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PSEUDO-MAGIC GRAPHS

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We characterize graphs for which there is a labeling of the edges by pairwise different integer labels such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. We generalize to mixed graphs, and to labelings with values in an integral domain.

1. Introduction

We consider finite, undirected, connected *graphs*, allowing loops and multiple edges. Let G be such a graph and E its edge set. If $\lambda \in \mathbb{Z}$, we call $s: E \to \mathbb{Z}$ a *labeling for* λ if for every vertex x:

$$\sum_{e \in E} i(x, e)s(e) = \lambda,$$

where $i(x, e) = 0$ if e is not incident with x, $=2$ if e is a loop at x, and $=1$ if e is an edge, not a loop, incident with x. $S(G)$ is the $\mathbb{Z}$-module of all such labelings, for any λ, and $Z(G)$ is the $\mathbb{Z}$-module of labelings for 0. A labeling $s \in S(G)$ is called *pseudo-magic* if the 'labels' $s(e)$ are pairwise different, *magic* if moreover they are all non-negative. We shall first mention some results that are necessary for understanding the remainder of this paper. Further details may be found in the references. We shall then prove what is in effect a special case of the main theorem; the generalization and its analogous proof are saved for the final sections.

If a submodule of $\mathbb{Z}^q$ has the property that for every i and j, $1 \leq i < j \leq q$, it contains an element $(x_1, \ldots, x_q)$ with $x_i \neq x_j$, then it contains an element with pairwise different components. In other words: if it is in none of the hyperplanes $x_i = x_j$, it is not in their union. The proof is easy, and the principle has been used before ([5, Theorem 5], [2, Theorem 3.1]). It follows that a graph G is not pseudo-magic iff it has a pair e, f of edges with $s(e) = s(f)$ for every $s \in S(G)$. These pairs we shall characterize. Note that a graph is magic iff if is pseudo-magic and there is for every edge e an $s \in S(G)$ with $s(e) > 0$ and $s(f) \geq 0$ for every edge f (cf. [5, Theorem 6]; the use of $\mathbb{R}$ instead of $\mathbb{Z}$ is not essential, see [2, Section 2] or [4, Section 1]). The latter condition is satisfied for instance if every edge is on a Hamiltonian cycle or an other spanning regular subgraph.

A connected bipartite graph with point-set $P_1 \cup P_2$, P_1 and P_2 stable, will be denoted by P_1P_2 (we admit $P_1 = \emptyset$, or $P_2 = \emptyset$, i.e. the trivial one-point graph is bipartite). It is called *balanced* if $|P_1| = |P_2|$, *unbalanced* if $|P_1| \neq |P_2|$. For a connected graph G that is non-bipartite or bipartite and balanced we have $S(G) \neq Z(G)$, i.e. there is a labeling for some $\lambda \neq 0$ ([1, Theorem 2.11], [3, Theorem 1']).

Let f be an edge of the connected graph G. We refer to f as an edge of

type s_a if G is non-bipartite, but $G - \{f\}$ is bipartite (so connected) and balanced,

type s_b if G consists of a non-bipartite graph and a balanced bipartite graph, connected by the bridge f,

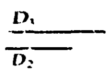
type s_c if G consists of two balanced bipartite graphs connected by the bridge f,

type s_d if G is an unbalanced bipartite graph and f a bridge.

It has been proved that $s(f) = 0$ for all $s \in S(G)$ iff s is of one of these four types [4, Theorem 1].

2. Pictograms

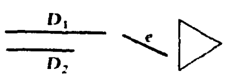
A few examples may suffice to explain the symbolism we found useful in formulating the theorem below. A connected bipartite graph D_1D_2 will be symbolized by

 or 

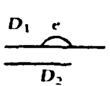
(which does not imply $|D_1| > |D_2|$). If we want to express that it is balanced we use

 .

A connected non-bipartite graph is symbolized by .
Further



means that e is a bridge from D_1D_2 to a connected non-bipartite graph, e having an endpoint in D_1. At last



means a connected bipartite graph D_1D_2 into which an extra edge is inserted with its endpoint(s) in D_1 (e may be a loop). When in doubt the reader may derive the precise meaning of a symbol from the proof of the theorem.

