A functional modulus of continuity for Brownian motion

Jicheng Liu a,b,c,*, Jiagang Ren a,c

a Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China
b GFMUL, Av. Prof. Gama Pinto, 2, 1649-003 Lisbon, Portugal
c School of Mathematics and Computational Science, Zhongshan University, Guangzhou, Guangdong 510275, PR China

Received 15 February 2006
Available online 18 April 2006

Abstract

In this paper, we prove a sharpening of large deviation for increments of Brownian motion in \((p, r)\)-capacity and Hölder norm case. As an application, we obtain a functional modulus of continuity for \((p, r)\)-capacity in the stronger topology.

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MSC: 60F10; 60F15

Keywords: Large deviation; Schilder’s theorem; Modulus of continuity; \((p, r)\)-capacity

1. Introduction

Schilder’s theorem can become a stronger statement if one uses \((p, r)\)-capacity and Hölder norm (with exponents \(p \geq 1, r \geq 0\) and \(\alpha < 1/2\)) instead of Wiener measure and sup-norm topology respectively. According to Baldi, Ben Arous and Kerkyacharian [1], we can choose appropriate spaces such that they form an abstract Wiener space, and thus it would be a consequence of Yoshida’s result in [8]. On the other hand, we will refine a large deviation principle for increments of Brownian motion to \((p, r)\)-capacity and Hölder norm case (cf. [2,7]).

* Corresponding author.
E-mail address: jcliu_hust@hotmail.com (J. Liu).

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doi:10.1016/j.bulsci.2006.03.009
The main points of this paper are the use of the capacity and Hölder large deviation estimates to improve the functional modulus of continuity, which can be considered as the stronger statement of [2, Theorem 3.1] and [7, Theorem 1.1].

2. Large deviations

Let \((B, H, \mu)\) be an abstract Wiener space. Denote by \(D^p_r\) the \((p, r)\)-Sobolev space over \(B\) endowed with the norm
\[
\|F\|_{p,2r} := \| (I - L)^r F \|_p,
\]
where \(L\) is the Ornstein–Uhlenbeck operator. When \(r\) is a natural number, this norm is, by Meyer’s inequality, equivalent to
\[
\|F\|_{p,2r} := \| D^{2r} F \|_p + \| F \|_p,
\]
where \(D\) is the gradient operator. Given an open set \(O\) of \(X\), its \((p, r)\)-capacity is defined by [4,6]
\[
C_{p,r}(O) = \inf \{ \|u\|_{p,r}; u \geq 0, u \geq 1 \mu\text{-a.e. on } O \},
\]
and for any subset \(A \subset X\),
\[
C_{p,r}(A) = \inf \{ C_{p,r}(O); O \supseteq A \}.
\]
Yoshida in [8] proved the following result.

**Theorem 2.1.** [8, Theorem 1.1] Let \(\{S_\varepsilon\}_{\varepsilon > 0}\) be a family of bijective, continuous linear operators on \(B\) such that
\[
\mu(S^{-1}_\varepsilon A) = \mu(\varepsilon^{-1/2} A)
\]
for all Borel subset \(A \subset B\) and \(\varepsilon > 0\). Then, for any \(A \subset B\) and \((p, r) \in (0, \infty) \times [0, \infty)\), it holds that
\[
- \inf_{f \in A_0} I(f) \leq \liminf_{\varepsilon \to 0} \varepsilon \log C_{p,r}(S^{-1}_\varepsilon A) \leq \limsup_{\varepsilon \to 0} \varepsilon \log C_{p,r}(S^{-1}_\varepsilon A) \leq - \inf_{A} I(f)
\]
where the function \(I : B \to [0, \infty]\) is given by
\[
I(z) = \begin{cases} 
\|z\|_H^2/2, & \text{if } z \in H; \\
\infty, & \text{if } z \in B \setminus H.
\end{cases}
\]

Let us denote by \(C = C([0, 1], R^d)\) the set of all continuous path \(\gamma : [0, 1] \to R^d\) which are continuous and such that \(\gamma(0) = 0\) and by \(H\) the subspace of \(C\) of all paths \(\gamma\) which are absolutely continuous and whose derivative is square integrable. \(C\) is a separable Banach space with respect to uniform norm and \(H\) is a Hilbert space with respect to the scalar product
\[
\langle \gamma_1, \gamma_2 \rangle_H = \int_0^1 \left( \dot{\gamma}_1(s), \dot{\gamma}_2(s) \right) \, ds.
\]
If \(\mu\) is Wiener measure, it is well known that \((C, H, \mu)\) forms an abstract Wiener space.

