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Representations of Toeplitz-plus-Hankel matrices using trigonometric transformations with application to fast matrix-vector multiplication

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Abstract

Representations of real Toeplitz and Toeplitz-plus-Hankel matrices are presented that involve real trigonometric transformations (DCT, DST, DHT) and diagonal matrices. These representations can be used for fast matrix-vector multiplication. In particular, it is shown that the multiplication of an $n \times n$ Toeplitz-plus-Hankel matrix by a vector requires only 4 transformations of length n plus $O(n)$ operations.

Keywords: Toeplitz matrix; Hankel matrix; Fast algorithm; Cosine transform

1. Introduction

Matrix-vector multiplication for general $n \times n$ matrices requires about $2n^2$ operations. It is desirable to reduce this number in case the matrix has a certain structure. In this paper we consider matrices with a Toeplitz $[a_{i-j}]$ or Toeplitz-plus-Hankel structure $[a_{i-j} + b_{i+j}]$. There are many motivations to consider the

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problem of fast matrix–vector multiplication by Toeplitz and Toeplitz-plus-Hankel matrices. One of them is the fact that for the solution of linear systems of equations with a Toeplitz or Toeplitz-plus-Hankel coefficient matrix iteration methods are turned to be very convenient, especially the combination of circulant or related preconditioning with the conjugate gradient method (see, for example [6]). In connection with the construction of an efficient iteration procedure the problem of fast matrix–vector multiplication emerges.

In order to multiply a vector by a circulant matrix $C = [a_{i-j}]_0^{n-1} (a_{-i} = a_{n-i})$ FFT can be applied because circulants can be diagonalized by the DFT (see e.g. [7]). Thus the multiplication by a circulant can be carried out with the help of 2 DFT's and multiplication by a diagonal matrix, and the costs are only $O(n \log n)$. The idea how to multiply a vector by a Toeplitz matrix is to reduce this problem to the circulant case. There are two possibilities to do this. The first one is to represent the Toeplitz matrix as the sum of a circulant and a skew-circulant, which is a Toeplitz matrix with $a_{-i} = -a_{n-i}$. We will refer to this way as *decomposition approach*. The second possibility is to extend the Toeplitz matrix to a circulant. We call this way *extension approach*. The decomposition and extension ideas are mentioned directly or indirectly at a number of places in the literature (see for example [10,11,16,19,26,27]) and can be considered as folklore among specialists. In [2] matrix–vector multiplication by Toeplitz matrices is proposed to carry out by embedding the Toeplitz matrix into a matrix of a multiplication operator. This is, however, less efficient than the circulant embedding.

In the case of a dense Toeplitz matrix the decomposition and the extension approaches are equivalent in the sense that they both require 4 DFT's of length n plus 2 DFT's of length n for preprocessing. In the case of a banded Toeplitz matrix the extension approach seems to be advantageous since in this case the amount can be further reduced to 2 transformations of length $n + s$ where $2s + 1$ is the bandwidth of the matrix. It is remarkable that for multiplication by a general Toeplitz-plus-Hankel matrix also only 4 DFT's of length n (or 2 DFT of length $2n$) are required plus 4 DFT for preprocessing. The extension approach for this was mentioned in [16] but a decomposition formula can also easily be found using some intertwining relation between the DFT and the counteridentity.

All operation counts above concern complex operations. If the matrix and the vector are real then it is desirable to use only real arithmetics. There are special algorithms for DFT with real data that require $n \log_2 n$ multiplications and $\frac{3}{2}n \log_2 n$ additions, provided that n is a power of 2. The same amount can be achieved for the Hartley transformation (see [8,21]). The calculation of discrete sine and cosine transform is very often reduced to real DFT, like, for example, in [27] (see also [18,20,28]). These versions require the same amount like real DFT. However, algorithms do exists with only $\frac{1}{2}n \log_2 n$, i.e. only half the number, multiplications and $\frac{3}{2}n \log_2 n$ additions (see, for example, [22,23,25]).

Therefore, it is reasonable to ask about representations of Toeplitz and Toeplitz-plus-Hankel matrices involving these transformations. Subject of the present paper is to discuss this problem. We give also representations involving the Hartley transformation. In one case, namely banded nonsymmetric Toeplitz matrices, we could get a better computational bound for the Hartley transformation compared with sine and cosine transformations.

Note that the representations with trigonometric transformations are of interest beyond the computational advantage. To mention one example, the kind of preconditioner used for a Toeplitz system may depend on the nature of the underlying continuous problem. In particular, this kind of boundary conditions determines the convenient preconditioner which is connected with a trigonometric transformation. For this reason it is desirable to have a library of representations for all trigonometric transformations.

A decomposition formula for symmetric Toeplitz matrices with the help of the sine-I and a modified cosine-I transformation, which is analogous to the circulant/skew-circulant representation for complex Toeplitz matrices, was found by Huckle and presented in [14]. The drawback of this formula is that the diagonal factor appearing there is not easily obtained by a transformation. The extension approach for symmetric Toeplitz matrices involving the sine-I transformation was discussed in [3] by Boman and Koltracht. In the unpublished note [17] of Olshevsky it is shown how triangular Toeplitz matrices can be multiplied by vectors using properties of Chebyshev polynomials. This ends up with an algorithm requiring 4 transformations and leads to a procedure to multiply general real Toeplitz matrices by vectors with the help of 8 transformations.

The structure of this paper is as follows. In Section 2 we introduce the real transformation we will use, which are the 4 common sine and cosine and the Hartley transformations. Besides the usual Hartley transformation we introduce another transformation which we call *skew-Hartley*.

In Sections 3–5 we discuss the decomposition approach, in Section 3 for the case of symmetric Toeplitz, in Section 4 for general Toeplitz, in Section 5 for general Toeplitz-plus-Hankel matrices. The main conclusion of our formulas is that a general $n \times n$ Toeplitz-plus-Hankel matrix can be multiplied by a vector with the help of only 4 transformations of length n plus $O(n)$ operations and 4 transformations of length n for preprocessing. Since the fast algorithms for trigonometric transformation are highly parallizable this would mean that in parallel computation the complexity is only $O(\log n)$. All proofs of the representations are completely elementary and use only trigonometric identities.

The Section 6 dedicated to the extension approach. This approach is in particular advantageous for banded Toeplitz matrices. The multiplication of an $n \times n$ Toeplitz matrix with bandwidth $2s + 1$ can be carried out with 2 transformations of length $n + s$ if the matrix is symmetric and with 3 transformations of this length in the general case. It is remarkable that if the Hartley transformation is used then also in the general case only 2 transformations are required.

In Section 7 we present some hybrid formulas involving both extension and decomposition. The application of these formulas involves higher costs. We nevertheless included them because some of them are quite nice and they give an indication how more formulas can be found which can be used for special purposes.

While writing this paper we took notice that Steidl and Tasche are also preparing papers [24] including representations for symmetric and triangular Toeplitz and centrosymmetric Toeplitz-plus-Hankel matrices. Furthermore, after this paper was almost completed we received a manuscript of the paper [15] by Kailath and Olshevsky which contains formulas for Toeplitz matrices using trigonometric transformations both for the decomposition and extension approaches. These formulas are derived there only for the case of symmetric matrices but the approach can be straightforwardly generalized. Note that the method in [15] is different from our approach. We thank all these authors for useful discussion on the subject. We also thank Olshevsky for providing us with the unpublished note [17].

The representation formulas discussed in this paper are not the only ones. For example, with the help of the formulas for transforming Toeplitz and Toeplitz-plus-Hankel matrices by trigonometric transformations into Cauchy matrices presented in [13] representations for Toeplitz and Toeplitz-plus-Hankel matrices can be derived, since the Cauchy matrices occuring there can be represented with the help of the same transformations. One formula of this type is presented in [14] (Theorem 7). This formula seems to be not very efficient for fast matrix–vector multiplication but there are more possibilities which could lead to more efficient formulas. Another way to obtain representations for Toeplitz and Toeplitz-plus-Hankel matrices involving sine and cosine transforms is the splitting approach described in [12] for the transformation of Toeplitz into paired Chebyshev–Vandermonde systems. The algorithms emerging from these representations are of the same complexity as those presented here. So far it is not clear to us whether this way leads to qualitatively new procedures. Let us furthermore mention the papers [5,4,9] where representations of more general Toeplitz-plus-Hankel-like matrices are derived from their displacement representation which can, of course, also be applied to simple Toeplitz-plus-Hankel matrices. This application, however, leads to algorithms with higher complexity than those contained in the present paper.

It is quite surprising that the problem of multiplication by inverses by Toeplitz and Toeplitz-plus-Hankel matrices found so far more attention in the literature than the same problem for the original matrices (see for example [10,11,5] and references therein). We are planning to discuss the corresponding representations of inverses of Toeplitz and related matrices using trigonometric transformations in a forthcoming paper.

2. Preliminaries

For convenience of notation, we define the common real trigonometric transformations in simplified form, without scaling factors:

DST-I and DCT-I:

$$\mathscr{S}_N^{\mathrm{I}} = \left[\sin\frac{(i+1)(j+1)\pi}{N+1}\right]_0^{N-1}, \qquad \mathscr{C}_N^{\mathrm{I}} = \left[\cos\frac{ij\pi}{N-1}\right]_0^{N-1},$$

DST-II and DCT-II

$$\mathscr{S}_N^{\mathrm{II}} = \left[\sin\frac{(i+1)(2j+1)\pi}{2N}\right]_0^{N-1}, \qquad \mathscr{C}_N^{\mathrm{II}} = \left[\cos\frac{i(2j+1)\pi}{2N}\right]_0^{N-1},$$

DST-III and DST-III

$$\mathscr{S}_N^{\mathrm{III}} = (\mathscr{S}_N^{\mathrm{II}})^{\mathrm{T}}, \quad \mathscr{C}_N^{\mathrm{III}} = (\mathscr{C}_N^{\mathrm{II}})^{\mathrm{T}},$$

DST-IV and DCT-IV

$$\mathscr{S}_N^{\mathrm{IV}} = \left[\sin\frac{(2i+1)(2j+1)\pi}{4N}\right]_0^{N-1}, \qquad \mathscr{C}_N^{\mathrm{IV}} = \left[\cos\frac{(2i+1)(2j+1)\pi}{4N}\right]_0^{N-1},$$

Hartley transformations

$$\mathscr{H}_N^{+} = \left[\operatorname{cas}\frac{2ij\pi}{N}\right]_0^{N-1}, \qquad \mathscr{H}_N^{-} = \left[\operatorname{cas}\frac{i(2j+1)\pi}{N}\right]_0^{N-1},$$

where $\operatorname{cas} x = \cos x + \sin x$.

