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Minimal Representations of Semiseparable Kernels and Systems with Separable Boundary Conditions

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The simplest representations of triangular parts of finite rank kernels are analysed. Criteria for uniqueness up to similarity are given. The results are applied to the problem of minimal realization of systems with separable boundary conditions. © 1987 Academic Press, Inc.

0. INTRODUCTION

Consider the integral equation

$$\varphi(t)-\int_a^b k(t,s)\,\varphi(s)\,ds=f(t),\qquad a\leqslant t\leqslant b,\tag{0.1}$$

and assume that the kernel $k(t, s)$ is an $m\times m$ matrix kernel of semiseparable type, i.e.,

$$k(t,s)=\begin{cases}F_1(t)\,G_1(s), & a\leqslant s<t\leqslant b,\\ -F_2(t)\,G_2(s), & a\leqslant t<s\leqslant b.\end{cases}\tag{0.2}$$

Here for $\nu=1, 2$ the functions $F_\nu(\cdot)$ and $G_\nu(\cdot)$ are matrix functions of sizes $m\times n_\nu$ and $n_\nu\times m$, respectively, and their entries are square integrable on $[a, b]$. In $L_2^m[a, b]$, Eq. (0.1) is equivalent to the following two point boundary value problem:

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$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix}=\begin{pmatrix}G_1(t)\\ G_2(t)\end{pmatrix}\varphi(t), \qquad a\leqslant t\leqslant b,$$
$$g(t)=-F_1(t)\,x_1(t)-F_2(t)\,x_2(t)+\varphi(t), \qquad a\leqslant t\leqslant b,$$
$$x_1(a)=0, \qquad x_2(b)=0,$$

and hence in (0.1) the given function g and the unknown φ are related as follows:

$$\begin{aligned} &\dot{x}(t)=A(t)\,x(t)+B(t)\,g(t), && a\leqslant t\leqslant b,\\ &\varphi(t)=C(t)\,x(t)+g(t), && a\leqslant t\leqslant b,\\ &(I-P)\,x(a)=0, && Px(b)=0. \end{aligned} \tag{0.3}$$

Here

$$A(\cdot)=\begin{pmatrix}G_1(\cdot)\,F_1(\cdot) & G_1(\cdot)\,F_2(\cdot)\\ G_2(\cdot)\,F_1(\cdot) & G_2(\cdot)\,F_2(\cdot)\end{pmatrix}, \qquad B(\cdot)=\begin{pmatrix}G_1(\cdot)\\ G_2(\cdot)\end{pmatrix},$$
$$C(\cdot)=(F_1(\cdot) \quad F_2(\cdot)), \qquad P=\begin{pmatrix}0 & 0\\ 0 & I_{n_2}\end{pmatrix}.$$

In this way the problem to solve (0.1) is reduced to the problem to determine the fundamental matrix $U(\cdot)$ of the differential equation $\dot{x}(t)=A(t)\,x(t)$ and to calculate the input–output behaviour of the linear system (0.3). For example (see [6, Sect. II.3]), Eq. (0.1) is uniquely solvable in $L_2^m[a,b]$ if and only if $\det((I-P)\,U(a)+PU(b))\neq 0$, and in that case the resolvent kernel is given by

$$\gamma(t,s)=\begin{cases} C(t)\,U(t)(I-Q)\,U(s)^{-1}\,B(s), & a\leqslant s<t\leqslant b,\\ -C(t)\,U(t)\,QU(s)^{-1}\,B(s), & a\leqslant t<s\leqslant b,\end{cases}$$

where $Q=[(I-P)\,U(a)+PU(b)]^{-1}\,PU(b)$.

In the above analysis the number of operations and their complexity depend in general on the sizes of the matrices involved and, in particular, on the number $n=n_1+n_2$, which is the dimension of the state space of the linear system (0.3). This brings us to the problem to represent the kernel k in the form (0.2) with n_1 and n_2 as small as possible and to analyse these minimal semiseparable representations. In system theoretical language we are dealing here with the problem of minimal realization of the input–output operator in the class of time varying systems with separable boundary conditions. Also for causal systems our theorems contain new elements, and our results may be viewed as an addition to the classical theory for causal time varying linear systems [10, 15, 16] (also [5, 9, 14]).