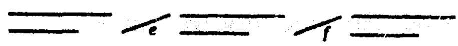
3. The Theorem

Theorem. *Let G be a connected, undirected, finite graph and let e and f be edges of G. Then $s(e)=s(f)$ for all $s \in S(G)$ iff e and f take one of the following positions in G (possibly after interchanging their names):*

(a) 

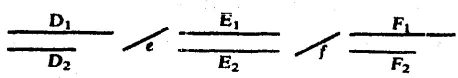
with G unbalanced.

(b) 

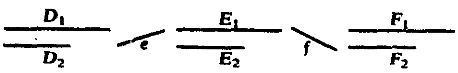
or 

both with G unbalanced.

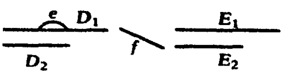
(c) 

or 

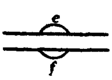
with $|D_1|+|F_1|=|D_2|+|F_2|$.

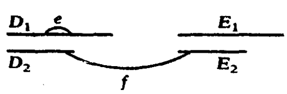
(d) 

with $|E_1|-|E_2|=2(|D_2|-|D_1|)=2(|F_2|-|F_1|)$.

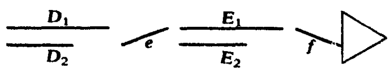
(e) 

with $|D_1|-|D_2|=3(|E_2|-|E_1|)$.

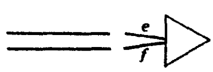
(f) 

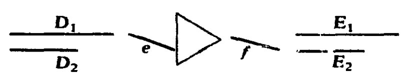
or 

with $|D_1|+|E_1|=|D_2|+|E_2|$.

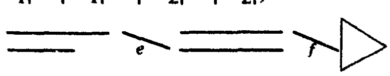
(g) 

with $|E_1|-|E_2|=2(|D_2|-|D_1|)$.

(h) 

or 

with $|D_1|+|E_1|=|D_2|+|E_2|$, *or*



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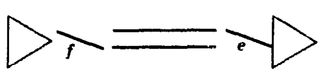
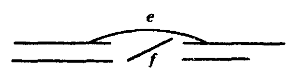


or 

Proof. The sufficiency can easily be proved for (groups of) separate cases, as follows. One takes a labeling s for λ and evaluates the sum of the labels of the edges of a bipartite subgraph in two ways. E.g. in case (e):

$$\lambda\,|D_2|=\lambda\,|D_1|-2s(e)-s(f) \quad \text{and} \quad \lambda\,|E_1|=\lambda\,|E_2|-s(f),$$

from which:

$$s(f)=\lambda(|E_2|-|E_1|) \quad \text{and} \quad s(e)=\tfrac{1}{2}\lambda(|D_1|-|D_2|+|E_1|-|E_2|),$$

or in cases (c) and (h): e and f arrive at different parts of a balanced subgraph P_1P_2, so $\lambda\ |P_1| - s(e) = \lambda\ |P_2| - s(f)$, yielding $s(e) = s(f)$. In cases (a) and (b) one also has to use that G has no labelings for a $\lambda \neq 0$ (take the sum of all labels to show this). It may seem strange that in cases (c) and (d) the given condition is not (fully) needed, but then if the remaining part doesn't hold we are in another case ((b), (a) respectively). The cases as given do not overlap (although of course various cases may occur in one graph). The left graphs in Fig. 1 illustrate the second case of (h) and the first of (f), respectively.

To prove the necessity we put $e = (x, y)$ and $f = (z, w)$. (None of the possible equalities between x, y, z and w are excluded, nor is the existence of other edges between x and y or z and w).

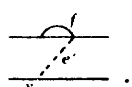
(**A**) Suppose in $G - \{e, f\}$ there is a walk W of odd length from an endpoint of e, x say, to an endpoint of f, z say (the repeated use of edges or points by W is not excluded). Let $m_1, \ldots, m_k$ be the edge-sequence of W. Construct G' by omitting e from G and inserting a new edge e' between y and w (see Fig. 1 for two examples). For $s \in S(G)$ with $s(e) = u$ we construct $s' \in S(G')$ as follows: $s'(e') = u$, $s'(f) = s(f) - u$, $s'(g) = s(g)$ if $g \neq e, f$ and g not on W, and if g is on W, then

$$s'(g) = s(g) + u(\alpha(g) - \beta(g))$$

where $\alpha(g)$ is the number of odd i with $m_i = g$, $\beta(g)$ the number of even i with $m_i = g$. Thus along W we alternatingly raise and diminish the labels by u. Now $s \mapsto s'$ defines an isomorphism $S(G) \to S(G')$. Therefore if $s(e) = s(f)$ for all $s \in S(G)$, $s'(f) = 0$ for all $s' \in S(G')$, i.e. f is of type $s_a, \ldots, s_d$ in G' (note that G' is connected).