The transformation $\mathscr{H}_N^{+}$ is the usual Hartley transformation DHT; $\mathscr{H}_N^{-}$ is possibly a new invention. We call it skew-Hartley transformation.

These transformations are almost unitary. More precisely,

$$\begin{aligned}
&(\mathscr{S}_N^{\mathrm{I}})^{-1} = (2/N+1)(\mathscr{S}_N^{\mathrm{I}})^{\mathrm{T}}, && (\mathscr{C}_N^{\mathrm{I}})^{-1} = (2/N-1)D(\mathscr{C}_N^{\mathrm{I}})^{\mathrm{T}}D,\\
&(\mathscr{S}_N^{\mathrm{II}})^{-1} = (2/N)(\mathscr{S}_N^{\mathrm{II}})^{\mathrm{T}}D_2, && (\mathscr{C}_N^{\mathrm{II}})^{-1} = (2/N)(\mathscr{C}_N^{\mathrm{II}})^{\mathrm{T}}D_1,\\
&(\mathscr{S}_N^{\mathrm{III}})^{-1} = (2/N)D_2(\mathscr{S}_N^{\mathrm{III}})^{\mathrm{T}}, && (\mathscr{C}_N^{\mathrm{I}})^{-1} = (2/N)D_1(\mathscr{C}_N^{\mathrm{III}})^{\mathrm{T}},\\
&(\mathscr{S}_N^{\mathrm{IV}})^{-1} = (2/N)(\mathscr{S}_N^{\mathrm{IV}})^{\mathrm{T}}, && (\mathscr{C}_N^{\mathrm{IV}})^{-1} = (2/N)(\mathscr{C}_N^{\mathrm{IV}})^{\mathrm{T}},\\
&(\mathscr{H}_N^{+})^{-1} = (1/N)(\mathscr{H}_N^{+})^{\mathrm{T}}, && (\mathscr{H}_N^{-})^{-1} = (1/N)(\mathscr{H}_N^{-})^{\mathrm{T}},
\end{aligned}$$

where

$$D_1 = \operatorname{diag}\left(\frac{1}{2}, 1, \ldots, 1\right), \quad D_2 = \operatorname{diag}\left(1, \ldots, 1, \frac{1}{2}\right), \quad D = D_1 D_2.$$

Furthermore, these transformations enjoy some intertwining relations listed next. Let J_N denote the matrix of the counteridentity,

$$J_N = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix},$$

and let Σ_N denote the matrix $\Sigma_N = \mathrm{diag}((-1)^k)_{k=0}^{N-1}$. Then

$$\begin{aligned}
&\mathscr{S}_N^{\mathrm{I}} J_N = \Sigma_N \mathscr{S}_N^{\mathrm{I}}, \quad \mathscr{C}_N^{\mathrm{I}} J_N = \Sigma_N \mathscr{C}_N^{\mathrm{I}},\\
&\mathscr{S}_N^{\mathrm{II}} J_N = \Sigma_N \mathscr{S}_N^{\mathrm{II}}, \quad \mathscr{C}_N^{\mathrm{II}} J_N = \Sigma_N \mathscr{C}_N^{\mathrm{II}},\\
&\mathscr{S}_N^{\mathrm{III}} J_N = \Sigma_N \mathscr{C}_N^{\mathrm{III}}, \quad \mathscr{C}_N^{\mathrm{III}} J_N = \Sigma_N \mathscr{S}_N^{\mathrm{III}},\\
&\mathscr{S}_N^{\mathrm{IV}} J_N = \Sigma_N \mathscr{C}_N^{\mathrm{IV}}, \quad \mathscr{C}_N^{\mathrm{IV}} J_N = \Sigma_N \mathscr{S}_N^{\mathrm{IV}}.
\end{aligned}$$

In order to describe the corresponding intertwining relations for the Hartley transformations we introduce the flip matrix

$$J'_N = \begin{bmatrix} 1 & 0 \\ 0 & J_{N-1} \end{bmatrix}.$$

Then

$$\mathscr{H}_N^+ J'_N = J'_N \mathscr{H}_N^+, \quad \mathscr{H}_N^- J_N = -J'_N \mathscr{H}_N^-.$$

Let us point out that the first J in the second equality has no prime.

For $q > p$, we denote by P_{pq} the $(q-p) \times N$ restriction matrix defined by

$$P_{pq}(x_i)_0^{N-1} = (x_i)_{i=p}^{q-1}.$$

We will use also some slight modifications of the transformations, namely

$$\mathring{\mathscr{S}}_N^{\mathrm{I}} = \left[\sin \frac{ij\pi}{N-1}\right]_0^{N-1}, \qquad \tilde{\mathscr{C}}_N^{\mathrm{I}} = \left[\cos \frac{(i+1)(j+1)\pi}{N+1}\right]_0^{N-1}.$$

The matrix $\mathring{\mathscr{S}}_N^{\mathrm{I}}$ is just the matrix $\mathscr{S}_{N-2}^{\mathrm{I}}$ bordered by zero columns and rows, i.e. $\mathring{\mathscr{S}}_N^{\mathrm{I}} = P_{1,N-1}^{\mathrm{T}} \mathscr{S}_{N-2}^{\mathrm{I}} P_{1,N-1}$. The matrix $\tilde{\mathscr{C}}_N^{\mathrm{I}}$ is a submatrix of $\mathscr{C}_{N+2}^{\mathrm{I}}$, viz. $\tilde{\mathscr{C}}_N^{\mathrm{I}} = P_{1N} \mathscr{C}_{N+2}^{\mathrm{I}} P_{1N}^{\mathrm{T}}$. The intertwining relations for these matrices are

$$\mathring{\mathscr{S}}_N^{\mathrm{I}} J_N = -\Sigma_N \mathring{\mathscr{S}}_N^{\mathrm{I}}, \qquad \tilde{\mathscr{C}}_N^{\mathrm{I}} J_N = -\Sigma_N \tilde{\mathscr{C}}_N^{\mathrm{I}}.$$

Throughout the paper, we denote by Z_N the $N \times N$ matrix of the forward shift

$$Z_N = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}.$$

Let us finally discuss some complexity issues. We denote by $\tau(n)$ the computational costs for a sine or cosine transformation of length n. At the present state of the art [24], $\tau(n) = ((1/2)n \log_2 n + cn)(\mathrm{M}) + (3/2)n \log_2 n(\mathrm{A}) + o(n)$, where (A) stands for additions and (M) for multiplications, and the factor c ranges between 0 and $\frac{5}{2}$, depending on the transformation. Furthermore, let $\phi(n)$ denote the amount for a real DFT or a Hartley transformation. One has $\phi(n) = n \log_2 n(\mathrm{M}) + (3/2)n \log_2 n(\mathrm{A}) + O(n)$.

3. Symmetric Toeplitz matrices

First we consider symmetric Toeplitz matrices $T_n = [a_{|i-j|}]_0^{n-1}$. We assume that n is already an integer which is convenient for the transformations, for example a power of 2. Otherwise T_n has to be embedded in a larger matrix which is easily done.

The vector $a = (a_j)_0^{n-1}$ is the cosine-I transform of some $\lambda = (\lambda_i)_0^{n-1}$, i.e.

$$a = \mathscr{C}_n^{\mathrm{I}} \lambda. \tag{3.1}$$

Theorem 3.1. *Let λ be given by Eq. (3.1). Then the Toeplitz matrix $T_n = [a_{|i-j|}]_0^{n-1}$ admits a representation*

$$T_n = \mathscr{C}_n^{\mathrm{I}} \Lambda \mathscr{C}_n^{\mathrm{I}} + \mathring{\mathscr{S}}_n^{\mathrm{I}} \Lambda \mathring{\mathscr{S}}_n^{\mathrm{I}}, \tag{3.2}$$

where $\Lambda = \operatorname{diag} \lambda$.

Proof. If a is given by Eq. (3.1) then

$$\begin{aligned} a_{|i-j|} &= \sum_{k=0}^{n-1} \lambda_k \cos \frac{(i-j)k\pi}{n-1} \\ &= \sum_{k=0}^{n-1} \lambda_k \left(\cos \frac{ik\pi}{n-1} \cos \frac{jk\pi}{n-1} + \sin \frac{ik\pi}{n-1} \sin \frac{jk\pi}{n-1} \right). \end{aligned}$$

Written in matrix form this means

$$\left[a_{|i-j|}\right]_0^{n-1} = \mathscr{C}_n^{\mathrm{I}} \Lambda \mathscr{C}_n^{\mathrm{I}} + \mathring{\mathscr{S}}_n^{\mathrm{I}} \Lambda \mathring{\mathscr{S}}_n^{\mathrm{I}}.$$

Now Eq. (3.2) follows immediately. $\square$

Corollary 3.2. *The multiplication of a symmetric $n \times n$ Toeplitz matrix by a vector can be carried out at the costs of* $4\tau(n) + 2n(\mathrm{M}) + n(\mathrm{A}) + o(n)$ *plus* $\tau(n)$ *for preprocessing the data.*

Let us show how Huckle's formula in [14] can be obtained from Theorem 3.1 We choose a_n and a_{n+1} in such a way that the first and last components of the cosine-I transform of the vector $a = (a_i)_0^{n+1}$ vanish. It can easily be checked that this can always be done and an explicit formula can be given. Suppose that $a = \mathscr{C}^{\mathrm{I}}_{n+2}[0\ \lambda\ 0]^{\mathrm{T}}$. We apply Theorem 3.1 for n replaced by $n+2$. Cancelling the first and last columns and rows we obtain the following.