The analysis of the minimal realization problem for systems with separable boundary conditions is given in the last section of this paper. It is based on a study of triangular parts of finite rank kernels and their most economical representations, which begins with a little theory of separable representations of kernels of finite rank. In the one but last section two open problems are discussed.

1. Kernels of Finite Rank

Throughout this paper Y and Z are finite dimensional linear spaces over $\mathbb{C}$. When necessary we assume Y and Z to be endowed with an inner product and a corresponding norm. In what follows Ω and Σ are (Lebesgue) measurable subsets of the real line of positive measure. By $L_2(\Omega; Y)$ we denote the Hilbert space of all (equivalence classes of) square (Lebesgue) integrable vector functions on Ω with values in Y; the space $L_2(\Sigma; Z)$ is defined analogously.

Let $k(t, s)\colon Z \to Y$, $(t, s) \in \Omega \times \Sigma$, be an operator-valued kernel. The kernel k is said to have a *separable representation* if k can be written in the form

$$k(t, s) = F(t)\, G(s), \quad (t, s) \in \Omega \times \Sigma, \quad \text{a.e.,} \tag{1.1}$$

where $G(s)\colon Z \to X$ and $F(t)\colon X \to Y$ are linear operators, the space X is a finite dimensional inner product space and the functions G and F are square (Lebesgue) integrable on Σ and Ω, respectively. The dimension of the internal space X is called the *order* of the separable representation (1.1).

If the kernel k admits a separable representation, then the corresponding integral operator

$$(K\varphi)(t) = \int_\Sigma k(t, s)\, \varphi(s)\, ds, \qquad t \in \Omega, \quad \text{a.e.,} \tag{1.2}$$

which has to be considered as an operator from $L_2(\Sigma; Z)$ into $L_2(\Omega; Y)$, is an operator of finite rank. Conversely, as is well known, if the integral operator (1.2) has finite rank, then its kernel admits a separable representation. We define the *rank* of k (notation: rank (k)) to be the rank of the operator (1.2). Thus the kernel k admits a separable representation if and only if k has finite rank. Note that the order of a separable representation is always larger than or equal to the rank of the corresponding kernel.

A finite rank kernel k on $\Omega \times \Sigma$ admits many different representations. To classify the various representations we use the concepts of similarity and dilation. Two separable representations $k(t, s) = F_1(t)\, G_1(s)$ and $k(t, s) = F_2(t)\, G_2(s)$ on $\Omega \times \Sigma$ are called *similar* if there exists an invertible operator S such that

$$G_1(s) = S^{-1} G_2(s), \qquad F_1(t) = F_2(t)\, S \tag{1.3}$$

almost everywhere for $t \in \Omega$ and $s \in \Sigma$, respectively. A separable representation $k(t, s) = F(t)\, G(s)$ on $\Omega \times \Sigma$ is said to be a *dilation* of the separable representation $k(t, s) = F_0(t)\, G_0(s)$ if the internal space of the first representation admits a direct sum decomposition, $X = X_1 \oplus X_0 \oplus X_2$, such that relative to this decomposition

$$G(s) = \begin{bmatrix} * \\ G_0(s) \\ 0 \end{bmatrix}, \qquad F(t) = (0 \quad F_0(t) \quad *) \tag{1.4}$$

almost everywhere for $t \in \Omega$ and $s \in \Omega$ and $s \in \Sigma$, respectively. We say that the dilation is *proper* if, in addition, $\dim X > \dim X_0$, i.e., the order of the representation $k(t, s) = F(t)\, G(s)$ is strictly larger than the order of the representation $k(t, s) = F_0(t)\, G_0(s)$. Finally, a separable representation of k on $\Omega \times \Sigma$ is called *irreducible* if this representation is not a proper dilation of another separable representation of k. From the definition of irreducibility it is clear that any separable representation of k is a dilation of an irreducible one.

