/2} H_n(x), \qquad n = 0, 1, 2, ... \tag{2.11}$$

Then according to (2.10), we have the orthonormality relation

$$\int_{-\infty}^{\infty} u_m(x)\, u_n(x)\, dx = \delta_{mn}. \tag{2.12}$$

We see easily from (2.7) and (2.8) that $u_n(x)$ is a solution of the differential equation

$$y'' - (x^2 - (2n+1))\, y = 0. \tag{2.13}$$

We set

$$c_n := (-1)^n\, (n!)^{1/2}\, 2^{n-1/4} \pi^{n/2}, \qquad n = 0, 1, 2, ... \tag{2.14}$$

Now we define the *Hermite function* $\mathscr{H}_n(x)$ by

$$e^{-2\pi(x+t)^2} e^{\pi x^2} = \sum_{n=0}^{\infty} c_n \mathscr{H}_n(x) \frac{t^n}{n!}. \tag{2.15}$$

Then we see easily from (2.1) that $\mathscr{H}_n(x)$ is given by

$$\mathscr{H}_n(x) = c_n^{-1} (2\pi)^{n/2}\, e^{-\pi x^2} H_n(-\sqrt{2\pi}\, x), \qquad n = 0, 1, 2, ... \tag{2.16}$$

Using the recursion formulas (2.7) and (2.8), we easily obtain

LEMMA 2.3. *For any positive integer* $n \in Z^+$, we have

$$c_{n+1}\mathscr{H}_{n+1}(x) + 4\pi c_n x\mathscr{H}_n(x) + 4\pi n c_{n-1}\mathscr{H}_{n-1}(x) = 0. \tag{2.17}$$

$$\mathscr{H}'_n(x) - 2\pi x\mathscr{H}_n(x) = \frac{c_{n+1}}{c_n}\mathscr{H}_{n+1}(x). \tag{2.18}$$

An easy computation and (2.13) *yields*

LEMMA 2.4.

$$\mathscr{H}_n(-x) = (-1)^n\,\mathscr{H}(x). \tag{2.19}$$

$$\hat{\mathscr{H}}_n(x) = (-i)^n\,\mathscr{H}_n(x). \tag{2.20}$$

$$\mathscr{H}''_n(x) - 4\pi^2 x^2\mathscr{H}_n(x) = -4\pi(n+\tfrac{1}{2})\,\mathscr{H}_n(x). \tag{2.21}$$

Here $\hat{f}(x)$ denotes the Fourier transform of $f(x)$. That is,

$$\hat{f}(x) := \int_{-\infty}^{\infty} f(y)\,e^{-2\pi xy}\,dy.$$

Thus $\mathscr{H}_n(x)$ is an eigenfunction of the differential operator $L = d^2/dx^2 - 4\pi^2x^2$. We set $p_n(x) := \mathscr{H}_n(x)\,e^{\pi x^2}$. Let $E := x(d/dx)$ be the Euler operator. Then (2.21) is equivalent to

$$\Delta p_n(x) = 4\pi(E-n)\,p_n(x), \tag{2.22}$$

where $\Delta = d^2/dx^2$ denotes the Laplacian operator on the real line. We set $h_n(x) := \mathscr{H}_n(x)\,e^{-\pi x^2}$. Then (2.21) is equivalent to

$$\Delta h_n(x) = -4\pi(E+n+1)\,h(x). \tag{2.23}$$

According to (2.10) and (2.16), the Hermite functions $\mathscr{H}_n(x)$ $(n = 0, 1, ...)$ satisfy the orthonormality relation

$$\int_{-\infty}^{\infty} \mathscr{H}_m(x)\,\mathscr{H}_n(x)\,dx = \delta_{mn}. \tag{2.24}$$