Corollary 3.3. *Let λ be as above. Then the Toeplitz matrix $T_n = [a_{|i-j|}]_0^{n-1}$ admits a representation*

$$T_n = \tilde{\mathscr{C}}^{\mathrm{I}}_n \Lambda \tilde{\mathscr{C}}^{\mathrm{I}}_n + \mathscr{S}^{\mathrm{I}}_n \Lambda \mathscr{S}^{\mathrm{I}}_n, \tag{3.3}$$

where $\Lambda = \operatorname{diag} \lambda$.

Note that there is also a representation of T_n using the vector given by Eq. (3.1) but involving type III transformations instead of type I. This representation can be derived using the identity

$$\cos\frac{(i-j)k\pi}{n-1} = \cos\frac{(2i+1)k\pi}{2n-2}\cos\frac{(2j+1)k\pi}{2n-2} + \sin\frac{(2i+1)k\pi}{2n-2}\sin\frac{(2j+1)k\pi}{2n-2}.$$

We refrain from presenting it because it is more complicated.

Next we mention decompositions involving transformations of type II and IV.

Theorem 3.4. *Let λ be given by*

$$a = \mathscr{C}^{\mathrm{II}}_n \lambda. \tag{3.4}$$

Then the Toeplitz matrix $T_n = [a_{|i-j|}]_0^{n-1}$ admits representations

$$T_n = \mathscr{C}^{\mathrm{II}}_n \Lambda \mathscr{C}^{\mathrm{III}}_n + Z_n \mathscr{S}^{\mathrm{II}}_n \Lambda \mathscr{S}^{\mathrm{III}}_n Z_n^T, \tag{3.5}$$

$$T_n = \mathscr{C}^{\mathrm{IV}}_n \Lambda \mathscr{C}^{\mathrm{IV}}_n + \mathscr{S}^{\mathrm{IV}}_n \Lambda \mathscr{S}^{\mathrm{IV}}_n, \tag{3.6}$$

where $\Lambda = \operatorname{diag} \lambda$.

Proof. If a is given by Eq. (3.4) then

$$\begin{aligned} a_{|i-j|} &= \sum_{k=0}^{n-1} \lambda_k \cos\frac{(i-j)(2k+1)\pi}{2n} \\ &= \sum_{k=0}^{n-1} \lambda_k \left(\cos\frac{i(2k+1)\pi}{2n}\cos\frac{j(2k+1)\pi}{2n} \right. \\ &\qquad \left. + \sin\frac{i(2k+1)\pi}{2n}\sin\frac{j(2k+1)\pi}{2n} \right), \end{aligned}$$

which leads to

$$[a_{|i-j|}]_0^{n-1} = \mathscr{C}^{\mathrm{II}}_n \Lambda \mathscr{C}^{\mathrm{III}}_n + Z_n \mathscr{S}^{\mathrm{II}}_n \Lambda \mathscr{S}^{\mathrm{III}}_n Z_n^{\mathrm{T}}.$$

Now Eq. (3.5) is immediate. Relation (3.6) is proved in the same way. $\square$

We derive now a representation involving the Hartley transformations. First let us agree upon some language. A vector $x \in \mathbb{R}^n$ is called *even* if $J_n'x = x$ and *odd* if $J_n'x = -x$. The vector $x \in \mathbb{R}^n$ is said to be *symmetric* if $J_n x = x$ and *skew-symmetric* if $J_n x = -x$.

We decompose the Hartley transformations into two parts $\mathscr{H}_n^{\pm} = \mathscr{C}_{\pm} + \mathscr{S}_{\pm}$, where $\mathscr{C}_{\pm}$ is the cosine part and $\mathscr{S}_{\pm}$ the sine part of $\mathscr{H}_n^{\pm}$. We need the following facts.

Lemma 3.5.
1. *If $\lambda \in \mathbb{R}^n$ is even then $\mathscr{S}_+\lambda = 0$ and $\mathscr{H}_n^+\lambda = \mathscr{C}_+\lambda$ is even. Vice versa, if $\mathscr{C}_+\lambda$ is even then λ is even.*
2. *If $\lambda \in \mathbb{R}^n$ is symmetric then $\mathscr{S}_-\lambda = 0$ and $\mathscr{H}_n^-\lambda = \mathscr{C}_-\lambda$ is odd. Vice versa, if $\mathscr{H}_n^-\lambda$ is odd then λ is symmetric.*

Proof. The rows of $\mathscr{S}_+$ are odd and the columns of $\mathscr{C}_+$ are even. The inner product of an even and odd vector vanishes. This implies the first assertion.

The rows of $\mathscr{S}_-$ are skew-symmetric and the columns of $\mathscr{C}_-$ are odd. The inner product of a symmetric and a skew-symmetric vector vanishes. This implies the second assertion. $\square$

Lemma 3.6.
1. *If $\lambda \in \mathbb{R}^n$ is even and $\Lambda = \operatorname{diag}\lambda$ then the matrix $T = \mathscr{H}_n^+\Lambda\mathscr{H}_n^-$ is a symmetric Toeplitz matrix, $T = [a_{|i-j|}]_0^{n-1}$, where $a = (a_i)_0^{n-1} = \mathscr{H}_n^+\lambda$ is even.*
2. *If $\lambda \in \mathbb{R}^n$ is symmetric and $\Lambda = \operatorname{diag}\lambda$ then the matrix $T = \mathscr{H}_n^-\Lambda(\mathscr{H}_n^-)^{\mathrm{T}}$ is a symmetric Toeplitz matrix, $T = [a_{|i-j|}]_0^{n-1}$, where $a = (a_i)_0^{n-1} = \mathscr{H}_n^-\lambda$ is odd.* [2]

Proof. Suppose that $T = [t_{ij}]_0^{n-1}$. In the first case we obtain, taking Lemma 3.5 into account,

$$t_{ij} = \sum_{k=0}^{n-1}\lambda_k \operatorname{cas}\frac{2ik\pi}{n}\operatorname{cas}\frac{2jk\pi}{n} = \sum_{k=0}^{n-1}\lambda_k\left(\cos\frac{2(i-j)k\pi}{n} + \sin\frac{2(i+j)k\pi}{n}\right)$$
$$= a_{|i-j|}.$$

Furthermore, in the second case we have

[2] Actually, T in Lemma 3.6 (1) is a circulant matrix, which agrees with a result in [5]. The T in Lemma 3.6 (2) is a skew-circulant matrix.

$$t_{ij} = \sum_{k=0}^{n-1} \lambda_k \operatorname{cas} \frac{i(2k+1)\pi}{n} \operatorname{cas} \frac{j(2k+1)\pi}{n}$$
$$= \sum_{k=0}^{n-1} \lambda_k \left(\cos \frac{(i-j)(2k+1)\pi}{n} + \sin \frac{(i+j)(2k+1)\pi}{n} \right) = a_{|i-j|}.$$

According to Lemma 3.5, the vector a is even in the first case and odd in the second one. □

We present a representation with the Hartley transformations. Let $T_n = [a_{|i-j|}]_0^{n-1}$ be the given matrix. We set $a_n = a_0$ and define vectors $a^{\pm} = (a_i^{\pm})_0^{n-1}$ by

$$a_i^{\pm} = \tfrac{1}{2}(a_i \pm a_{n-i}).$$

Then a^+ is even and a^- is odd and $a = a^+ + a^-$. According to Lemma 3.5 there exists an even vector λ^+ and a symmetric vector λ^- such that

$$a^+ = \mathscr{H}_n^+ \lambda^+, \qquad a^- = \mathscr{H}_n^- \lambda^-. \tag{3.7}$$

The following is now immediate.

Theorem 3.7. *Let $\lambda_{\pm}$ be given by Eq. (3.5). Then the Toeplitz matrix T_n admits a representation*

$$T_n = \mathscr{H}_n^+ \Lambda^+ \left(\mathscr{H}_n^+\right)^{\mathrm{T}} + \mathscr{H}_n^- \Lambda^- \left(\mathscr{H}_n^-\right)^{\mathrm{T}},$$

where $\Lambda^{\pm} = \operatorname{diag} \lambda^{\pm}$.

Theorem 3.7 can also be obtained from the representation of T as the sum of a circulant and a skew-circulant matrix. The circulant part can be diagonalized with the Hartley transformation according to [1], and the skew-circulant part can be diagonalized according to an analogous result.

4. General Toeplitz matrices

We consider now general Toeplitz matrices $T_n = [a_{i-j}]_0^{n-1}$. First we deal with transformations involving sine-I and cosine-I transformations. We split the vector $a = (a_i)_{1-n}^{n-1}$ into its symmetric and skew-symmetric parts $a = a^+ + a^-$, $a^{\pm} = (a_i^{\pm})_{1-n}^{n-1}$, $a_i^{\pm} = (1/2)(a_i \pm a_{-i})$. Suppose that

$$(a_i^+)_0^{n-1} = \mathscr{C}_n^{\mathrm{I}} (\lambda_k^+)_0^{n-1}, \qquad (a_i^-)_1^{n-2} = \mathscr{S}_{n-2}^{\mathrm{I}} (\lambda_k^-)_1^{n-2}. \tag{4.1}$$

Then we have

$$a_{i-j} = \sum_{k=0}^{n-1} \lambda_k^+ \left(\cos \frac{ik\pi}{n-1} \cos \frac{jk\pi}{n-1} + \sin \frac{ik\pi}{n-1} \sin \frac{jk\pi}{n-1} \right)$$
$$+ \sum_{k=1}^{n-2} \lambda_k^- \left(\sin \frac{ik\pi}{n-1} \cos \frac{jk\pi}{n-1} - \cos \frac{ik\pi}{n-1} \sin \frac{jk\pi}{n-1} \right).$$

Translating this into matrix language we obtain the following.