1.1. THEOREM. *Let the finite rank kernel k be represented by* (1.1). *Then the following are equivalent*:

(i) *the separable representaton* (1.1) *is irreducible*;

(ii) *the representation* (1.1) *is of minimal order among all separable representations of k*;

(iii) *the order of* (1.1) *is equal to rank of k*;

(iv) *the following operators are invertible*:

$$\int_\Sigma G(s)\, G(s)^*\, ds, \qquad \int_\Omega F(t)^*\, F(t)\, dt. \tag{1.5}$$

Proof. Introduce the following auxiliary operators

$$\Gamma\colon\ L_2(\Sigma; Z) \to X, \qquad \Gamma\varphi = \int_\Sigma G(s)\, \varphi(s)\, ds;$$

$$\Lambda\colon\ X \to L_2(\Omega; Y), \qquad (\Lambda\varphi)(t) = F(t)\, x.$$

Obviously, $\Gamma\Gamma^*$ is equal to the first operator in (1.5) and $\Lambda^*\Lambda$ is the second operator in (1.5). Further, $\Lambda\Gamma = K$, where K is the integral operator (1.2) with kernel (1.1). Assume that (iv) holds. Then Λ is one-to-one and Γ is onto. So rank $K = \dim X$. Thus (iv) implies (iii).

Since rank(k) is less than or equal to the order of the representation (1.1), it is clear that (iii) implies (ii). Further, if (ii) holds, then the representation (1.1) cannot be a proper dilation of another separable representation of k. So (i) follows from (ii). It remains to prove that (i) implies (iv). To do this we make the following general construction.

Let Γ and Λ be as in the first paragraph of the proof. Put $X_1 = \operatorname{Ker} \Lambda$, and define X_0 to be a direct complement of $\operatorname{Ker} \Lambda \cap \operatorname{Im} \Gamma$ in $\operatorname{Im} \Gamma$. Let X_2 be a direct complement of $X_1 + X_0$ in X. Then $X = X_1 \oplus X_0 \oplus X_2$. Consider the partitioning of $F(t)$ and $G(s)$ relative to this decomposition:

$$G(s) = \begin{bmatrix} G_1(s) \\ G_0(s) \\ G_2(s) \end{bmatrix}, \qquad F(t) = (F_1(t) \quad F_0(t) \quad F_2(t)).$$

Since $\operatorname{Im} \Gamma \subset X_1 + X_0$, we have $\operatorname{Im} G(s) \subset X_1 \oplus X_0$ for almost all $s \in \Sigma$. So $G_2(s) = 0$ almost everywhere on Σ. Further, Λ is zero on X_1, which implies that $F_1(t) = 0$ almost everywhere on Ω. It follows that

$$k(t, s) = F(t)\, G(s) = F_0(t)\, G_0(s), \qquad (t, s) \in \Omega \times \Gamma, \quad \text{a.e.} \tag{1.6}$$

Thus (cf., formula (1.4)) the separable representation $k(t, s) = F(t)\, G(s)$ is a dilation of the separable representation $k(t, s) = F_0(t)\, G_0(s)$. Note that this dilation is proper whenever $X_1 \neq (0)$ and/or $X_2 \neq (0)$.

Now, assume that (i) holds. Then the representation (1.1) is not a proper dilation of another separable representation of k. So in the above construction the spaces X_1 and X_2 consist of the zero element only. Hence $\operatorname{Ker} \Lambda = (0)$ and $\operatorname{Im} \Gamma = X$. But this means that $\Lambda^* \Lambda$ and $\Gamma \Gamma^*$ are invertible, and hence the operators (1.5) are invertible. ▮

Since similarity does not change the order of a separable representation, the equivalence of (i) and (ii) in Theorem 1.1 implies that irreducibility is preserved under similarity.