In the introduction, we mentioned that the Hermite polynomials are solutions of the hypergeometric equations. Indeed, Hermite polynomials are expressed in terms of hypergeometric functions (cf. [S], p. 97). Precisely,

$$H_n(x) = \frac{n!\,(-1)^{-n/2}}{(n/2)!}\,{}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; x^2\right) \qquad \text{for even } n \tag{2.25a}$$

and

$$H_n(x) = \frac{2n!\,(-1)^{(1-n)/2}}{((n-1)/2)!}\,x\,{}_1F_1\left(-\frac{n-1}{2}; \frac{3}{2}; x^2\right) \qquad \text{for odd } n. \tag{2.25b}$$

3. An Orthonormal Basis of $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$

For any positive integers g and h, we consider the Heisenberg group

$$H_R^{(g,h)} := \{ [(\lambda, \mu), \kappa] \mid \lambda, \mu \in R^{(h,g)}, \kappa \in R^{(h,h)}, \kappa + \mu\,{}^t\lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] = [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda'].$$

We denote by $H_Z^{(g,h)}$ the discrete subgroup of $H_R^{(g,h)}$ consisting of integral elements.

From now on, we fix an element Ω of the Siegel upper half plane H_g of degree g. Now for a positive definite symmetric even integral matrix $\mathcal{M}$ of degree h and $J \in Z_{\geqslant 0}^{(h,g)}$, we define a function on $H_R^{(g,h)}$

$$\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]) := e^{\pi i \sigma(\mathcal{M}(\kappa - \lambda\,{}^t\mu))}$$
$$\times \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J \, e^{\pi i \sigma\{\mathcal{M}((\lambda + N + A)\,\Omega\,{}^t(\lambda + N + A) + 2(\lambda + N + A)\,{}^t\mu)\}}, \tag{3.1}$$

where $A \in \mathcal{M}^{-1} Z^{(h,g)}/Z^{(h,g)}$.

Let $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ be the completion of the vector space spanned by $\Phi_J^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right](\Omega \mid [(\lambda, \mu), \kappa])(J \in Z_{\geqslant 0}^{(h,g)})$. Then by Theorem 2 in [Y], $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ is an irreducible invariant subspace of $L^2(H_Z^{(h,g)} \backslash H_R^{(h,g)})$ with respect to the right regular representation of the Heisenberg group $H_R^{(g,h)}$.

Now we will construct an orthonormal basis for $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ using the Hermite polynomials. For $J = (J_{kl}) \in Z_{\geqslant 0}^{(h,g)}$ and $x = (x_{kl}) \in R^{(h,g)}$, we define the *Hermite polynomial* $H_J(x)$ in several variables

$$H_J(x) := H_{J_{11}}(x_{11})\, H_{J_{12}}(x_{12}) \cdots H_{J_{hg}}(x_{hg}). \tag{3.2}$$

Then according to (2.10), Hermite polynomials $H_J(x)$ $(J \in Z_{\geqslant 0}^{(h,g)})$ satisfy the following orthogonality relation

$$\int_{R^{(h,g)}} H_J(x)\, H_K(x)\, e^{-\sigma(x\,{}^tx)}\, dx = \begin{cases} 2^{|J|} J!\, \pi^{hg/2} & \text{if } J = K \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We set $Y = \operatorname{Im} \Omega = (1/2i)(\Omega - \bar{\Omega})$. Since Y is positive definite, we may define the unique square root $Y^{1/2}$. Let $\mathcal{M}$ be a positive definite, symmetric even integral matrix of degree h and let $\mathcal{M}^{1/2}$ be its unique square root. Then by an easy computation we see that the functions

$$H_J(\sqrt{2\pi}\, \mathcal{M}^{1/2} x Y^{1/2}), \qquad J \in Z_{\geqslant 0}^{(h,g)}$$

satisfy the orthogonality relation

$$\int_{R^{(h,g)}} H_J(\sqrt{2\pi}\,\mathcal{M}^{1/2}xY^{1/2})\,H_K(\sqrt{2\pi}\,\mathcal{M}^{1/2}xY^{1/2})\,e^{-2\pi\sigma(\mathcal{M}[x]Y)}\,dx$$

$$=\begin{cases} 2^{|J|-hg/2}J!\,(\det\mathcal{M})^{-g/2}\,(\det Y)^{-h/2} & \text{if } J=K \\ 0 & \text{otherwise.}\end{cases} \tag{3.4}$$