Theorem 4.1. *Let $\lambda_k^\pm$ be given by Eq. (4.1). Then the Toeplitz matrix $T_n = [a_{i-j}]_0^{n-1}$ admits a representation*

$$T_n = \mathscr{C}_n^{\mathrm{I}} \left(\Lambda^+ \mathscr{C}_n^{\mathrm{I}} - \Lambda^- \mathring{\mathscr{S}}_n^{\mathrm{I}} \right) + \mathring{\mathscr{S}}_n^{\mathrm{I}} \left(\Lambda + \mathring{\mathscr{S}}_n^{\mathrm{I}} + \Lambda^- \mathscr{C}_n^{\mathrm{I}} \right),$$

$\Lambda^\pm = \mathrm{diag}(\lambda_i^\pm)_0^{n-1}$, *where* $\lambda_0^- = \lambda_{n-1}^- = 0$.

Similarly the following is derived.

Theorem 4.2. *Let $\lambda_i^\pm$ be given by*

$$(a_i^+)_0^{n-1} = \mathscr{C}_n^{\mathrm{II}} (\lambda_k^+)_0^{n-1}, \qquad (a_i^-)_1^n = \mathscr{S}_n^{\mathrm{II}} (\lambda_k^-)_0^{n-1},$$

where $a_i^\pm$ are defined as above and a_n^- is arbitrary. Then the Toeplitz matrix $T_n = [a_{i-j}]_0^{n-1}$ admits a representation

$$T_n = \mathscr{C}_n^{\mathrm{II}} (\Lambda^+ \mathscr{C}_n^{\mathrm{III}} - \Lambda^- Z_n^{\mathrm{T}} \mathscr{S}_n^{\mathrm{III}}) + Z_n \mathscr{S}_n^{\mathrm{II}} (\Lambda^+ Z_n^{\mathrm{T}} \mathscr{S}_n^{\mathrm{III}} + \Lambda^- \mathscr{C}_n^{\mathrm{III}}),$$

where $\Lambda^\pm = \mathrm{diag}(\lambda_i^\pm)_0^{n-1}$.

If one wants to use transformations of type III and IV one has to decompose the vector $a = (a_i)_{1-n}^{n-1}$ generating the Toeplitz matrix T_n in a different way, namely as $a = a_+ + a_-$, $a_\pm = (a_{\pm,i})_{1-n}^{n-1}$, $a_{\pm,i} = (1/2)(a_i \pm a_{-i-1}), a_{-n} = 0$. Then $a_{\pm,i} = \pm a_{\pm,-i-1}$. Now we represent

$$(a_{+,i})_0^{n-1} = \mathscr{C}_n^{\mathrm{III}} \lambda_+, \qquad (a_{-,i})_0^{n-1} = \mathscr{S}_n^{\mathrm{III}} \lambda_-. \tag{4.2}$$

A similar calculation as for the type-I transformations then leads to the following.

Theorem 4.3. *Let $\lambda_\pm$ be given by (4.2). Then the Toeplitz matrix $T_n = [a_{i-j}]_0^{n-1}$ admits a representation*

$$T_n = \Big[\mathscr{C}_n^{\mathrm{III}} \left(\Lambda_+ P_{0n} \mathscr{C}_{n+1}^{\mathrm{I}} - Z_n \Lambda_- P_{1,n+1} \mathring{\mathscr{S}}_{n+1}^{\mathrm{I}} \right)$$
$$+ \mathscr{S}_n^{\mathrm{III}} \left(Z_n^{\mathrm{T}} \Lambda_+ P_{0n} \mathring{\mathscr{S}}_{n+1}^{\mathrm{I}} + \Lambda_- P_{1,n+1} \mathscr{C}_{n+1}^{\mathrm{I}} \right) \Big] P_{0n}^{\mathrm{T}},$$

where $\Lambda_\pm = \mathrm{diag}\, \lambda_\pm$.

For a representation with type-IV transformations we assume that

$$(a_{+,i})_0^{n-1} = \mathscr{C}_n^{\mathrm{IV}}\lambda_+, \qquad (a_{-,i})_0^{n-1} = \mathscr{S}_n^{\mathrm{IV}}\lambda_-. \tag{4.3}$$

An elementary calculation leads to the following.

Theorem 4.4. *Let $\lambda_\pm$ be given by Eq. (4.3). Then the Toeplitz matrix $T_n = [a_{i-j}]_0^{n-1}$ admits a representation*

$$T_n = \mathscr{C}_n^{\mathrm{IV}}\left(\Lambda_+\mathscr{C}_n^{\mathrm{III}} - \Lambda_-\mathscr{S}_n^{\mathrm{III}}\right) + \mathscr{S}_n^{\mathrm{IV}}\left(\Lambda_+\mathscr{S}_n^{\mathrm{III}} + \Lambda_-\mathscr{C}_n^{\mathrm{III}}\right),$$

where $\Lambda_\pm = \operatorname{diag}\lambda_\pm$.

The main conclusion of this section is the following.

Corollary 4.5. *An $n \times n$ Toeplitz matrix can be multiplied by a vector at the costs of $4\tau(n) + 4n(\mathrm{M}) + 3n(\mathrm{A}) + \mathrm{o}(n)$ plus $2\tau(n)$ for preprocessing.*

In order to get representations with Hartley transformations we need the following counterpart of Lemma 3.5, which can be proved in the same way.

Lemma 4.6.
1. *If $\lambda \in \mathbb{R}^n$ is odd then $\mathscr{C}_+\lambda = 0$ and $\mathscr{H}_n^+\lambda = \mathscr{S}_+\lambda$ is odd. Vice versa, if $\mathscr{S}_+\lambda$ is odd then λ is odd.*
2. *If $\lambda \in \mathbb{R}^n$ is skew-symmetric then $\mathscr{C}_-\lambda = 0$ and $\mathscr{H}_n^-\lambda = \mathscr{S}_-\lambda$ is even. Vice versa, if $\mathscr{H}_n^-\lambda$ is even then λ is skew-symmetric.*

The counterpart of Lemma 3.6 is the following.

Lemma 4.7.
1. *If $\lambda \in \mathbb{R}^n$ is odd $\Lambda = \operatorname{diag}\lambda$ then the matrix $T = \mathscr{H}_n^+\Lambda J_n'\mathscr{H}_n^+$ is a skew-symmetric Toeplitz matrix, $T = [a_{i-j}]_0^{n-1}$, where $a - (a_i)_0^{n-1} = \mathscr{H}_n^+\lambda$ is odd.*
2. *If $\lambda \in \mathbb{R}^n$ is skew-symmetric and $\Lambda = \operatorname{diag}\lambda$ then the matrix $T = \mathscr{H}_n^-\Lambda J_n'(\mathscr{H}_n^-)^{\mathrm{T}}$ is a skew-symmetric Toeplitz matrix, $T = [a_{i-j}]_0^{n-1}$, where $a = (a_i)_0^{n-1} = \mathscr{H}_n^-\lambda^-$ is even.*

Proof. First let us recall that

$$J_n'\mathscr{H}_n^+ = \mathscr{H}_n^+J_n' = \left[\cos\frac{2ij\pi}{n} - \sin\frac{2ij\pi}{n}\right]_0^{n-1}$$

$$J_n'\mathscr{H}_n^- = -\mathscr{H}_n^-J_n = -\left[\cos\frac{i(2j+1)\pi}{n} - \sin\frac{i(2j+1)\pi}{n}\right]_0^{n-1}.$$

Suppose that $T = [t_{ij}]_0^{n-1}$. In the first case we obtain, taking Lemma 4.6 into account,

$$t_{ij} = \sum_{k=0}^{n-1} \lambda_k \left(\cos \frac{2(i+j)k\pi}{n} + \sin \frac{2(i-j)k\pi}{n} \right) = a_{i-j}.$$

In the second case we have

$$t_{ij} = \sum_{k=0}^{n-1} \lambda_k \left(\cos \frac{(i+j)(2k+1)\pi}{n} + \sin \frac{(i-j)(2k+1)\pi}{n} \right) = a_{i-j}.$$

According to Lemma 4.6 the vector a is even in the first case and odd in the second one. □

Combining Lemmas 3.6 and 4.7 we obtain a representation of general Toeplitz matrices by Hartley transformations. Suppose $T_n = [a_{i-j}]_0^{n-1}$ be the given matrix and $a = (a_i)_{1-n}^{n-1}$. We split a into its symmetric and skew-symmetric parts $a = a^+ + a^-$ and the halfs of these vectors into even and odd parts. That means we define the following vectors $a_{\pm\pm} = (a_{\pm\pm,i})_{1-n}^{n-1}$ by

$$\begin{aligned} a_{+\pm,i} &= \tfrac{1}{4}(a_i \pm a_{n-i} + a_{-i} \pm a_{i-n}), \\ a_{-\pm,i} &= \tfrac{1}{4}(a_i \pm a_{n-i} - (a_{-i} \pm a_{i-n})). \end{aligned} \tag{4.4}$$

Furthermore, let $\lambda_{\pm\pm}$ be given by

$$\begin{aligned} (a_{++,i})_0^{n-1} &= \mathscr{H}_n^+ \lambda_{++}, \qquad & (a_{+-,i})_0^{n-1} &= \mathscr{H}_n^- \lambda_{+-}, \\ (a_{-+,i})_0^{n-1} &= \mathscr{H}_n^- \lambda_{-+}, \qquad & (a_{-,i})_0^{n-1} &= \mathscr{H}_n^+ \lambda_{--}. \end{aligned} \tag{4.5}$$

note that λ_{++} is even, λ_{+-} is symmetric, λ_{-+} is skew-symmetric, and λ_{--} is odd.