The construction carried out in the one but last paragraph of the proof of Theorem 1.1 yields an explicit way to reduce a separable representation of k to an irreducible one. To see this, we show that the separable representation $k(t, s) = F_0(t)\, G_0(s)$ in (1.6) is irreducible. Using the notation introduced in the proof of Theorem 1.1, let

$$\Gamma_0 \colon\ L_2(\Sigma; Z) \to X_0, \qquad \Gamma_0 \varphi = \int_\Sigma G_0(s)\, \varphi(s)\, ds,$$

$$\Lambda_0 \colon\ X_0 \to L_2(\Omega; Y), \qquad (\Lambda_0 x)(t) = F_0(t)\, x.$$

Note that Λ_0 is the restriction of Λ to X_0. So Λ_0 is injective, and hence $\Lambda_0^* \Lambda_0$ is invertible. Furthermore, $\Gamma_0 \varphi$ is equal to the component of $\Gamma \varphi$ in X_0 relative to the decomposition $X_1 \oplus X_0 \oplus X_2$. It follows that Γ_0 is surjective, and hence $\Gamma_0 \Gamma_0^*$ is invertible. But then we can apply Theorem 1.1 to show that the representation $k(t, s) = F_0(t)\, G_0(s)$ is irreducible.

1.2. THEOREM. *Two irreducible separable representations of the same kernel are similar.*

Proof. Let $k(t, s) = F_1(t)\, G_1(s)$ and $k(t, s) = F_2(t)\, G_2(s)$ be irreducible separable representations of k on $\Omega \times \Sigma$. For $\nu = 1, 2$ assume that $F_\nu(t)$: $X_\nu \to Y$ and $G_\nu(s)$: $Z \to X_\nu$ and consider the auxiliary operators

$$\Gamma_\nu:\quad L_2(\Sigma; Z) \to X_\nu, \qquad \Gamma_\nu \varphi = \int_\Sigma G_\nu(s)\, \varphi(s)\, ds,$$

$$\Lambda_\nu:\quad X_\nu \to L_2(\Omega; Y), \qquad (\Lambda_\nu x)(t) = F_\nu(t)\, x.$$

Because of irreducibility we know that Λ_1 and Λ_2 are injective and the operators Γ_1 and Γ_2 are surjective. Note that $\Lambda_1 \Gamma_1 = \Lambda_2 \Gamma_2$, and so $\operatorname{Im} \Lambda_1 = \operatorname{Im} \Lambda_1 \Gamma_1 = \operatorname{Im} \Lambda_2 \Gamma_2 = \operatorname{Im} \Lambda_2$. Since X_2 is finite dimensional, the operator Λ_2 has a left inverse, Λ_2^+ say. Put $S = \Lambda_2^+ \Lambda_1$. Then S: $X_1 \to X_2$ is invertible and

$$\Lambda_2 S = \Lambda_2 \Lambda_2^+ \Lambda_1 = \Lambda_1, \qquad S\Gamma_1 = \Lambda_2^+ \Lambda_1 \Gamma_1 = \Lambda_2^+ \Lambda_2 \Gamma_2 = \Gamma_2.$$

From $\Lambda_2 S = \Lambda_1$ it follows that $F_2(t)\, S = F_1(t)$ almost everywhere for $t \in \Omega$ and $S\Gamma_1 = \Gamma_2$ implies that $SG_1(s) = G_2(s)$ almost everywhere for $s \in \Sigma$. So we have proved (1.3) and the two representations are similar. ▌

Since any separable representation of k is a dilation of an irreducible one, Theorem 1.2 yields the following classification theorem.

1.3. THEOREM. *Any two separable representations of a finite rank kernel k are dilations of similar irreducible separable representations of k.*

The separable representation (1.1) of k is said to be *analytic* if in (1.1) the operators $G(s)$ and $F(t)$ depend analytically on the variables s and t, respectively. Analyticity of a separable representation is preserved under similarity, and if a dilation of a separable representation is analytic, then the original representation is analytic. From the latter statement it follows that a kernel with an analytic separable representation also has an irreducible separable representation which is analytic.