We define

$$H_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) := 2^{hg/4-|J|/2}(J!)^{-1/2}\,(\det\mathcal{M})^{g/4}\,(\det Y)^{h/4}$$

$$\times\, e^{\pi i\sigma(\mathcal{M}(\kappa-\lambda\,{}^t\mu))}\sum_{N\in Z^{(h,g)}} H_J(\sqrt{2\pi}\,\mathcal{M}^{1/2}(\lambda+N+A)\,Y^{1/2})$$

$$\times\, e^{\pi i\sigma\{\mathcal{M}((\lambda+N+A)\,\Omega\,{}^t(\lambda+N+A)+2(\lambda+N+A)\,{}^t\mu)\}}, \tag{3.5}$$

where $A\in\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$.

LEMMA 3.1. *The functions* $H_J^{(\mathcal{M})}[{}^A_0](\Omega\,|\,[\lambda,\mu),\kappa])$ $(J\in J_{\geq 0}^{(h,g)})$ *satisfy the orthonormality relation*

$$\int_{R^{(h,g)}} H_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\,\overline{H_K^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])}\,d\lambda$$

$$=\begin{cases} 1 & \textit{if } J=K \\ 0 & \textit{otherwise}\end{cases} \tag{3.6}$$

Proof. It follows easily from (3.4) and Lemma 3.1 in [Y]. Q.E.D.

From Lemma 2.2, we obtain

THEOREM 1. *The functions* $H_J^{(\mathcal{M})}[{}^A_0](\Omega\,|\,[(\lambda,\mu),\kappa])$ $(J\in Z_{\geq 0}^{(h,g)})$ *form an orthonormal basis for* $H_\Omega^{(\mathcal{M})}[{}^A_0]$.

4. THETA SERIES ASSOCIATED TO INDEFINITE QUADRATIC FORMS

Let $q(\xi)$ be an indefinite quadratic form on R^h $(h\in Z^+)$ of signature (s,t) with $s+t=h$. Let L be a lattice in R^h such that $q(L)\subset Z$. The bilinear form $\langle\,,\rangle$ on R^h associated to the quadratic form $q(\xi)$ is given by $\langle\xi,\eta\rangle := q(\xi+\eta)-q(\xi)-q(\eta)$ $(\xi,\eta\in R^h)$. We recall that the dual L^* of a lattice L relative to $q(\xi)$ is defined by

$$L^* := \{a\in R^h\,|\,\langle a,k\rangle\in Z \text{ for all } k\in L\}.$$

We choose a basis $\{e_1, ..., e_h\}$ for the real vector space R^h such that for the coordinate $\xi = {}^t(\xi_1, ..., \xi_h) \in R^h$ with respect to this basis $\{e_1, ..., e_h\}$

$$q(\xi) = \tfrac{1}{2}(\xi_1^2 + \cdots + \xi_s^2 - \xi_{s+1}^2 - \cdots - \xi_h^2).$$

For $x = (x_1, ..., x_h) \in R^h \times \cdots \times R^h = R^{(h,g)}$ with the column vectors $x_i = {}^t(x_{1i}, x_{2i}, ..., x_{h_i})$ $(1 \leqslant i \leqslant g)$, we define

$$\tilde{q}(x) := \tfrac{1}{2}(x_{11}^2 + \cdots + x_{sg}^2 - x_{s+1,1}^2 - \cdots - x_{hg}^2),$$