Theorem 4.8. *Let $\lambda_{\pm\pm}$ be given by Eq. (4.5). Then the Toeplitz matrix $T_n = [a_{i-j}]_0^{n-1}$ admits a representation*

$$T_n = \mathscr{H}_n^+ (\Lambda_{++} + \Lambda_- J_n') \mathscr{H}_n^+ + \mathscr{H}_n^- (\Lambda_{+-} + \Lambda_{-+} J_n)(\mathscr{H}_n^-)^{\mathrm{T}}, \quad \Lambda_{\pm\pm} = \operatorname{diag}\lambda_{\pm\pm}.$$

5. General Toeplitz-plus-Hankel matrices

We consider now general $n \times n$ Toeplitz-plus-Hankel matrices R_n. These matrices can be represented in the form

$$R_n = T_n^{(1)} + T_n^{(2)} J_n, \quad T_n^{(1)} = [a_{i-j}^{(l)}]_0^{n-1}, \quad (l = 1, 2). \tag{5.1}$$

Applying the theorems of Section 4 and the intertwining relations for the transformations listed in Section 2 we obtain the following.

Theorem 5.1. *1. Suppose that $a_{\pm,i}^{(l)} = \frac{1}{2}(a_i^{(l)} \pm a_{-i}^{(l)})$,*

$$(a_{+,i}^{(l)})_0^{n-1} = \mathscr{C}_n^{\mathrm{I}} (\lambda_{+,k}^{(l)})_0^{n-1}, \qquad (a_{-,i}^{(l)})_1^{n-2} = \mathscr{S}_{n-2}^{\mathrm{I}} (\lambda_{-,k}^{(l)})_1^{n-2} \quad (l = 1, 2).$$

Then the Toeplitz-plus-Hankel matrix R_n admits a representation

$$R_n = \mathscr{C}_n^{\mathrm{I}}((\Lambda_+^{(1)} + \Sigma_n \Lambda_+^{(2)})\mathscr{C}_n^{\mathrm{I}} - (\Lambda_-^{(1)} - \Sigma_n \Lambda_-^{(2)} \overset{\circ}{\mathscr{S}}{}_n^{\mathrm{I}}) + \overset{\circ}{\mathscr{S}}{}_n^{\mathrm{I}}((\Lambda_+^{(1)} - \Sigma_n \Lambda_-^{(2)})\overset{\circ}{\mathscr{S}}{}_n^{\mathrm{I}} + (\Lambda_-^{(1)} + \Sigma_n \Lambda_-^{(2)})\mathscr{C}_n^{\mathrm{I}}),$$

where $\Lambda_\pm^{(l)} = \mathrm{diag}(\lambda_{\pm,i}^{(l)})_0^{n-1}, \lambda_{-,0}^{(l)} = \lambda_{-,n-1}^{(l)} = 0.$

2. *Suppose that* $a_{\pm,i}^{(l)} = \frac{1}{2}(a_i^{(l)} \pm a_{-i}^{(l)})$,

$$(a_{+,i}^{(l)})_0^{n-1} = \mathscr{C}_n^{\mathrm{II}}(\lambda_{-,k}^{(l)})_0^{n-1}, (a_{-,i}^{(1)})_1^{n} = \mathscr{S}_n^{\mathrm{II}}(\lambda_{-,k}^{(l)})_0^{n-1} \quad (l = 1, 2),$$

where $a_{-,n}^{(l)}$ *is arbitrary. Then the matrix* R_n *admits a representation*

$$R_n = \mathscr{C}_n^{\mathrm{II}}(\Lambda_-^{(1)} - \Lambda_-^{(2)} Z_n^{\mathrm{T}} \Sigma_n)\mathscr{C}_n^{\mathrm{III}} - (\Lambda_-^{(1)} Z_n^{\mathrm{T}} - \Lambda_+^{(2)} \Sigma_n)\mathscr{S}_n^{\mathrm{III}}) + Z_n \mathscr{S}_n^{\mathrm{II}}((\Lambda_+^{(1)} Z_n^{\mathrm{T}} + \Lambda_-^{(2)} \Sigma_n)\mathscr{S}_n^{\mathrm{III}} + (\Lambda_-^{(1)} + \Lambda_+^{(2)} Z_n^{\mathrm{T}} \Sigma_n)\mathscr{C}_n^{\mathrm{III}}),$$

where $\Lambda_\pm^{(l)} = \mathrm{diag}(\lambda_{\pm,i}^{(l)})_0^{n-1}$.

3. *Suppose that* $a_{\pm,i}^{(l)} = \frac{1}{2}(a_i^{(l)} \pm a_{-i-1}^{(l)})$,

$$(a_{+,i}^{(l)})_0^{n-1} = \mathscr{C}_n^{\mathrm{IV}}(\lambda_{+,k}^{(l)})_0^{n-1}, (a_{-,i}^{(l)})_0^{n-1} = \mathscr{S}_n^{\mathrm{IV}}(\lambda_{-,k}^{(l)})_0^{n-1} \quad (l = 1, 2),$$

where $a_{-,n}^{(l)}$ *is arbitrary. Then the matrix* R_n *admits a representation*

$$R_n = \mathscr{C}_n^{\mathrm{IV}}((\Lambda_+^{(1)} - \Sigma_n \Lambda_-^{(2)})\mathscr{C}_n^{\mathrm{III}} - (\Lambda_-^{(1)} - \Sigma_n \Lambda_+^{(2)})\mathscr{S}_n^{\mathrm{III}}) + \mathscr{S}_n^{\mathrm{IV}}((\Lambda_+^{(1)} + \Sigma_n \Lambda_-^{(2)})\mathscr{S}_n^{\mathrm{III}} + (\Lambda_-^{(1)} + \Sigma_n \Lambda_+^{(2)})\mathscr{C}_n^{\mathrm{III}}),$$

where $\Lambda_\pm^{(l)} = \mathrm{diag}(\lambda_{\pm,i}^{(l)})_0^{n-1}$.

These formulas simplify significantly if the matrix R_n is centrosymmetric, i.e. if both $T_n^{(1)}$ and $T_n^{(2)}$ are symmetric. Let us mention the corresponding formula for the type-I case.

Corollary 5.2. *Suppose that* R_n *is as above where* $T_n^{(l)} (l = 1, 2)$ *are symmetric and*

$$(a_i^{(l)})_0^{n-1} = \mathscr{C}_n^{\mathrm{I}}(\lambda_k^{(l)})_0^{n-1}, \quad (l = 1, 2).$$

Then R_n *admits a representation*

$$R_n = \mathscr{C}_n^{\mathrm{I}}((\Lambda^{(1)} + \Sigma_n \Lambda^{(2)})\mathscr{C}_n^{\mathrm{I}} + \overset{\circ}{\mathscr{S}}{}_n^{\mathrm{I}}((\Lambda^{(1)} - \Sigma_n \Lambda^{(2)})\overset{\circ}{\mathscr{S}}{}_n^{\mathrm{I}},$$

where $\Lambda^{(l)} = \mathrm{diag}(\lambda_i^{(l)})_0^{n-1}$.

The main conclusion of this paper is the following.

Corollary 5.3. *An $n \times n$ Toeplitz-plus-Hankel matrix R_n can be multiplied by a vector at the costs of* $4\tau(n) + 8n(\mathrm{M}) + 7n(\mathrm{A})$ *plus* $4\tau(n)$ *for preprocessing. If R_n is centrosymmetric then the costs for preprocessing reduces to* $2\tau(n)$.

We present now a representation of general Toeplitz-plus-Hankel matrices with the Hartley transformations. Suppose that $R_n = T_n^{(1)} + T_n^{(2)} J_n$ is as above. We extend the matrix R_n to an $(n+1) \times (n+1)$ matrix $\tilde{R}$ adding one column on the left and one row at the top via

$$\tilde{R} = T_{n+1}^{(1)} + T_{n+1}^{(2)} J'_{n+1},$$

where $T_{n+1}^{(l)} = [a_{i-j}^{(l)}]_0^n$ and $a_n^{(l)}$ is arbitrary $(l = 1,2)$. Then we have $R_n = P_{1,n+1} \tilde{R} P_{1,n+1}^{\mathrm{T}}$. We can use now the intertwining relations for the Hartley transforms in order to get a representation for $\tilde{R}$.

We define, like in Section 4,

$$\begin{aligned} a_{+\pm,i}^{(l)} &= \tfrac{1}{4}(a_i^{(l)} \pm a_{n+1-i}^{(l)} + a_{-i}^{(l)} \pm a_{i-n-1}^{(l)}), \\ a_{-\pm,i}^{(l)} &= \tfrac{1}{4}(a_i^{(l)} \pm a_{n+1-i}^{(l)} - (a_{-i}^{(l)} \pm a_{i-n-1}^{(l)})), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} (a_{++,i}^{(l)})_0^n &= \mathscr{H}_n^{+} \lambda_{++}^{(l)}, \quad (a_{+-,i}^{(l)})_0^n = \mathscr{H}_n^{-} \lambda_{+-}^{(l)}, \\ (a_{-+,i}^{(l)})_0^n &= \mathscr{H}_n^{-} \lambda_{-+}^{(l)}, \quad (a_{--,i}^{(l)})_0^n = \mathscr{H}_n^{+} \lambda_{--}^{(l)}, \end{aligned} \tag{5.3}$$

where $l = 1, 2$.