2. MINIMAL FINITE RANK EXTENSIONS

Let $k(t, s)$: $Z \to Y$ be an operator valued kernel on the square $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$. The *lower triangular part* of the kernel k will be denoted by k_ℓ and the *upper triangular part* by k_u. Thus, by definition,

$$k_\ell(t, s) = k(t, s), \qquad a \leqslant s < t \leqslant b,$$

$$k_u(t, s) = k(t, s), \qquad a \leqslant t < s \leqslant b.$$

We say that k_ℓ (resp. k_u) admits a *finite rank extension* if there exists a finite rank kernel $k(t, s)$: $Z \to Y$, $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$, such that $h_\ell = k_\ell$ (resp. $h_u = k_u$).

If the lower triangular part of k has a finite rank extension, then it has infinitely many different ones. In fact, if h is a finite rank extension of k_ℓ, then the kernel

$$h^\gamma(t, s) = h(t, s) + \chi_{[a,\gamma]}(t)\, \chi_{(\gamma,b]}(s)\, K, \tag{2.1}$$

where $a < \gamma < b$ and K: $Z \to Y$ is an arbitrary nonzero linear operator, is again a finite rank extension of k_ℓ and, obviously, h^γ differs for different γ's. (In (2.1) the symbol χ_E stands for the characteristic function of the set E.)

A *minimal rank extension* of k_ℓ is a finite rank extension h of k_ℓ with the extra property that among all finite rank extensions of k_ℓ the rank of h is as small as possible. By replacing in the previous sentence the index ℓ by u one obtains the definition of a minimal rank extension of k_u. We say that the (lower triangular part of the) kernel k is *lower unique* if k_ℓ has precisely one minimal rank extension. Similarly, the (upper triangular part of the) kernel k will be called *upper unique* if k_u has precisely one minimal rank extension.

It may happen that k is upper unique, but not lower unique (and conversely). To illustrate this, consider the (scalar) finite rank kernel $h(t, s) = f(t)\, g(s)$ on $[0, 1] \times [0, 1]$, where

$$f(t) = t - \tfrac{1}{2}, \qquad g(t) = 1, \qquad 0 \leqslant t < \tfrac{1}{2}, \tag{2.2}$$

$$f(t) = 0, \qquad g(t) = t + \tfrac{1}{2}, \qquad \tfrac{1}{2} \leqslant t \leqslant 1. \tag{2.3}$$

The kernel h is upper unique, but not lower unique. To see this, note that trivially, h_ℓ can be extended to a finite rank kernel and the minimal rank of such an extension is equal to one. Put $h_o(t, s) = f(t)$ for $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant 1$. Then $(h_0)_\ell = h_\ell$, and thus h_0 and h are two different minimal rank extensions of h_ℓ, which shows that h is not lower unique. The upper uniqueness of h one can check directly or obtain as a consequence of Theorem 3.3 in the next section.

3. Criteria for Lower and Upper Uniqueness

In this section $k(t, s)$: $Z \to Y$ is an operator-valued kernel on $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$. Let Ω and Σ be measurable subsets of $[a, b]$ of positive measure. Define the *restriction* of the kernel k to $\Omega \times \Sigma$ to be the kernel

$$k_{\Omega \times \Sigma}(t, s) = k(t, s), \quad (t, s) \in \Omega \times \Sigma.$$

If the kernel k has finite rank, then the same is true for the restriction and, obviously, $\operatorname{rank}(k) \geqslant \operatorname{rank}(k_{\Omega \times \Sigma})$.

Now, let h be a finite rank extension of k_{ℓ}, and choose Ω and Σ such that $\Omega \times \Sigma$ is a subset of $a \leqslant s < t \leqslant b$. Then $k_{\Omega \times \Sigma} = h_{\Omega \times \Sigma}$. Thus $k_{\Omega \times \Sigma}$ is a kernel of finite rank and

$$\operatorname{rank}(k_{\Omega \times \Sigma}) = \operatorname{rank}(h_{\Omega \times \Sigma}) \leqslant \operatorname{rank}(h). \tag{3.1}$$

This leads to the following proposition.