$$\tilde{q}_+(x) := \tfrac{1}{2}(x_{11}^2 + \cdots + x_{hg}^2).$$

For $J = (J_{kl}) \in Z_{\geqslant 0}^{(h,g)}$, $\lambda \in R$, and $a = (a_{kl}) \in R^{(h,g)}$, we define

$$J! = J_{11}! \cdots J_{hg}!, \qquad \lambda^J = \lambda^{|J|}, \qquad a^J = a_{11}^{J_{11}} \cdots a_{hg}^{J_{hg}},$$

$$\varepsilon(J) = J_{11} + \cdots + J_{sg} - J_{s+1,1} - \cdots - J_{hg}.$$

For any $J \in Z_{\geqslant 0}^{(h,g)}$, we set

$$c_J := (-1)^J (J!)^{1/2} \, 2^{J-1/4} \pi^{J/2}. \tag{4.1}$$

We define the *Hermite functions* $\mathscr{H}_J(x)$ $(J \in Z_{\geqslant 0}^{(h,g)})$ in several variables by the relation

$$e^{-4\pi \tilde{q}_+(x+t)} \cdot e^{2\pi \tilde{q}_+(x)} = \sum_{J \in Z_{\geqslant 0}^{(h,g)}} c_J \mathscr{H}_J(x) \frac{t^J}{J!}. \tag{4.2}$$

For a function f on $R^{(h,g)}$, we define the Fourier transform by

$$\hat{f}(x) := \int_{R^{(h,g)}} f(y)\, e^{-2\pi i \langle x, y \rangle} \, dy, \qquad x \in R^{(h,g)},$$

where $\langle x, y \rangle = x_{11} y_{11} + \cdots + x_{sg} y_{sg} - x_{s+1,1} y_{s+1,1} - \cdots - x_{hg} y_{hg}$ for $x = (x_{kl})$, $y = (y_{kl}) \in R^{(h,g)}$ and dy is the normalized Haar measure so that $\mathrm{vol}(L^g) := \mathrm{vol}(R^{(h,g)}/L^g) = 1$.

LEMMA 4.1. *For $J = (J_{kl}) \in J_{\geqslant 0}^{(h,g)}$ and $x = (x_{kl}) \in R^{(h,g)}$, we have*

$$\mathscr{H}_J(x) = \mathscr{H}_{J_{11}}(x_{11})\, \mathscr{H}_{J_{12}}(x_{12}) \cdots \mathscr{H}_{J_{hg}}(x_{hg}). \tag{4.3}$$

$$\mathscr{H}_J(-x) = (-1)^J \mathscr{H}_J(x). \tag{4.4}$$

$$\hat{\mathscr{H}}_J(x) = \hat{\mathscr{H}}_{J_{11}}(x_{11}) \cdots \hat{\mathscr{H}}_{J_{sg}}(x_{sg}) \times \hat{\mathscr{H}}_{J_{s+1,1}}(-x_{s+1,1}) \cdots \hat{\mathscr{H}}_{J_{hg}}(-x_{hg}). \tag{4.5}$$

$$\hat{\mathscr{H}}_J(x) = (-1)^{J_{11} + \cdots + J_{sg}} \, i^{|J|} \mathscr{H}_J(x). \tag{4.6}$$

Proof. (4.3) follows easily from (2.15). (4.4) follows immediately from (2.19). (4.5) follows from (4.3) and the definition of the Fourier transform. (4.6) follows from (2.19), (2.20), and (4.5). Q.E.D.

LEMMA 4.2. *Let* $\Delta = \sum_{k=1}^{s} \sum_{l=1}^{g} (\partial^2/\partial x_{kl}^2) - \sum_{k=s+1}^{h} \sum_{l=1}^{g} (\partial^2/\partial x_{kl}^2)$ *be the Laplacian on* $R^{(h,g)}$ *associated with the quadratic form* $\tilde{q}(x)$. *Then we have*

$$(\Delta - 8\pi^2 \tilde{q}(x))\, \mathscr{H}_J(x) = -4\pi \left(\varepsilon(J) + \frac{(s-t)\, g}{2} \right) \mathscr{H}_J(x). \tag{4.7}$$

Proof. It follows immediately from (2.21). Q.E.D.