Let us denote by \(C^\alpha\) the Banach space of all \(\alpha\)-Hölder path \(\gamma : [0, 1] \to R^d\), such that \(\gamma(0) = 0\), endowed with the norm
\[
\|\gamma\|_\alpha = \sup_{s, t \in [0, 1]} \frac{|\gamma(t) - \gamma(s)|}{|t - s|^{\alpha}}.
\]
For every $\sigma > 0$, let us set
\[
\omega_\gamma (\sigma) = \sup_{s,t \in [0,1], |t-s| \leq \sigma} \frac{|\gamma(t) - \gamma(s)|}{|t-s|^\alpha}
\]
so that the modulus of continuity of $\gamma$ is $\sigma^\alpha \omega_\gamma (\sigma)$. We shall denote by $C^{\alpha,0}$ the subspace of $C^\alpha$ of all path such that $\lim_{\sigma \to 0} \omega_\gamma (\sigma) = 0$. It is well known that $C^{\alpha,0}$ is a closed convex subspace of $C^\alpha$, so that it is a Banach space with norm $\| \cdot \|_{[\alpha]}$, and that it is separable. By [1, Theorem 2.4], it is easy to check that $(C^{\alpha,0}, H, \mu)$ is also an abstract Wiener space. By Theorem 2.1, we have the following results.

**Theorem 2.2** (Schilder’s theorem in $(p,r)$-capacity and Hölder case). For any subset $F \subset C^{\alpha,0}$, we have
\[
- \inf_{f \in F^0} I(f) \leq \liminf_{\epsilon \to 0} \epsilon \log C_{p,r} (\epsilon^{-1/2} F) \leq \limsup_{\epsilon \to 0} \epsilon \log C_{p,r} (\epsilon^{-1/2} F) \leq - \inf_{F} I(f).
\]

Let $\{w(t), t \in [0,1]\}$ be a standard Brownian motion. To establishing modulus of continuity for Brownian motion, we need the following result of large deviation.

**Theorem 2.3.** For any closed set $F \subset C^{\alpha,0}$, the inequality
\[
\limsup_{\epsilon \to \infty} \epsilon \left( \log C_{p,r} \left( \bigcup_{0 \leq t \leq 1-h} \left\{ \frac{\epsilon}{h} \left( \omega(t+h) - \omega(t) \right) \in F \right\} \right) + \log h \right) \leq - \inf_{F} I(f)
\]
holds for every positive $h < 1$.

To prove Theorem 2.3, the following lemmas are used.

**Lemma 2.4.** For $\alpha < 1/2$ and $u > 0$, there exist constants $C_4$ and $C_5$ so that
\[
\mu \left( \sup_{0 \leq \tau \leq h} \sup_{0 \leq s,t \leq 1-h} \frac{|\omega(\tau + t) - \omega(t) - [\omega(\tau + s) - \omega(s)]|}{|t-s|^\alpha} \geq u \right) \leq \frac{1}{h} C_4 \exp \{-C_5 u^2 h^{2\alpha-1}\}.
\]

**Lemma 2.5.** For $\alpha < 1/2$ and $u > 0$, there exist constants $C_4$ and $C_6$ so that
\[
\mu \left( \sup_{0 \leq s \leq 1-h} \sup_{0 \leq \tau \leq h} \frac{|\omega(s + \tau) - \omega(s)|}{\tau^\alpha} \geq u \right) \leq \frac{1}{h} C_4 \exp \{-C_6 u^2 h^{2\alpha-1}\}
\]
with $C_6 = 4C_5$.

**Proof.** By Fernique’s theorem (cf. [3, Theorem 1.3.24] or [5, p. 402]), there exists a positive constant $C_6 = C_6(\alpha)$ such that
\[
C_4 := E(\exp \{ C_6 \| \omega \|^2 \}) < \infty,
\]
where $\| \omega \| := \sup_{0 \leq \tau \leq 1} \sup_{0 \leq t \leq 1} \frac{|\omega(\tau + t) - \omega(t)|}{\tau^\alpha}$. 