3.1. Proposition. *Let h be a finite rank extension of k_{ℓ}, and assume that for some $\Omega \times \Sigma$ in $a \leqslant s < t \leqslant b$*

$$\operatorname{rank}(h) = \operatorname{rank}(k_{\Omega \times \Sigma}). \tag{3.2}$$

Then h is a minimal rank extension of k_{ℓ}.

Proof. For any finite rank extension h of k_{ℓ} we have $\operatorname{rank}(h) \geqslant \operatorname{rank}(k_{\Omega \times \Sigma})$ (see (3.1)). So, if, in addition, (3.2) holds, then h must be a minimal rank extension of k_{ℓ}. ∎

3.2. Theorem. *Let h be a finite rank extension of k_{ℓ}, and assume that for each $a < \gamma < b$,*

$$\operatorname{rank}(h) = \operatorname{rank}(k_{[\gamma, b] \times [a, \gamma]}). \tag{3.3}$$

Then k is lower unique and h is the unique minimal rank extension of k_{ℓ}.

Proof. From the previous proposition we know that h is a minimal rank extension of k_{ℓ}. Let h_1 be a second minimal rank extension of k_{ℓ}. We have to prove that $h = h_1$. Write

$$h(t, s) = F(t)\, G(s), \qquad a \leqslant t \leqslant b, \quad a \leqslant s \leqslant b, \tag{3.4}$$

$$h_1(t, s) = F_1(t) G_1(s), \qquad a \leqslant t \leqslant b, \quad a \leqslant s \leqslant b, \tag{3.5}$$

and assume that the representations (3.4) and (3.5) are irreducible (see Sect. 1). Let X and X_1 be the corresponding internal spaces. Since (3.4) and (3.5) are irreducible representations, we have $\operatorname{rank}(h) = \dim X$ and $\operatorname{rank}(h_1) = \dim X_1$ (cf. Theorem 1.1).

Let us write k_γ for the restriction of k to $[\gamma, b] \times [a, y]$. By taken restrictions in (3.4) and (3.5) one sees that for each $a < \gamma < b$

$$k_\gamma(t, s) = F(t)\, G(s), \qquad \gamma \leqslant t \leqslant b, \quad a \leqslant s \leqslant \gamma, \tag{3.6}$$

$$k_\gamma(t, s) = F_1(t)\, G_1(s), \qquad \gamma \leqslant t \leqslant b, \quad a \leqslant s \leqslant \gamma. \tag{3.7}$$

Formula (3.3) implies that $\operatorname{rank}(k_\gamma) = \operatorname{rank}(h) = \dim X$, and thus the order of the separable representation (3.6) is equal to $\operatorname{rank}(k_\gamma)$, which implies that (3.6) is an irreducible separable representation (cf. Theorem 1.1). Similarly, $\operatorname{rank}(k_\gamma) = \operatorname{rank}(h) = \operatorname{rank}(h_1) = \dim X_1$, and hence (3.7) is an irreducible representation of k_γ.

So (3.6) and (3.7) are irreducible separable representations of the same finite rank kernel. Hence there exists an invertible operator $S_\gamma: X \to X_1$ such that

$$G_1(s) = S_\gamma^{-1} G(s), \qquad F_1(t) = F(t)\, S_\gamma \tag{3.8}$$

almost everywhere on $a \leqslant s \leqslant \gamma$ and $\gamma \leqslant t \leqslant b$, respectively. We shall prove that S_γ does not depend on γ. Take $a < \gamma_1 < \gamma_2 < b$. Then

$$F(t)\, S_{\gamma_1} = F_1(t) = F(t)\, S_{\gamma_2}, \qquad \gamma_2 \leqslant t \leqslant b, \quad \text{a.e.}$$