Now from this knowledge about f, we try to reconstruct G from G'. Note that e' has an endpoint in common with f and that x has to be found at the end of an odd walk starting from the other endpoint of f (if f is not a loop) and not using f or e'.

(a) f of type s_a. G' looks like



x has to be located in the lower part. Depending on whether e' is a bridge or not in $G' - \{f\}$ we find the second or first case of (f).

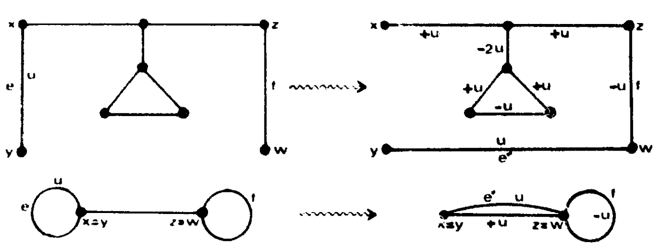


Fig. 1.

(b) f of type s_b. If e' is in the bipartite component of $G-\{f\}$, we find the second or first case of (h), depending on whether e' is a bridge in that component or is not. If e' is in the non-bipartite component of $G-\{f\}$ and is not a bridge in that component we find the first case of (h) or the fifth, depending on whether the component stays non-bipartite or becomes bipartite if e' is deleted. If e' is a bridge in that component we find the third or fourth case of (h).

(c) f of type s_c. We find the cases (c).

(d) f of type s_d. We find the cases (b).

(**B**) Suppose there is no walk W as above.

(a) Let $G''=G-\{e,f\}$ be connected. Then it is bipartite, since the existence of an odd cycle would permit the construction of an odd walk between any two points. Also the points x, y, z and w must all belong to the same 'part' of the bipartition of G''. But then we could construct an (even) path from x to z (possibly of length 0) and one from w to y, and connect these paths by e and f to a closed walk. Assign alternatingly $+1$ and -1 to its edges (adding if an edge occurs twice) and 0 to all other edges, thus constructing a labeling s for 0 with $s(e)=+1$, $s(f)=-1$. This excludes this possibility.

(b) Let $G''=G-\{e,f\}$ be disconnected and have two components G_1 and G_2.

(b_1) Let e nor f be a bridge. Then we may suppose that x and z are points in G_1, y and w points in G_2. Again G_1 has to be bipartite with x and z in one part, and the same goes for G_2, y and w. Now there is an even closed path containing e and f, which enables us to find a labeling for 0 with label $+1$ for e and label -1 for f. So we have:

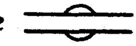
(b_2) Precisely one of e and f is a bridge. We may suppose that e is the bridge and that x, z, and w are points of G_1, y of G_2. Then $G_1-\{f\}$ is connected, and (as above) bipartite with x, z and w in one 'part'. Thus in G_1 we have an odd walk from x to x using f once. If there is also an odd walk from y to y in G_2 we can construct a labeling s for 0 with $s(e)=2$ and $s(f)=-1$. Thus G_2 is also bipartite. Now take a labeling s for $\lambda\neq 0$ (G is non-bipartite) and put $G_1-\{f\}=D_1D_2$, $G_2=E_1E_2$ with $x, y, z\in D_1$, $y\in E_2$. Then

$$\lambda\,|D_1|-2s(f)-s(e)=\lambda\,|D_2| \quad\text{and}\quad \lambda\,|E_1|=\lambda\,|E_2|-s(e).$$

From $s(e)=s(f)$ it follows that $|D_1|-|D_2|=3(|E_2|-|E_1|)$. We have found case (e).