Proof of Lemma 2.4.

where the last inequality is due to the Chebyshev’s inequality and we complete the proof. □

Proof of Lemma 2.4.

\[
\mu\left(\sup_{0 \leq s \leq 1-h} \sup_{0 \leq \tau \leq h} \frac{|\omega(s + \tau) - \omega(s)|}{\tau^\alpha} \geq u\right)
\]

\[
\leq \mu\left(\sup_{0 \leq t/h \leq 1/h-1} \sup_{0 \leq \tau/h \leq 1} \sqrt{h} \frac{|\omega(\tau/h + t/h) - \omega(t/h)|}{h^\alpha(\tau/h)^\alpha} \geq u\right)
\]

\[
= \mu\left(\sup_{0 \leq s \leq 1/h-1} \sup_{0 \leq x \leq 1} \frac{|\omega(s + x) - \omega(s)|}{x^\alpha} h^{1/2-\alpha} \geq u\right)
\]

(-setting \(s := t/h\) and \(x := \tau/h\))

\[
\leq \mu\left(\bigcup_{k=1}^{[1/h]} \sup_{k-1/s \leq s \leq k} \sup_{0 \leq x \leq 1} \frac{|\omega(s + x) - \omega(s)|}{x^\alpha} \geq uh^{\alpha-1/2}\right)
\]

\[
\leq \sum_{k=1}^{[1/h]} \mu\left(\sup_{k-1/s \leq s \leq k} \sup_{0 \leq x \leq 1} \frac{|\omega(s + x) - \omega(s)|}{x^\alpha} \geq uh^{\alpha-1/2}\right)
\]

\[
\leq \frac{1}{h} \mu\left(\sup_{0 \leq s \leq 1} \sup_{0 \leq x \leq 1} \frac{|\omega(s + x) - \omega(s)|}{x^\alpha} \geq uh^{\alpha-1/2}\right)
\]

\[
\leq \frac{1}{h} C_4 \exp\{-C_6 u^2 h^{2\alpha-1}\},
\]

where the last inequality is due to the Chebyshev’s inequality and we complete the proof. □

Proof of Lemma 2.4.

\[
\sup_{0 \leq \tau \leq h} \sup_{0 \leq s, t \leq 1-h} \frac{|\omega(\tau + t) - \omega(t) - [\omega(\tau + s) - \omega(s)]|}{|t - s|^\alpha}
\]

\[
\leq \sup_{0 \leq \tau \leq h} \sup_{0 \leq s, t \leq 1-h} \left[\frac{|\omega(\tau + t) - \omega(t \lor (\tau + s))|}{|t - s|^\alpha} + \frac{|\omega(t \land (\tau + s)) - \omega(s)|}{|t - s|^\alpha}\right]
\]

\[
=: I_1 \lor I_2,
\]

where

\[
I_1 = \sup_{0 \leq \tau \leq h} \sup_{0 \leq s, t \leq 1-h} \left[\frac{|\omega(\tau + t) - \omega(t)|}{|t - s|^\alpha} + \frac{|\omega(\tau + s) - \omega(s)|}{|t - s|^\alpha}\right]
\]

\[
\leq \sup_{0 \leq \tau \leq h} \sup_{0 \leq s, t \leq 1-h} \left[\frac{|\omega(\tau + t) - \omega(t)|}{\tau^\alpha} + \frac{|\omega(\tau + s) - \omega(s)|}{\tau^\alpha}\right]
\]

\[
\leq \sup_{0 \leq \tau \leq h} \sup_{0 \leq t \leq 1-h} 2 \frac{|\omega(\tau + t) - \omega(t)|}{\tau^\alpha},
\]

i.e., when \(t \geq \tau + s\).

\[
I_2 = \sup_{0 \leq \tau \leq h} \sup_{0 \leq s, t \leq 1-h} \left[\frac{|\omega(\tau + t) - \omega(\tau + s)|}{|t - s|^\alpha} + \frac{|\omega(t) - \omega(s)|}{|t - s|^\alpha}\right]
\]

\[
\leq \sup_{0 \leq \tau \leq h} \sup_{0 \leq s, t \leq 1-h} 2 \frac{|\omega(\tau + t) - \omega(\tau + s)|}{|t - s|^\alpha}
\]