It follows that

$$\left(\int_{\gamma_2}^{b} F^*(t)\, F(t)\, dt \right) S_{\gamma_1} = \left(\int_{\gamma_2}^{b} F^*(t)\, F(t)\, dt \right) S_{\gamma_2}. \tag{3.9}$$

Now use that for each $a < \gamma < b$ the representation (3.6) is irreducible. So (cf. Theorem 1.1) the integral in (3.9) is an invertible operator, and thus $S_{\gamma_1} = S_{\gamma_2}$. It follows that $S = S_\gamma$ is independent of γ, and we may conclude that $G_1(s) = S^{-1} G(s)$ and $F_1(t) = F(t)\, S$ almost everywhere on $a \leqslant s \leqslant b$ and $a \leqslant t \leqslant b$. But then $h = h_1$. ▮

The next theorem is the upper triangular version of Theorem 3.2; its proof is similar to that of Theorem 3.2.

3.3. THEOREM. *Let h be a finite rank extension of k_u, and assume that for each $a < \delta < b$,*

$$\operatorname{rank}(h) = \operatorname{rank}(h_{[a,\delta] \times [\delta,b]}).$$

Then k is upper unique and h is the unique minimal rank extension of k_u.

4. Minimal Lower Separable Representations and Lower Order

Let $k(t, s): Z \to Y$ be an operator-valued kernel on $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$. A pair $\{F, G\}$ is called a *lower separable representation* of k if

$$k(t, s) = F(t)\, G(s), \qquad a \leqslant s < t \leqslant b, \quad \text{a.e.} \tag{4.1}$$

Here $F(t): X \to Y$ and $G(s): Z \to X$ are linear operators, the space X is a finite dimensional inner product space, which will be called the *internal space* of the representation, and as functions F and G are square integrable on $[a, b]$. If (4.1) holds, then we say that k is *lower separable.* Of course, k is lower separable if and only if its lower triangular part has a finite rank extension.

The dimension of the internal space X will be called the *order* of the representation $\{F, G\}$. We say that the pair $\{F, G\}$ is a *minimal* lower separable representation of k if (4.1) holds and among all lower separable representations of k the order of the pair $\{F, G\}$ is as small as possible. We define the *lower order* of k (notation: $\mathrm{l}(k)$) to be the order of a minimal lower separable representation of k.

Using the similarity and dilation theory of Section 1 the following proposition is easy to prove. We omit the details.

4.1. PROPOSITION. *A lower separable representation $\{F, G\}$ of k is minimal if and only if the finite rank kernel $h(t, s) = F(t)\, G(s)$ on $[a, b] \times [a, b]$ is a minimal rank extension of k_ℓ and the following operators are invertible*:

$$\int_a^b G(s)\, G(s)^*\, ds, \qquad \int_a^b F(t)^*\, F(t)\, dt. \tag{4.2}$$

From Proposition 4.1 it is clear that the lower order of a lower separable kernel k is equal to the rank of a minimal rank extension of k_ℓ. We also have

4.2. COROLLARY. *The lower separable representation $\{F, G\}$ of k is minimal if for some $a < \gamma < b$ the following operators are invertible*:

$$\int_a^\gamma G(s)\, G(s)^*\, ds, \qquad \int_\gamma^b F(t)^*\, F(t)\, dt. \tag{4.3}$$

Proof. Let X be the internal space of the pair $\{F, G\}$. Put $h(t, s) = F(t)\, G(s)$, $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$. The invertibility of the operators (4.3) implies that the operators (4.2) are also invertible. So, by Theorem 1.1, $\dim X = \operatorname{rank}(h)$. Next, let k_γ be the restriction of k to the rectangle $[\gamma, b] \times [a, \gamma]$. According to Theorem 1.1 the invertibility of the operators (4.3) implies that rank $(k_\gamma) = \dim X$. So $\operatorname{rank}(h) = \operatorname{rank}(k_\gamma)$, and we can apply Proposition 3.1 to show that h is a minimal rank extension of k_γ. We know already that the operators (4.2) are invertible. So Proposition 4.1 shows that $\{F, G\}$ is a minimal lower separable representation of k. ∎