LEMMA 4.3. *Let* $E := \sum_{k=1}^{h} \sum_{l=1}^{g} x_{kl}(\partial/\partial x_{kl})$ *be the Euler operator on* $R^{(h,g)}$. *We set* $P_J(x) = \mathscr{H}_J(x)\, e^{2\pi \tilde{q}(x)}$ $(J \in Z_{\geqslant 0}^{(h,g)})$. *Then we have*

$$\Delta P_J(x) = 4\pi(E - \varepsilon(J) + \mu)\, P_J(x), \qquad \mu = (h-s)\, g. \tag{4.8}$$

Proof. (4.8) follows from (2.22), (2.23), and (4.7). Q.E.D.

For the present time being, we fix an element $\Omega = X + iY \in H_g$. We define the function $f_{J,\Omega}(x)$ on $R^{(h,g)}$ by

$$f_{J,\Omega}(x) := (\det Y)^{-\lambda/2}\, \mathscr{H}_J(xY^{1/2})\, e^{2\pi i \sigma(Q[x]X)}, \qquad x \in R^{(h,g)}, \tag{4.9}$$

where $\lambda = \varepsilon(J) - \mu$. Here $2Q = \operatorname{diag}(1, ..., 1, -1, ..., -1)$ is the symmetric matrix of degree h associated with the quadratic form $2q(x)$.

LEMMA 4.4. *For any* $\Omega \in H_g$, $J \in Z_{\geqslant 0}^{(h,g)}$, *and* $x \in R^h$, *we have*

$$\hat{f}_{J,-\Omega^{-1}}(x) = (-i)^a (\det \Omega)^{\lambda + h/2} f_{J,\Omega}(x), \qquad a = \varepsilon(J) + \left(\lambda + \frac{h}{2}\right) g. \tag{4.10}$$

Proof. It suffices to show (4.10) for $\Omega = iY$, $Y > 0$. Then we have

$$\begin{aligned}
\hat{f}_{J,iY^{-1}}(x) &= (\det Y)^{\lambda/2} \int_{R^{(h,g)}} \mathscr{H}_J(\xi Y^{-1/2})\, e^{-2\pi i \langle x,\xi\rangle}\, d\xi \\
&= (\det Y)^{\lambda/2 + h/2} \int_{R^{(h,g)}} \mathscr{H}_J(\xi)\, e^{-2\pi i \langle x, \xi Y^{1/2}\rangle}\, d\xi \\
&= (\det Y)^{\lambda/2 + h/2} \int_{R^{(h,g)}} \mathscr{H}_J(\xi)\, e^{-2\pi i \langle x Y^{1/2}, \xi\rangle}\, d\xi \\
&= (\det Y)^{\lambda/2 + h/2}\, \hat{\mathscr{H}}_J(xY^{1/2}) \\
&= (-1)^{J_{11} + \cdots + J_{sg}}\, i^{|J|} (\det Y)^{\lambda/2 + h/2}\, \mathscr{H}_J(xY^{1/2}) \qquad \text{by (4.6)} \\
&= (-1)^{J_{11} + \cdots + J_{sg}}\, i^{|J|} (\det Y)^{\lambda + h/2}\, f_{J,iY}(x) \qquad \text{by (4.9)}.
\end{aligned}$$

By an easy calculation, we obtain the desired result (4.10). Q.E.D.