(setting \(x := t - s\))
\[
= \sup_{0 \leq x \leq h} \sup_{0 \leq \tau + s \leq 1 - h} \frac{2|\omega(\tau + s + x) - \omega(\tau + s)|}{x^\alpha},
\]
(setting \(x := \tau\) and \(t := \tau + s\))
\[
= \sup_{0 \leq \tau \leq h} \sup_{0 \leq \tau \leq 1 - h} \frac{2|\omega(\tau + t) - \omega(t)|}{\tau^\alpha},
\]
i.e., when \(t \leq \tau + s\). So
\[
\sup_{0 \leq \tau \leq h} \sup_{0 \leq s, t \leq 1 - h} \frac{|\omega(\tau + t) - \omega(t) - [\omega(\tau + s) - \omega(s)]|}{|t - s|^\alpha}
\leq 2 \sup_{0 \leq s \leq 1 - h} \sup_{0 \leq \tau \leq h} \frac{|\omega(s + \tau) - \omega(\tau)|}{\tau^\alpha}
\]
which show that Lemma 2.5 implies Lemma 2.4. \(\square\)

**Lemma 2.6.** (Cf. [8, Lemma 3.3]) If \(B\) is separable real Branch space with norm \(\rho := \| \cdot \|\) and \(\mu\) is a mean zero Gaussian measure. Then, for \(k \geq 0\), \(p \geq 1\) and \(\delta \in (0, 1)\), there exists a constant \(C(k, p)\) such that
\[
C_{p,k} \left( \bigcap_{i=1}^{N} (a_i < \rho(z) < b_i) \right) \leq C(p, k) \left( N \delta \right)^{kp} \left( \| \rho \|_{2kp} + 1 \right)^{pk}
\times \mu \left( \bigcap_{i=1}^{N} (a_i - \delta < \rho(z) < b_i + \delta) \right)^{\frac{1}{p}}.
\]
In particular, for \((p, r) \in [1, \infty) \times [0, \infty)\),
\[
C_{p,r} \left( \bigcap_{i=1}^{N} (a_i < \rho(z) < b_i) \right) \leq C(p, r) \left( N \delta \right)^{([r]+1)p} \left( \| \rho \|_{2p([r]+1)} + 1 \right)^{p([r]+1)}
\times \mu \left( \bigcap_{i=1}^{N} (a_i - \delta < \rho(z) < b_i + \delta) \right)^{\frac{1}{2[p(r)+1]}}.
\]

**Proof.** By Fernique’s theorem, \(\rho \in L^q\) for any \(q \in [1, \infty)\). Since \(\rho\) is a norm, for all \(z, y \in B\) and \(t > 0\), we have
\[
\rho(e^{-t}z + \sqrt{1 - e^{-2t}}y) \leq e^{-t} \rho(z) + \sqrt{1 - e^{-2t}} \rho(y)
\]
and
\[
\rho(z) \leq e^{-t} \rho(e^{-t}z + \sqrt{1 - e^{-2t}}y) + \sqrt{e^{2t} - 1} \rho(y).
\]
Notice that, for all \(F \in L^p\),
\[
(T_t F)(z) = \int_{B} F(e^{-t}z + \sqrt{1 - e^{-2t}}y) \mu(dy).
\]
Thus integration with respect to \(y\) yields
\[ T_t \rho(z) \leq e^{-t} \rho(z) + \sqrt{1 - e^{-2t}} \| \rho \|_1, \]
\[ \rho(z) \leq e^{-t} T_t \rho(z) + \sqrt{e^{2t} - 1} \| \rho \|_1. \]

Hence \( \rho \) satisfies [8, Corollary 3.2] with \( A = e^{-t} \), \( B = e^t \) and \( C_t = \sqrt{e^{2t} - 1} \). So applying [8, Corollary 3.2], \( q_1 = q_2 = 2p_k \), we obtain

\[ C_{p,k} \left( \bigcap_{i=1}^{N} (a_i < \rho(z) < b_i) \right) \leq C(p,k) \left( \frac{N}{\delta} \right)^{kp} (\| \rho \|_{2kp} + 1)^{pk} \]
\[ \times \mu \left( \bigcap_{i=1}^{N} (a_i - \delta < \rho(z) < b_i + \delta) \right). \]

If taking \( k = [r] + 1 \) in the right of the above inequality, we have

\[ C_{p,r} \left( \bigcap_{i=1}^{N} (a_i < \rho(z) < b_i) \right) \leq C(p,r) \left( \frac{N}{\delta} \right)^{p([r]+1)} (\| \rho \|_{2p([r]+1)} + 1)^{p([r]+1)} \]
\[ \times \mu \left( \bigcap_{i=1}^{N} (a_i - \delta < \rho(z) < b_i + \delta) \right)^{\frac{1}{2^{[r]+1}}} \]

as desired. Lemma 2.6 is thus established.