Theorem 5.4. *Let $\lambda_{\pm\pm}^{(l)}$ be given by Eqs. (5.2) and (5.3). Then the Toeplitz-plus-Hankel matrix $R_n = T_n^{(1)} + T_n^{(2)} J_n$ admits a representation*

$$\begin{aligned} R_n = P_{1,n+1}(&\mathscr{H}_{n+1}^{-}(\Lambda_{++}^{(1)} + \Lambda_{--}^{(2)} + (\Lambda_{--}^{(1)} + \Lambda_{++}^{(2)})J'_{n+1})\mathscr{H}_{n+1}^{+} \\ &+ \mathscr{H}_{n+1}^{-}(\Lambda_{+-}^{(1)} - \Lambda_{-+}^{(2)} + (\Lambda_{-+}^{(1)} - \Lambda_{+-}^{(2)})J_{n+1})(\mathscr{H}_{n+1}^{-})^{\mathrm{T}})P_{1,n+1}^{\mathrm{T}}, \end{aligned}$$

$\Lambda_{\pm\pm}^{(l)} = \operatorname{diag} \lambda_{\pm\pm}^{(l)} (l = 1, 2)$.

6. Extension approach

The idea of the extension approach is to identify given $n \times n$ Toeplitz, Hankel or Toeplitz-plus-Hankel matrices as submatrices of $N \times N$ matrices, $N > n$, of the form $R_N = \mathscr{T}_N^{(1)} \Lambda (\mathscr{T}_N^{(2)})^{\mathrm{T}}$, where $\mathscr{T}_N^{(l)} (l = 1,2)$ are real trigonometric transformations and Λ is a diagonal matrix. Matrices of this form are for many choices of the transformations special Toeplitz-plus-Hankel matrices, as we see at once.

6.1. Symmetric Toeplitz matrices

First we want to embed symmetric Toeplitz matrices $T_n = [a_{|i-j|}]_0^{n-1}$ into matrices of this form. It is reasonable to assume in this case $\mathcal{T}_1 = \mathcal{T}_2$.

Lemma 6.1. *Suppose that* $a = (a_i)_0^{N-1}, \Lambda = \operatorname{diag} \lambda$.

1. $$\mathscr{C}_N^{\mathrm{I}} \Lambda \mathscr{C}_N^{\mathrm{I}} = \tfrac{1}{2}[a_{|i-j|} + a_{i+j}]_0^{N-1},$$

where $a = \mathscr{C}_N^{\mathrm{I}}\lambda, a_{2N-2-i} = a_i$.

2. $$\mathring{\mathscr{S}}_N^{\mathrm{I}} \Lambda \mathring{\mathscr{S}}_N^{\mathrm{I}} = \tfrac{1}{2}[a_{|i-j|} - a_{i+j}]_0^{N-1},$$

where $a = \mathscr{C}_N^{\mathrm{I}}\lambda, a_{2N-2-i} = a_i$.

3. $$\mathscr{C}_N^{\mathrm{II}} \Lambda (\mathscr{C}_N^{\mathrm{II}})^{\mathrm{T}} = \tfrac{1}{2}[a_{|i-j|} + a_{i+j}]_0^{N-1},$$

where $a = \mathscr{C}_N^{\mathrm{II}}\lambda, a_{2N-i} = -a_i$.

4. $$\mathscr{S}_N^{\mathrm{II}} \Lambda (\mathscr{S}_N^{\mathrm{II}})^{\mathrm{T}} = \tfrac{1}{2}[a_{|i-j|} - a_{i+j+2}]_0^{N-1},$$

where $a = \mathscr{C}_N^{\mathrm{II}}\lambda, a_{2N-i} = -a_i$.

5. $$\mathscr{C}_N^{\mathrm{III}} \Lambda (\mathscr{C}_N^{\mathrm{III}})^{\mathrm{T}} = \tfrac{1}{2}[a_{|i-j|} + a_{i+j+1}]_0^{N-1},$$

where $\begin{bmatrix} a \\ a_N \end{bmatrix} = \mathscr{C}_{N+1}^{\mathrm{I}} \begin{bmatrix} \lambda \\ 0 \end{bmatrix}, a_{2N-i} = a_i$.

6. $$\mathscr{S}_N^{\mathrm{III}} \Lambda (\mathscr{S}_N^{\mathrm{III}})^{\mathrm{T}} = \tfrac{1}{2}[a_{|i-j|} - a_{i+j+1}]_0^{N-1},$$

where $\begin{bmatrix} a \\ a_N \end{bmatrix} = \mathscr{C}_{N+1}^{\mathrm{I}} \begin{bmatrix} 0 \\ \lambda \end{bmatrix}, a_{2N-i} = a_i$.

7. $$\mathscr{C}_N^{\mathrm{IV}} \Lambda \mathscr{C}_N^{\mathrm{IV}} = \tfrac{1}{2}[a_{|i-j|} + a_{i+j+1}]_0^{N-1},$$

where $a = \mathscr{C}_N^{\mathrm{II}}\lambda, a_{2N-i} = -a_i$.

8. $$\mathscr{S}_N^{\mathrm{IV}} \Lambda \mathscr{S}_N^{\mathrm{IV}} = \tfrac{1}{2}[a_{|i-j|} - a_{i+j+1}]_0^{N-1},$$

where $a = \mathscr{C}_N^{\mathrm{II}}\lambda, a_{2N-i} = -a_i$.

9. $$\mathscr{H}_N^{+} \Lambda \mathscr{H}_N^{+} = [c_{i-j} + s_{i+j}]_0^{N-1},$$

where $c = \mathscr{C}_{+}\lambda, s = \mathscr{S}_{+}\lambda, s_{i+N} = s_i$.

10. $$\mathscr{H}_N^{-} \Lambda (\mathscr{H}_N^{-})^{\mathrm{T}} = [c_{i-j} + s_{i+j}]_0^{N-1},$$

where $c = \mathscr{C}_{-}\lambda, s = \mathscr{S}_{-}\lambda, s_{i+N} = -s_i$.

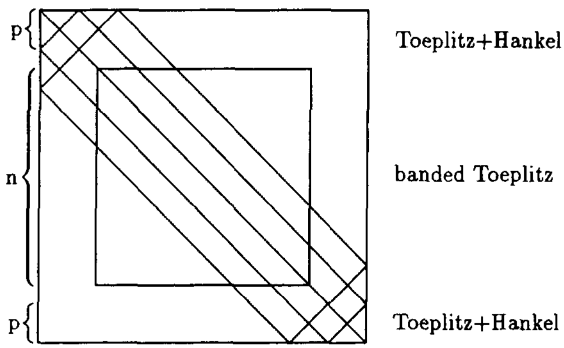
11. $$(\mathscr{H}_N^{-})^{\mathrm{T}} \Lambda \mathscr{H}_N^{-} = [c_{i-j} + s_{i+j}]_0^{N-1},$$

where $c = \mathscr{C}_{-}^{\mathrm{T}}\lambda, s = \mathscr{S}_{-}^{\mathrm{T}}\lambda, s_{i+N} = -s_i$.

In Lemma 6.1 (5) and (6) the number a_N has to be chosen in such a way that the first or the last component of $(\mathscr{C}^{\mathrm{I}}_{N+1})^{-1}[a\ a_N]^{\mathrm{T}}$ vanishes. But this can easily be achieved.

In the following theorem we are looking for the smallest possible N for which a symmetric banded Toeplitz matrix can be embedded into a matrix of the form $R_N = \mathscr{T}_N \Lambda \mathscr{T}_N^{\mathrm{T}}$. Of course a embedding into matrices of larger size is also possible. Note also that in the case of a dense matrix sometimes this number N can still be reduced by 1.

Let us illustrate the structure of the matrix R_N in the following figure.



Theorem 6.2. *Let $T_n = [a_{|i-j|}]_0^{n-1}$ be a symmetric Toeplitz matrix such that $a_i = 0$ for $i \geqslant s$ and let $(a_i)_0^{M-1} = \mathscr{U}_M \lambda$, where $\mathscr{U}_M$ is one of the trigonometric transformations. Then*

$$T_n = 2P_{p,p+n}\mathscr{T}_N \Lambda \mathscr{T}_N^{\mathrm{T}} P^{\mathrm{T}}_{p,p+n}, \tag{6.1}$$

where $N = n + 2p$, $\Lambda = \mathrm{diag}\lambda$ and one of the following is valid:

$$\mathscr{T}_N = \mathscr{C}^{\mathrm{I}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{I}}_M,\quad M = N,\quad p = [(s+1)/2],$$

$$\mathscr{T}_N = \mathring{\mathscr{S}}^{\mathrm{I}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{I}}_M,\quad M = N,\quad p = [(s+1)/2],$$

$$\mathscr{T}_N = \mathscr{C}^{\mathrm{II}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{II}}_M,\quad M = N,\quad p = [(s+1)/2],$$

$$\mathscr{T}_N = \mathscr{S}^{\mathrm{II}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{II}}_M,\quad M = N,\quad p = [(s-1)/2],$$

$$\mathscr{T}_N = \mathscr{C}^{\mathrm{III}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{I}}_M,\quad M = N+1,\quad p = [s/2],$$

$$\mathscr{T}_N = \mathscr{S}^{\mathrm{III}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{I}}_M,\quad M = N+1,\quad p = [s/2],$$

$$\mathscr{T}_N = \mathscr{C}^{\mathrm{IV}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{II}}_M,\quad M = N,\quad p = [s/2],$$

$$\mathscr{T}_N = \mathscr{S}^{\mathrm{IV}}_N,\quad \mathscr{U}_M = \mathscr{C}^{\mathrm{II}}_M,\quad M = N,\quad p = [s/2].$$

Here $[\cdot]$ denotes the integer part.

Corollary 6.3. *An $n \times n$ banded symmetric Toeplitz matrix with bandwidth $2s+1$ can be multiplied by a vector at the costs of $2\tau(n+s)+(n+s)(\mathrm{M})+\mathrm{o}(n)$ plus $\tau(n+s)$ for preprocessing.*

For the Hartley transformations the extension approach is slightly different. We do not extend the a_i by zeros but in such a way that the extended vector is even is odd. In this situation the matrix R_N is, according to Lemma 3.6, purely Toeplitz and an extension of the original matrix.