For any $\alpha \in (L^*)^g$ and $J \in Z^{(h,g)}_{\geqslant 0}$, we define the *theta series* on H_g by

$$\vartheta_{\alpha,J}(\Omega) := \sum_{x \in L^g + \alpha} f_{J,\Omega}(x), \qquad \Omega = X + iY \in H_g. \tag{4.11}$$

It is known that the Siegel modular group $\Gamma_g := \mathrm{Sp}(g, Z)$ is generated by

$$\begin{pmatrix} E_g & S \\ 0 & E_g \end{pmatrix}, \qquad S = {}^tS \text{ integral} \qquad \text{and} \qquad \begin{pmatrix} 0 & -E_g \\ E_g & 0 \end{pmatrix}.$$

Therefore in order to investigate the transformation behaviour of the theta series $\vartheta_{\alpha,J}(\Omega)$ $(\Omega \in H_g)$ for the action of the Siegel modular group, it suffices to investigate the transformation law of $\vartheta_{\alpha,J}(\Omega)$ under the two actions $\Omega \mapsto \Omega + S$ with $S = {}^tS$ integral and $\Omega \mapsto -\Omega^{-1}$ $(\Omega \in H_g)$.

If $S = {}^tS$ is a symmetric integral matrix of degree g, by an easy computation we have

$$\vartheta_{\alpha,J}(\Omega + S) = e^{2\pi i \sigma(Q[\alpha]S)} \vartheta_{\alpha,J}(\Omega). \tag{4.12}$$

The Poisson formula says that for a function f on $R^h \times \cdots \times R^h = R^{(h,g)}$

$$\sum_{\alpha \in L^g} f(\alpha) = \sum_{\alpha \in (L^*)^g} \hat{f}(\alpha).$$

THEOREM 2. *Let $a = \varepsilon(J) + (\lambda + h/2)g$. Then for any $\alpha \in (L^*)^g$, we have the transformation law*

$$\vartheta_{\alpha,J}(-\Omega^{-1}) = (-i)^a (\det \Omega)^{\lambda + h/2} \sum_{\substack{k \in (L^*)^g \\ k (\mathrm{mod}\ L^g)}} e^{2\pi i \langle \alpha, k \rangle} \vartheta_{k,J}(\Omega). \tag{4.13}$$

Proof. Using the Poisson formula, we obtain

$$\vartheta_{\alpha,J}(-\Omega^{-1}) = \sum_{\substack{k \in (L^*)^g \\ k (\mathrm{mod}\ L^g)}} e^{2\pi i \langle \alpha, k \rangle} \sum_{x \in L^g + k} \hat{f}_{J,-\Omega^{-1}}(x).$$

Therefore by (4.10), we obtain (4.13). Q.E.D.

We now define the theta series

$$\vartheta_J(\Omega) = (\det Y)^{-\lambda/2} \sum_{\alpha \in L^g} \mathscr{H}_J(\alpha Y^{1/2})\, e^{2\pi\sigma(Q[\alpha]Y)} e^{2\pi i \sigma(Q[\alpha]\Omega)}, \tag{4.14}$$

where $\Omega = X + iY \in H_g$. According to (4.12) and Theorem 2, we have

$$\vartheta_J(\Omega + S) = \vartheta_J(\Omega), \qquad \text{for any integral} \quad S = {}^tS \in Z^{(g,g)}. \tag{4.15}$$

$$\vartheta_J(-\Omega^{-1}) = (-i)^a (\det \Omega)^{\lambda + h/2} \sum_{\substack{k \in (L^*)^g \\ k (\mathrm{mod}\ L^g)}} \vartheta_{k,J}(\Omega). \tag{4.16}$$

Therefore we obtain

THEOREM 3. *We assume that a lattice L is self-dual with respect to the quadratic form $q(x)$, that is, $L = L^*$. Then the theta series $\vartheta_J(\Omega)$ is a nonholomorphic modular form of weight $\lambda + h/2$ with respect to a certain congruence subgroup of Γ_g. Its level is the same as that of the quadratic form $\tilde{q}(x)$ on $R^{(h,g)}$.*

Remark. In [F], using the pluriharmonic forms, Freitag constructed the vector-valued theta series of a certain type and proved that this theta series is a vector-valued modular form of half integral weight with respect to a certain congruence subgroup.

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