(c) Let e and f both be bridges. let G_1, G_2, G_3 be the components of $G-\{e,f\}$, with x a point of G_1, y and z points of G_2, and w a point of G_3. G_2 is bipartite, E_1E_2 say, with y and z in E_1. Existence of odd cycles in G_1 and G_3 would make possible a labeling s for 0 with $s(e)=2$, $s(f)=-2$. So we assume that G_1 is bipartite, D_1D_2 say, with x in D_2. If G (so G_3) is non-bipartite or if G is bipartite and balanced, then there is a labeling s for some $\lambda\neq 0$. We then find $\lambda\,|D_1|=\lambda\,|D_2|-s(e)$, $\lambda\,|E_1|-s(e)-s(f)=\lambda\,|E_2|$, and from $s(e)=s(f)$ it follows that $|E_1|-|E_2|=2(|D_2|-|D_1|)$. This gives cases (g) and (d). We have left: G unbalanced bipartite, which is case (a). □

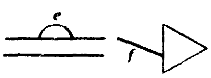
Corollary 1. *A connected graph is pseudo-magic iff it is not of one of the types pictured in the theorem.*

Corollary 2. *A triply line-connected graph is pseudo-magic or of type* .

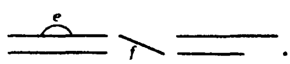
Corollary 3. *If a connected graph is not pseudo-magic it has two edges whose simultaneous deletion yields a graph with at least one balanced bipartite component or at least two bipartite components.*

4. Labelings over an integral domain

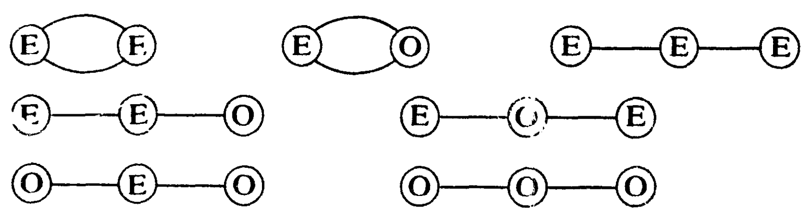
In the definitions of labeling and pseudo-magic we replace $\mathbb{Z}$ by an integral domain F, the modules of labelings being now called $S(G, F)$ and $Z(G, F)$. A bipartite graph P_1P_2 is now called balanced if $|P_1| \equiv |P_2|$ (mod char F). If char $F \neq 2$, then still $S(G, F) \neq Z(G, F)$ iff G is bipartite and balanced or non-bipartite and also $s(f) = 0$ for all $s \in S(G, F)$ iff f is of one of the four types s_a, s_b, s_c, s_d defined in Section 1 ('balanced' now read in the above sense; references as in Section 1). The proof of the theorem goes through without changes, except when char $F = 3$, which gives a small difficulty in part (B)(b_2). It turns out that for char $F = 3$ we should replace case (e) in the theorem by:



or



If char $f = 2$ there are more changes (references as before). First of all the bipartite graphs play no special role, for $S(G, F) \neq Z(G, F)$ if and only if the number of points of G is even. Secondly the four types $s_a, \ldots, s_d$ reduce to two: f a bridge between two graphs with even point-sets and f a bridge in a graph with an odd point-set. The necessity part of the proof is much easier: in A an odd as well as an even walk does the trick, so B is superfluous. We find the following cases (Ⓔ and Ⓞ depicting connected graphs with an even and an odd number of points respectively):



The sufficiency is again easy: twice the sum of the labels of the edges in a 'balloon' plus the labels of the edges attached to it equals λ times the number of points in it (everything in F).

The validity of the corollaries depends on that of the separation-principle of Section 1. It is an easy exercise to show that it holds provided F is infinite, or finite with $|F|>\binom{q}{2}$, q the number of edges of G. Let us assume that this condition is satisfied. Then Corollary 1 stays true if 'types' is adapted for char $F=2,3$. Corollary 2 also stays true (one may omit the part 'or...' if char $F=2$), and Corollary 3 holds for all characteristics $\neq 2$.

5. Labelings of mixed graphs

We now allow (some of) the edges to be directed. In the definition of 'labeling' $i(x,e)$ is defined as before for e undirected. If e is directed we put $i(x,e)=0$ if e is not incident with x or is a loop at x, $=+1$ if e is not a loop and x its endpoint, $=-1$ if e is not a loop and x its initial point. We suppose char $F\neq 2$ (if char $F=2$, direction of edges is irrelevant).