**Proof of Theorem 2.3.** For any \( 0 \leq t \leq 1 \), we have

\[ t = \sum_{j=0}^{\infty} \frac{\epsilon_j(t)}{2^j}, \quad \epsilon_j(t) = 0, 1, j = 0, 1, 2, \ldots. \]

Set \( t_n := \sum_{j=0}^{n} \frac{\epsilon_j(t)}{2^j}, n = 0, 1, 2, \ldots \). For any \( \delta > 0 \), let

\[ F^\delta = \left\{ g \in C^{\alpha,0}; \inf_{f \in F} \| f - g \|_\alpha < \delta \right\}. \]

Then, for any positive integer \( k \) and \( \delta > 0 \), we have

\[ C_{p,r} \left( \bigcup_{0 \leq t \leq 1-h} \left\{ \sqrt{\frac{\epsilon}{h}} (\omega(t + h \cdot) - \omega(t)) \in F \right\} \right) \]
\[ \leq C_{p,r} \left( \bigcup_{0 \leq t \leq 1-h} \left\{ \sqrt{\frac{\epsilon}{h}} (\omega(t_k + h \cdot) - \omega(t_k)) \in F^\delta \right\} \right) \]
\[ + C_{p,r} \left( \bigcup_{0 \leq t \leq 1-h} \left\{ \sqrt{\frac{\epsilon}{h}} \| \omega(t + h \cdot) - \omega(t_k + h \cdot)\|_\alpha \geq \delta \right\} \right) \]
\[ =: A + B. \]

Take \( k \) such that \( 2^{k+1} > R/h \geq 2^k \), where \( R \) is a positive constant and will be specified later on. Let \( C(F^\delta) \) be the closure of \( F^\delta \), then

\[ F^\delta \subset C(F^\delta) \subset F^{2\delta}. \]

Then, we have
Hence, by Theorem 2.1 we obtain
\[
\limsup_{\omega \to 0} \varepsilon (\log A + \log h - \log R) \leq - \inf_{f \in C(F^\delta)} I(f) \leq - \inf_{f \in F^{2\delta}} I(f).
\]
For \( \alpha < 1/2 \), let
\[
D(\alpha) := \sum_{j=0}^{\infty} \left( \sum_{j=0}^{\infty} \frac{j}{2j(1-2\alpha)} + \frac{1}{\sqrt{2j(1-2\alpha)}} \right) \quad \text{and} \quad x_j := \frac{1}{D} \sum_{j=0}^{\infty} \frac{j+1}{2j(1-2\alpha)}, \varepsilon < 1,
\]
then
\[
\sum_{j=0}^{\infty} x_j \leq \frac{1}{D} \sum_{j=0}^{\infty} \sqrt{\frac{j}{2j(1-2\alpha)}} + \frac{1}{\sqrt{2j(1-2\alpha)}} \leq \frac{1}{D} \sum_{j=0}^{\infty} \sqrt{\frac{j}{2j(1-2\alpha)}} + \frac{1}{\sqrt{2j(1-2\alpha)}} \leq 1.
\]
For any \( u > 0 \), by Lemma 2.4 and Lemma 2.6, we have
\[
C_{p,r} \left( \sup_{0 \leq t \leq h} \sup_{0 \leq s, \tau \leq 1} \frac{|\omega(\tau + t) - \omega(t) - [\omega(\tau + s) - \omega(s)]|}{|t - s|^\alpha} \right) \geq 2u
\]
\[
\leq C(p,r) \cdot \frac{1}{u^{\beta_{[r]+1}}} \cdot \frac{1}{h} \exp(-C_1u^2h^{2\alpha-1}).
\]
Thus
\[
B \leq C_{p,r} \left( \sum_{0 \leq \tau \leq h} \sum_{j=0}^{\infty} \left( \sum_{j=0}^{\infty} \frac{\sqrt{\varepsilon}}{\tau} \|\omega(t_{k+j+1} + \cdot) - \omega(t_{k+j} + \cdot)\|_{p} \right) \leq \delta \sum_{j=0}^{\infty} x_j \right)
\]
\[
\leq C_{p,r} \left( \sum_{0 \leq \tau \leq h} \sum_{j=0}^{\infty} \left( \sum_{j=0}^{\infty} \frac{\sqrt{\varepsilon}}{\tau} \|\omega(t_{k+j+1} + \cdot) - \omega(t_{k+j} + \cdot)\|_{p} \right) \leq \delta x_j \right)
\]
\[
\leq \sum_{j=0}^{\infty} C(p,r) \cdot \frac{2^{(k+j+1)}}{\delta x_j} \left( \frac{2\varepsilon}{\delta x_j} \right)^{(\beta_{[r]+1})} \cdot \exp \left\{ -C_1 \delta^2 x_j^2 (h \cdot 2^{k+j+1})^{1-2\alpha} \frac{1}{\varepsilon} \right\}
\]
\[
\leq C_2R^2 \frac{1}{h} \sum_{j=0}^{\infty} \frac{2^{2k+j+1}}{\delta x_j x_{\beta_{[r]+1}}} \exp \left\{ -C_1 \delta^2 R^{1-2\alpha} \frac{x_j^2 + 1}{D^2 \varepsilon} \right\}
\]
\[
\leq C_2R^2 \frac{1}{h} \exp \left\{ -C_1 \delta^2 R^{1-2\alpha} \frac{1}{D^2 \varepsilon} \right\} \sum_{j=0}^{\infty} \frac{2^{2k+j+1}}{x_{\beta_{[r]+1}}} \exp \left\{ -C_1 \delta^2 R^{1-2\alpha} \frac{1}{D^2} \right\}
\]
\[
\leq C_2ER^2 \frac{1}{h} \exp \left\{ -C_1 \delta^2 R^{1-2\alpha} \frac{1}{D^2 \varepsilon} \right\}.
\]
where \( C_2 := 4C(p, r)(2/\delta)^{[(r+1)/p]} \) and \( E := \sum_{j=0}^{\infty} 2^j \frac{1}{x_j^{[(r+1)/p]}} \exp\{-\frac{C_2 \delta^2 R^{1-2\alpha}}{D^2}\} < \infty \). Thus for \( R \) large enough, we obtain