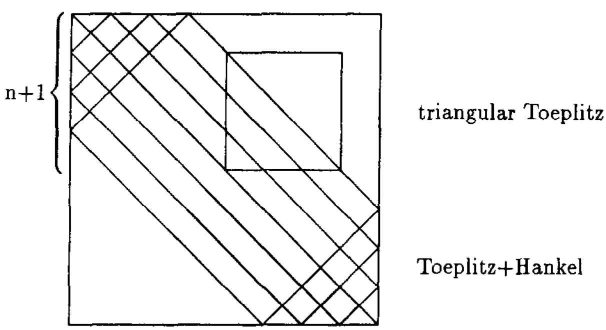
Theorem 6.4. *Let $T_n = [a_{|i-j|}]_0^{n-1}$ be a symmetric Toeplitz matrix such that $a_i = 0$ for $i \geqslant s$. Assume that $N = n+s-1$ and the a_i for $i \geqslant n$ are defined by $a_{N-i} = \pm a_i$. Then*

$$T_n = P_{0n}\mathscr{H}_N^{\pm}\Lambda_{\pm}(\mathscr{H}_N^{\pm})^{\mathrm{T}}P_{0n}^{\mathrm{T}},$$

where $(a_i)_0^{N-1} = \mathscr{H}_N^{\pm}\lambda_{\pm}$, $\Lambda_{\pm} = \mathrm{diag}\,\lambda_{\pm}$.

6.2. Triangular Toeplitz matrices

We discuss now the extension approach for triangular Toeplitz matrices. Suppose $T_n = [a_{i-j}]_0^{n-1}$ where $a_i = 0$ for $i < 0$. We set $N = 2n+1$ and $b = (b_i)_0^{N-1}$ with $b_i = a_{n-i-1}$. Let μ be defined by $b = \mathscr{C}_N^{\mathrm{T}}\mu$. The matrix T_n can now be found in the upper right corner of the matrix $R_N = \mathscr{C}_N^{\mathrm{I}}\,\mathrm{diag}\,\mu\mathscr{C}_N^{\mathrm{I}}$. Only the first row and the last column have to be cancelled.



More precisely, the following is true.

Theorem 6.5. *Let $M = \mathrm{diag}\,\mu$, where μ is as above. Then the lower triangular Toeplitz matrix T_n can be represented in the form*

$$T_n = 2P_{1,n+1}\mathscr{C}_N^{\mathrm{I}}M\mathscr{C}_N^{\mathrm{I}}P_{n,2n-1}^{\mathrm{T}}.$$

Remark 6.6. It is also possible to choose $N = 2n - 1$ and b and μ as above. Then T_n is the upper right corner of R_N except for two 2 entries.

Analogous representations can be deduced for the other trigonometric transformations, except for the Hartley transformations. We refrain from presenting them all because for triangular Toeplitz matrices no gain is achieved with the extension approach in comparison with decomposition approach.

6.3. General Toeplitz and Toeplitz-plus-Hankel matrices

In order to get representations for general Toeplitz matrices we need expressions generating skew-symmetric Toeplitz matrices in an analogous manner like symmetric Toeplitz matrices were generated in Lemma 6.1. This can be done combining sine and cosine transforms and is outlined in the next lemma.

Lemma 6.7.

1. $$\mathring{\mathscr{S}}^{\mathrm{I}}_N \Lambda \mathscr{C}^{\mathrm{I}}_N = \frac{1}{2}[a_{i-j} + a_{i+j}]_0^{N-1},$$

where $(a_i)_1^{N-2} = \mathscr{S}^{\mathrm{I}}_{N-2}(\lambda_k)_1^{N-2}, \lambda_0 = \lambda_{N-1} = 0, a_{2N-2-i} = a_{-i} = -a_i.$

2. $$Z_n \mathscr{S}^{\mathrm{II}}_N \Lambda \mathscr{C}^{\mathrm{III}}_N = \frac{1}{2}[a_{i-j} + a_{i+j+1}]_0^{N-1},$$

where $(a_i)_1^N = \mathscr{S}^{\mathrm{II}}_N(\lambda_k)_0^{N-1}, a_{2N-i} = -a_{-i} = a_i.$

3. $$\mathscr{S}^{\mathrm{III}}_N Z_N^{\mathrm{T}} \Lambda \mathscr{C}^{\mathrm{II}}_N = \frac{1}{2}[a_{i-j} + a_{i+j+1}]_0^{N-1},$$

where $(a_i)_1^{N-1} = \mathscr{S}^{\mathrm{I}}_{N-1}(\lambda_k)_1^{N-1}, \lambda_0 = 0, a_{2N-i} = a_{-i} = -a_i.$

4. $$\mathscr{S}^{\mathrm{II}}_N \Lambda \mathscr{C}^{\mathrm{III}}_N Z_n = \frac{1}{2}[a_{i-j} + a_{i+j+2}]_0^{N-1},$$

where $(a_i)_1^N = \mathscr{S}^{\mathrm{II}}_N(\lambda_k)_0^{N-1}, a_{2N-i} = -a_{-i} = a_i.$

5. $$\mathscr{S}^{\mathrm{IV}}_N \Lambda \mathscr{C}^{\mathrm{IV}}_N = \frac{1}{2}[a_{i-j} + a_{i+j+1}]_0^{N-1},$$

where $(a_i)_1^N = \mathscr{S}^{\mathrm{II}}_N(\lambda_k)_0^{N-1}, a_{2N-i} = -a_{-i} = a_i.$

6. $$\mathscr{S}^{\mathrm{IV}}_N \Lambda \mathscr{C}^{\mathrm{III}}_N = \frac{1}{2}[a_{i-j} + a_{i+j}]_0^{N-1},$$

where $(a_i)_0^{N-1} = \mathscr{S}^{\mathrm{IV}}_N(\lambda_k)_0^{N-1}, a_{2N-i} = a_{i-1} = -a_{-i}.$

In the next theorem we formulate only a few of all possible representations of banded Toeplitz matrices with trigonometric transformations.

Theorem 6.8. *Let $T_n = [a_{i-j}]_0^{n-1}$ be a Toeplitz matrix such that $a_i = 0$ for $|i| \geqslant s$ and $a_i^{\pm} = (1/2)(a_i + a_{-i})$. Then*:

1. *If*

$$(a_i^+)_0^{N-1} = \mathscr{C}_N^{\mathrm{I}} \lambda_+, \qquad (a_i^-)_1^{N-2} = \mathscr{S}_{N-2}^{\mathrm{I}} (\lambda_k^-)_1^{N-2}, \quad \lambda_0^- = \lambda_{N-1}^- = 0$$

then

$$T_n = P_{p,p+n}(\mathscr{C}_N^{\mathrm{I}} \Lambda_+ + \mathring{\mathscr{S}}_N^{\mathrm{I}} \Lambda_-)\mathscr{C}_N^{\mathrm{I}} P_{p,p+n}^{\mathrm{T}}.$$

2. *If*

$$(a_i^+)_0^{N-1} = \mathscr{C}_N^{\mathrm{II}} \lambda_+, \quad (a_i^-)_1^{N} = \mathscr{S}_N^{\mathrm{II}} (\lambda_k^-)_1^{N},$$

then

$$T_n = P_{p,p+n}(\mathscr{C}_N^{\mathrm{II}} \Lambda_+ + Z_N \mathscr{S}_N^{\mathrm{II}} \Lambda_-)\mathscr{C}_N^{\mathrm{IV}} P_{p,p+n}^{\mathrm{T}}.$$

3. *If*

$$(a_i^+)_0^{N-1} = \mathscr{C}_N^{\mathrm{II}} \lambda_+, \qquad (a_i^-)_1^{N} = \mathscr{S}_N^{\mathrm{II}} (\lambda_k^-)_1^{N},$$

then

$$T_n = P_{p,p+n}(\mathscr{C}_N^{\mathrm{IV}} \Lambda_+ + \mathscr{S}_N^{\mathrm{IV}} \Lambda_-)\mathscr{C}_N^{\mathrm{IV}} P_{p,p+n}^{\mathrm{T}}.$$

In the first case $p = [(s+1)/2]$, in the other cases $p = [s/2]$; in all cases $\Lambda_{\pm} = \operatorname{diag} \lambda_{\pm}$.

Corollary 6.9. *An $n \times n$ banded Toeplitz matrix with bandwidth $2s+1$ can be multiplied by a vector at the costs of* $3\tau(n+s) + 2(n+s)(\mathrm{M}) + (n+s)(\mathrm{A}) + \mathrm{o}(n)$ *plus* $2\tau(n+s)$ *for preprocessing if sine and cosine transforms are used.*

Thus, the extension approach is advantageous compared with the decomposition approach if $s < n/3$.

If the Hartley transform are used one gets always a gain if the matrix is banded. The approach to derive the formulas is quite different. Note that, differently to the sine/cosine transformations the combination of different Hartley transformations leads again to symmetric and not to skew-symmetric matrices. The skew-symmetry is achieved with the help of the middle factor J_N' (see Lemma 4.6). Applying Lemmas 3.6 and 4.6 we obtain the following.

Theorem 6.10. *Let $T_n = [a_{i-j}]_0^{n-1}$ be a Toeplitz matrix such that $a_i = 0$ for $|i| \geqslant s$. Assume that $N = n + s - 1$ and $a_i^{\pm} = (1/2)(a_i \pm a_{-i})$.*

1. *Let $a_i^{\pm}$ for $i = n, \ldots, N-1$ be defined by $a_{N-i}^{\pm} = \pm a_i^{\pm}$, $(a_i^{\pm})_0^{N-1} = \mathscr{H}_N^+ \lambda_{\pm}$, $\Lambda_{\pm} =$* $\operatorname{diag} \lambda_{\pm}$. *Then*

$$T_n = P_{0n} \mathscr{H}_N^+ (\Lambda_+ + \Lambda_- J_N')(\mathscr{H}_N^+)^{\mathrm{T}} P_{0n}^{T}.$$

2. *Let* $a_i^{\pm}$ *for* $i \geqslant n$ *be defined by* $a_{N-i}^{\pm} = \mp a_i^{\pm}, (a_i^{\pm})_0^{N-1} = \mathscr{H}_N^{+}\lambda_{\pm}, \Lambda_{\pm} = \operatorname{diag} \lambda_{\pm}$. *Then*

$$T_n = P_{0n}\mathscr{H}_N^{-}(\Lambda_+ + \Lambda_- J_N')(\mathscr{H}_N^{-})^{\mathrm{T}} P_{0n}^{\mathrm{T}}.$$

Corollary 6.11. *An* $n \times n$ *banded Toeplitz matrix with bandwidth* $2s+1$ *can be multiplied by a vector at the costs of* $2\phi(n+s) + 2(n+s)(\mathrm{M}) + (n+s)(\mathrm{A}) +$ $\mathrm{o}(n)$ *plus* $2\phi(n+s)$ *for preprocessing if Hartley transformations are used.*

All formula presented in this section can now be generalized to Toeplitz-plus-Hankel matrices using the intertwining relations listed in Section 2.