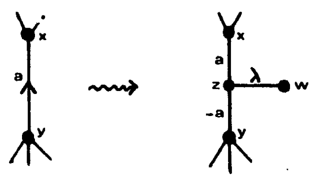
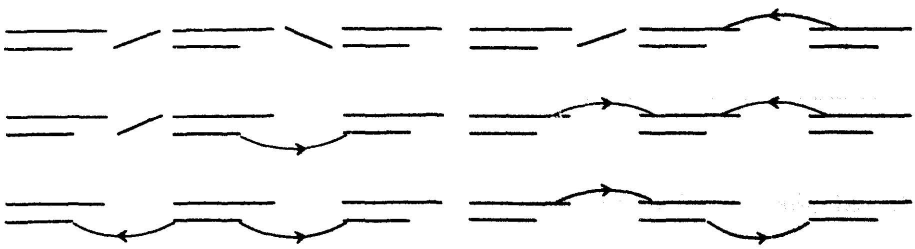


Fig. 2.

As is shown by Fig. 2 one can associate with a mixed graph G an undirected graph G' such that there is an isomorphism between $S(G,F)$ and $S(G',F)$, in which labelings of G for λ correspond to such of G' for λ. Note that G' is bipartite iff the point-set of G can be partitioned as $P_1\cup P_2$ in such a way that undirected edges have an endpoint in P_1 and one in P_2, whereas directed edges have their initial point and their endpoint both in P_1 or both in P_2. Let us call G 'bipartite' if G' is bipartite. It is not difficult now to translate our theorem for the case of mixed graphs. Starting with the necessity-part: if e and f are edges of G with $s(e)=s(f)$ for all $s\in S(G,F)$, then in G' they yield edges e' and f' with $s(e')=s(f')$ for all $s\in S(G',F)$, so for G' with e',f' we have one of the eight cases of the theorem. For each case we reconstruct G from G', taking into account the possibilities: e (un)directed, f (un)directed. Note that the orientation of $(x,y)=e$ (see Fig. 2) is determined in the reconstruction as being towards x, and that x is one of the endpoints of $e'=(x,z)$. See also below, under case (a). The pictograms stay useful if we keep in mind to replace 'bipartite' for bipartite. Note that the notion of 'balanced' for G' can easily be carried over to G: it still means $|P_1|\equiv|P_2|$ (mod char F) for the partition $P_1\cup P_2$ of G as above. If the reconstruction of G from G' is done carefully, i.e. yields precisely those G whose G' is of one of the types of the theorem and no more, then a sufficiency proof is superfluous. It would be, however, again easy and is done in the same way as before, adding the labels of the undirected edges in 'bipartite' subgraphs. Moreover it is useful as a check, see below.

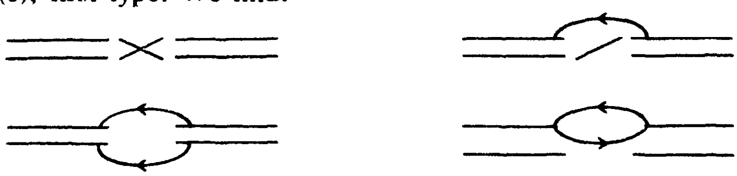
Let us illustrate the result for a few cases of the theorem.

Case (a). We find six possibilities:

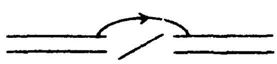
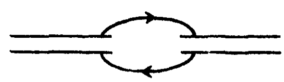


each with G 'bipartite' and unbalanced. In checking this the reader may have found five other cases, with certain changes in the directions of e and f, if he has erronously taken e' or f' in G' to be (z, y) instead of (z, x) (see Fig. 2). But the proof that in our six cases indeed $s(e) = s(f)$ for all $s \in S(G, F)$ doesn't use the orientation, so the five cases should turn up. In fact they are found from the second type of case (b). That the error was harmless here is due to the fact that in case (a) the edges e and f get label 0 for all labelings so in Fig. 2 the label a is always 0 and (z, y) can indeed be taken as e'. Things are different however in:

Case (c), first type. We find:



and neither

 nor .

In fact the latter turn up nowhere. Their corresponding G' looks like

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