\[
\limsup_{\varepsilon \to \infty} \varepsilon (\log B + \log h - 2 \log R) \leq -\frac{C_1 \delta^2 R^{1-2\alpha}}{D^2}.
\]

Combining the above inequalities, for \( R \) large enough, we obtain

\[
\limsup_{\varepsilon \to \infty} \varepsilon (\log (A + B) + \log h)
\leq \limsup_{\varepsilon \to \infty} \varepsilon (\log (Ah) - \log R + \log R) \vee \limsup_{\varepsilon \to \infty} \varepsilon (\log (Bh) - 2 \log R + 2 \log R)
\leq (-\inf_{f \in F_{2\delta}} I(f)) \vee \left( -\frac{C_1 \delta^2 R^{1-2\alpha}}{D^2} \right) + \limsup_{\varepsilon \to 0} 2 \varepsilon \log R
\leq -\inf_{f \in F_{2\delta}} I(f).
\]

Noting that \( F \) is a closed set, it is easy to show that

\[
\lim_{\delta \to 0} \inf_{f \in F_{2\delta}} I(f) = \inf_{f \in F} I(f).
\]

Therefore the proof of Theorem 2.3 is complete. \( \square \)

3. A functional modulus of continuity

For any \( h \in (0, 1) \) and \( t \in [0, 1-h] \), let

\[
M_{t, h}(x) = \frac{\omega(t + hx) - \omega(t)}{\sqrt{2h \log 1/h}}, \quad 0 \leq x \leq 1.
\]

Define the sets \( V_h \) and \( K \) as follows

\[
V_h = \{ M_{t, h}(\cdot) \in C^{\alpha, 0}; 0 \leq t \leq 1-h \}
\]

and

\[
K = \left\{ f(\cdot) \in H; I(f) := \frac{1}{2} \int_0^1 (\dot{f}(t))^2 \, dt \leq 1/2 \right\}.
\]

**Theorem 3.1.** For any \( \varepsilon > 0 \) and \( C_{p, r-q.s.} \omega \), there exists an \( h_0 = h_0(\varepsilon) \) such that

\[
V_h \subseteq K^\varepsilon
\]

and

\[
K \subseteq V_h^\varepsilon
\]

if \( h \leq h_0 \), where the following notation is used

\[
E^\varepsilon := \{ g \in C^{\alpha, 0}; \inf_{f \in E} \| f(\cdot) - g(\cdot) \|_\alpha < \varepsilon \}.
\]