7. Hybrid formulas

It is also possible to combine the decomposition and extension approaches to get some hybrid formulas. First we present such formulas for Hankel matrices $H_n = [h_{i+j}]_0^{n-1}$. We set $N = 2n - 1$ and $h = (h_i)_0^{N-1}$. Suppose that

$$h = \mathscr{C}_N^{\mathrm{X}}\lambda^{\mathrm{X}}, \qquad h = \mathscr{S}_N^{\mathrm{X}}\mu^{\mathrm{X}}$$

and

$$\Lambda^{\mathrm{X}} = \operatorname{diag} \lambda^{\mathrm{X}}, \qquad M^{\mathrm{X}} = \operatorname{diag} \mu^{\mathrm{X}},$$

where X = I, II, III, IV. Now h_{i+j} can be represented as a linear combination of cosines or sines in which the sum of arguments appear. For example, in the cosine-I case we have

$$h_{i+j} = \sum_{k=0}^{N-1} \lambda_k^{\mathrm{I}} \cos\frac{(i+j)k\pi}{N-1} = \sum_{k=0}^{N-1} \lambda_k^{\mathrm{I}} \cos\frac{ik\pi}{N-1}\cos\frac{jk\pi}{N-1}$$
$$- \sum_{k=0}^{N-1} \lambda_k^{\mathrm{I}} \sin\frac{ik\pi}{N-1}\sin\frac{jk\pi}{N-1}.$$

Written in matrix form this leads to the following.

Theorem 7.1. *The Hankel matrix* H_n *admits the following representations:*

1. *DCT-I*

$$H_n = P_{0,n}\left[\mathscr{C}_N^{\mathrm{I}}\Lambda^{\mathrm{I}}\mathscr{C}_N^{\mathrm{I}} - \mathring{\mathscr{S}}_N^{\mathrm{I}}\Lambda^{\mathrm{I}}\mathring{\mathscr{S}}_N^{\mathrm{I}}\right]P_{0,n}^{\mathrm{T}},$$

2. *DST-I*

$$H_n = P_{0,n}\left[\mathscr{S}_N^{\mathrm{I}}M^{\mathrm{I}}(\hat{\mathscr{C}}_N^{\mathrm{I}})^{\mathrm{T}} + \check{\mathscr{C}}_N^{\mathrm{I}}M^{\mathrm{I}}\mathscr{S}_N^{\mathrm{I}}Z_N^{\mathrm{T}}\right]P_{1,n+1}^{\mathrm{T}},$$

where

$$\hat{\mathscr{C}}_N^{\mathrm{I}} = P_{0,N}\mathscr{C}_{N+2}^{\mathrm{I}}P_{1,N-1}^{\mathrm{T}}, \quad \check{\mathscr{C}}_N^{\mathrm{I}} = P_{1,N+1}\mathscr{C}_{N+2}^{\mathrm{I}}P_{1,N+1}^{\mathrm{T}}.$$

3. *DCT-II*

$$\begin{aligned} H_n = P_{0,n}[&\mathscr{C}_N^{\mathrm{II}}\Lambda^{\mathrm{II}}(\mathscr{C}_N^{\mathrm{II}})^{\mathrm{T}} \\ &- Z_N\mathscr{S}_N^{\mathrm{II}}\Lambda^{\mathrm{II}}(\mathscr{S}_N^{\mathrm{II}})^{\mathrm{T}}Z_N^{\mathrm{T}}]P_{0,n}^{\mathrm{T}}; \end{aligned}$$

4. *DST-II*

$$\begin{aligned} H_n = P_{0,n}[&Z_N\mathscr{S}_N^{\mathrm{II}}M^{\mathrm{II}}(\mathscr{C}_N^{\mathrm{II}})^{\mathrm{T}}Z_N \\ &+ \mathscr{C}_N^{\mathrm{II}}M^{\mathrm{II}}(\mathscr{S}_N^{\mathrm{II}})^{\mathrm{T}}]P_{0,n}^{\mathrm{T}}. \end{aligned}$$

5. *DCT-III*

$$\begin{aligned} H_n = P_{0,n}[&\mathscr{C}_N^{\mathrm{III}}\Lambda^{\mathrm{III}}(\overline{\mathscr{C}}_N^{\mathrm{I}})^{\mathrm{T}} \\ &- \mathscr{S}_N^{\mathrm{III}}Z_N^{\mathrm{T}}\Lambda^{\mathrm{III}}(\overline{\mathscr{S}}_N^{\mathrm{I}})^{\mathrm{T}}]P_{0,n}^{\mathrm{T}}, \end{aligned}$$

where here

$$\overline{\mathscr{C}}_N^{\mathrm{I}} = P_{0N}\mathscr{C}_{N+1}^{\mathrm{I}}P_{0N}^{\mathrm{T}}, \quad \overline{\mathscr{S}}_N^{\mathrm{I}} = P_{0N}\mathring{\mathscr{S}}_{N+1}^{\mathrm{I}}P_{0N}^{\mathrm{T}}.$$

6. *DST-III*

$$\begin{aligned} H_n = P_{0,n}[&\mathscr{S}_N^{\mathrm{III}}M^{\mathrm{III}}(\overline{\mathscr{C}}_N^{\mathrm{II}})^{\mathrm{T}}Z_N \\ &+ Z_N^{\mathrm{T}}\mathscr{C}_N^{\mathrm{III}}M^{\mathrm{II}}(\overline{\mathscr{S}}_N^{\mathrm{II}})^{\mathrm{T}}]P_{0,n}^{\mathrm{T}}, \end{aligned}$$

where here

$$\bar{\mathscr{C}}_N^{\mathrm{I}} = P_{1,N+1}\mathscr{C}_{N+1}^{\mathrm{I}}P_{0,N}^{\mathrm{T}}, \quad \bar{\mathscr{S}}_N^{\mathrm{I}} = P_{1,N+1}\mathring{\mathscr{S}}_{N+1}^{\mathrm{I}}P_{0N}^{\mathrm{T}},$$

7. *DCT-IV*

$$H_n = P_{0,n}\left[\mathscr{C}_N^{\mathrm{II}}\Lambda^{\mathrm{IV}}\mathscr{C}_N^{\mathrm{IV}} - Z_N\mathscr{S}_N^{\mathrm{II}}\Lambda^{\mathrm{IV}}\mathscr{S}_N^{\mathrm{IV}}\right]P_{0,n}^{\mathrm{T}},$$

8. *DST-IV*

$$H_n = P_{0,n}\left[Z_N\mathscr{S}_N^{\mathrm{II}}M^{IV}\left(\mathscr{C}_N^{\mathrm{IV}}\right)^{\mathrm{T}} + \mathscr{C}_N^{\mathrm{II}}M^{\mathrm{IV}}\left(\mathscr{S}_N^{\mathrm{IV}}\right)^{\mathrm{T}}\right]P_{0,n}^{\mathrm{T}}.$$

We consider now general Toeplitz-plus-Hankel matrices $R_n = H_n + G_nJ_n$, where $H_n = [h_{i+j}]_0^{n-1}$ and $G_n = [g_{i-j}]_0^{n-1}$.

We intend to combine Theorem 7.1 with the intertwining relations listed in Section 2. Below we mention two cases where such a combination leads to simple formulas.

Theorem 7.2. *1. Suppose that*

$$h = \mathscr{C}_N^{\mathrm{I}} \lambda^{\mathrm{I}}, \quad g = \mathscr{C}_N^{\mathrm{I}} \gamma.$$

Then

$$R_n = P_{0,n} \left[\mathscr{C}_N^{\mathrm{I}} (\Lambda^{\mathrm{I}} + \Gamma) \mathscr{C}_N^{\mathrm{I}} - \mathring{\mathscr{S}}_N^{\mathrm{I}} (\Lambda^{\mathrm{I}} - \Gamma) \mathring{\mathscr{S}}_N^{\mathrm{I}} \right] P_{0,n}^{\mathrm{T}},$$

where $\Lambda^{\mathrm{I}} = \mathrm{diag}\, \lambda^{\mathrm{I}}$, $\Gamma = \Sigma_N \,\mathrm{diag}\, \gamma$.

2. Suppose that

$$h = \mathscr{C}_N^{\mathrm{IV}} \lambda^{\mathrm{IV}}, \quad g = \mathscr{S}_N^{\mathrm{IV}} \mu.$$

Then

$$R_n = P_{0,n} \left[\mathscr{C}_N^{\mathrm{II}} (\Lambda^{\mathrm{IV}} + M) \mathscr{C}_N^{\mathrm{IV}} - Z_N \mathscr{S}_N^{\mathrm{II}} (\Lambda^{\mathrm{IV}} - M) \mathscr{S}_N^{\mathrm{IV}} \right] P_{0,n}^{\mathrm{T}},$$

where $\Lambda^{\mathrm{IV}} = \mathrm{diag} \lambda^{\mathrm{IV}}$ *and* $M = \Sigma_N \,\mathrm{diag}\, \mu$.

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