An equivalent result is the following theorem.
Theorem 3.2. We have
\[ \lim_{h \to 0} \sup_{0 \leq t \leq 1-h} \inf_{f \in K} \| M_t, h(\cdot) - f(\cdot) \|_\alpha = 0, \quad C_{p,r}-q.s. \omega \]
and for any \( f \in K \)
\[ \lim_{h \to 0} \inf_{0 \leq t \leq 1-h} \| M_t, f(\cdot) \|_\alpha = 0, \quad C_{p,r}-q.s. \omega. \]

Proof. For any \( \theta > 1 \) and integer \( n \geq 1 \), let \( h_n = \theta^{-n} \). First of all, we show that
\[ \lim_{n \to \infty} \sup_{0 \leq t \leq 1-h_n+1} \inf_{f \in K} \| M_t, h_n(\cdot) - f(\cdot) \|_\alpha \geq \varepsilon \]
\[ \leq C_{p,r} \left( \sum_{0 \leq t \leq 1-h_n+1} \left\{ M_{t, h_n}(\cdot) \in (K^\varepsilon)^c \right\} \right) \]
\[ \leq C_{p,r} \left( \sum_{0 \leq t \leq 1-h_n} \left\{ M_{t, h_n}(\cdot) \in (K^\varepsilon)^c \right\} \right) \]
\[ + C_{p,r} \left( \sum_{1-h_n \leq t \leq 1-h_n+1} \left\{ M_{t, h_n}(\cdot) \in (K^\varepsilon)^c \right\} \right) \]
\[ \leq 2C_{p,r} \left( \sum_{0 \leq t \leq 1-h_n} \left\{ M_{t, h_n}(\cdot) \in (K^\varepsilon)^c \right\} \right) \]
\[ = 2C_{p,r} \left( \sum_{0 \leq t \leq 1-h_n} \left\{ \frac{\omega(t+h_n \cdot) - \omega(t)}{\sqrt{2h_n \log 1/h_n}} \in (K^\varepsilon)^c \right\} \right) \]
\[ \leq \frac{2}{h_n} \exp \left\{ -2(1 - v) \inf_{f \in (K^\varepsilon)^c} I(f) \log \left( \frac{1}{h_n} \right) \right\} \]
\[ = 2\theta^{-2(1-v)\inf_{f \in (K^\varepsilon)^c} I(f) \log(1/h_n)} \]
for sufficiently large \( n \). Take \( v \) small enough such that
\[ \eta := 2(1 - v) \inf_{f \in (K^\varepsilon)^c} I(f) > 1. \]
Hence we obtain
\[ \sum_n C_{p,r} \left( \sum_{0 \leq t \leq 1-h_n+1} \inf_{f \in K} \| M_{t, h_n}(\cdot) - f(\cdot) \|_\alpha \geq \varepsilon \right) < \infty, \]
which, by the Borel–Cantelli lemma, implies (3.3).

Next, for all \( h \in (0, 1) \), there exists \( n \) such that \( h_{n+1} \leq h \leq h_n \). Then we have
\[ \sup_{0 \leq t \leq 1-h} \inf_{f \in K} \| M_t, h(\cdot) - f(\cdot) \|_\alpha \]
\[ \leq \sup_{0 \leq t \leq 1-h_n} \inf_{f \in K} \| M_t, h_n \left( \frac{h}{h_n} \right) - f \left( \frac{h}{h_n} \right) \|_\alpha \]
\[ + \left( \sqrt{\frac{h_n \log 1/h_n}{h \log 1/h}} - 1 \right) \sup_{0 \leq t \leq 1-h_{n+1}} \left\| \frac{\omega(t + h_n \left( \frac{h}{h_n} \right) \cdot) - \omega(\cdot)}{\sqrt{2h_n \log 1/h_n}} \right\|_{a} \]

\[ + \sup_{f \in K} \left\| f \left( \frac{h}{h_n} \right) \right\|_{a} \]

\[ =: I_1 + I_2 + I_3. \]

Via the already established (3.3), we obtain

\[ \lim_{n \to \infty} I_1 = 0, \quad C_{p,r}-q.s. \omega \]

and for \( n \) large enough we get

\[ \sup_{0 \leq t \leq 1-h_{n+1}} \left\| \frac{\omega(t + h_n \left( \frac{h}{h_n} \right) \cdot) - \omega(\cdot)}{\sqrt{2h_n \log 1/h_n}} \right\|_{a} \leq 2, \quad C_{p,r}-q.s. \omega. \]

Since

\[ \sqrt{\frac{h_n \log 1/h_n}{h \log 1/h}} - 1 \leq \sqrt{\theta} - 1 \]

thus

\[ \lim_{n \to \infty} I_2 \leq 2(\sqrt{\theta} - 1), \quad C_{p,r}-q.s. \omega. \]

Concerning \( I_3 \), we have

\[ I_3^2 \leq \sup_{f \in K} \left\| f \left( \frac{h}{h_n} \right) - f(\cdot) \right\|_{a}^2 \leq 2(\theta - 1). \]

Combining \( I_1, I_2 \) and \( I_3 \), we obtain

\[ \lim_{h \to 0} \sup_{0 \leq t \leq 1-h} \inf_{f \in K} \left\| M_t,h(\cdot) - f(\cdot) \right\|_{a} \leq 4(\theta - 1), \quad C_{p,r}-q.s. \omega. \]

Since we can take any \( \theta > 1 \), therefore (3.1) is true.

Next, we show that (3.2) holds for any \( f \in K \). For all \( h \in (0, 1) \), there exists \( n \) such that \( h_{n+1} \leq h \leq h_n \). Thus, for any \( f \in K \), we have

\[ \inf_{0 \leq t \leq 1-h} \left\| M_{t,h}(\cdot) - f(\cdot) \right\|_{a} \]

\[ \leq \sup_{0 \leq t \leq 1-h} \left\| M_{t,h_n} \left( \frac{h}{h_n} \right) - f \left( \frac{h}{h_n} \right) \right\|_{a} \]

\[ + \left( \sqrt{\frac{h_n \log 1/h_n}{h \log 1/h}} - 1 \right) \sup_{0 \leq t \leq 1-h_{n+1}} \left\| \frac{\omega(t + h_n \left( \frac{h}{h_n} \right) \cdot) - \omega(\cdot)}{\sqrt{2h_n \log 1/h_n}} \right\|_{a} \]

\[ + \sup_{f \in K} \left\| f \left( \frac{h}{h_n} \right) \right\|_{a} \]

\[ := I_4 + I_2 + I_3. \]

For any \( \varepsilon > 0 \) and \( \nu > 0 \), by Lemma 2.6 and Theorem 2.2, we have
\[ C_{p,r} \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq 2 \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq C (p, r)^{q+1} \mu \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq \varepsilon \right) \]
\[ \leq 2 C (p, r)^{q+1} \mu \exp \left( - \frac{1}{2} \inf_{g \in B(f, \varepsilon)} I (g) \right) \]

for sufficiently large \( n \). Since \( \inf_{g \in B(f, \varepsilon)} I (g) < 1/2 \), taking \( v \) small enough, we have
\[ 1 - 2(1 + v) \inf_{g \in B(f, \varepsilon)} I (g) > 0. \]

Hence
\[ \sum_n C_{p,r} \left( \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \geq 2 \varepsilon \right) < \infty. \]

Therefore, by the Borel–Cantelli lemma, we obtain
\[ \lim_{h \to 0} \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha = 0, \quad C_{p,r} \text{-q.s. } \omega \]

which implies that
\[ \lim_{n \to \infty} I_4 = 0, \quad C_{p,r} \text{-q.s. } \omega. \]

Combining \( I_2 \) and \( I_4 \), we have
\[ \lim_{h \to 0} \inf_{0 \leq t \leq 1 - h_n} \| M_{t,h_n} (\cdot) - f (\cdot) \|_\alpha \leq 4(\theta - 1), \quad C_{p,r} \text{-q.s. } \omega \]

for any \( f \in K \). Since we can take any \( \theta > 1 \), therefore (3.2) is true. We complete the proof. \( \square \)

**Acknowledgements**

Part of the work was done when the first named author was a postdoctoral fellowship, at the Group of Mathematical Physics of the University of Lisbon, POCT1/MAT/34924/2000. He would like to express his deepest gratitude to Professor Jean-Claude Zambrini and Professor Ana Bela Cruzeiro for their warm hospitality and stimulating discussions. Both authors thank support from Project 973 and NSF (No. 10526020) of China.
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