Upper separable representations and the *upper order* of k can be defined in a similar way. To get the analogue of Corollary 4.2 for upper separable representations one has to replace the operators (4.3) by

$$\int_\gamma^b G(s)\,G(s)^*\,ds, \qquad \int_a^\gamma F(t)^*\,F(t)\,dt.$$

Two lower separable representations $\{F_1, G_1\}$ and $\{F_2, G_2\}$ of k are said to be *similar* if there exists an invertible operator $S\colon X_1 \to X_2$ (where X_1 and X_2 are the corresponding internal spaces) such that

$$F_2(t) = F_1(t)\,S, \qquad G_2(t) = S^{-1}G_1(t), \quad a \leqslant t \leqslant b, \quad \text{a.e.}$$

Obviously, similar lower separable representations have the same order.

In general, two minimal lower separable representations of the same kernel k do not have to be similar. To see this let $h(t, s) = f(t)\,g(s)$ be the finite rank kernel on $[0, 1] \times [0, 1]$ considered at the end of Section 2 with f and g as in (2.2) and (2.3). Then $\{f, g\}$ and $\{f, 1\}$ are minimal lower separable representations of h, which, trivially, are not similar. The point here is that the lower triangular part of h has different minimal rank extensions. In fact, from Proposition 4.1 and Theorem 1.1 it is clear that any two minimal lower separable representations of k are similar if and only if k is lower unique. This leads to

4.3. COROLLARY. *Let $\{F, G\}$ be a lower separable representation of k and assume that for each γ $(a < \gamma < b)$ the following operators are invertible*:

$$\int_a^\gamma G(s)\,G(s)^*\,ds, \qquad \int_\gamma^b F(t)^*\,F(t)\,dt. \tag{4.4}$$

Then k is lower unique and up to similarity $\{F, G\}$ is the unique minimal lower separable representation of k.

Proof. From Corollary 4.2 we know that $\{F, G\}$ is a minimal lower separable representation of k. So it suffices to show that k is lower unique. Put $h(t, s) = F(t)\,G(s)$ for $a \leqslant t \leqslant b$ and $a \leqslant s \leqslant b$. Then h is a finite rank extension of k_ℓ and the invertibility of the operators (4.4) imply that

$$\operatorname{rank}(k_{[\gamma,b] \times [a,\gamma]}) = \dim X = \operatorname{rank}(h),$$

where X is the internal space of $\{F, G\}$. But then we can apply Theorem 3.2 to show that k is lower unique. ∎

We conclude this section with an example showing that the invertibility of the operators (4.3) is only a sufficient condition for the minimality of $\{F, G\}$ and not a necessary condition. Put

$$F(t)=G(t)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad 0\leqslant t\leqslant \tfrac{1}{2},$$

$$F(t)=G(t)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \qquad \tfrac{1}{2}< t\leqslant 1,$$

and consider the lower separable kernel k given by

$$k(t, s)=\begin{cases}F(t)\, G(s), & 0\leqslant s<t\leqslant 1,\\ 0, & 0\leqslant t<s\leqslant 1.\end{cases} \tag{4.5}$$

Obviously, $\mathrm{l}(k)\leqslant 2$. We shall prove that the lower order $\mathrm{l}(k)=2$, which implies that $\{F, G\}$ is a minimal lower separable representation of k.

Assume that $\mathrm{l}(k)=1$. Then there exist scalar functions f_1, f_2, g_1, and g_2 such that

$$F(t)\, G(s)=\begin{pmatrix}f_1(t)\\ f_2(t)\end{pmatrix}[g_1(s)\, g_2(s)],$$

almost everywhere on $0\leqslant s\leqslant t\leqslant 1$. In particular

$$f_1(t)\, g_1(s)=1, \qquad 0\leqslant s<t\leqslant \tfrac{1}{2}, \quad \text{a.e.},$$

$$f_2(t)\, g_1(s)=0, \qquad 0\leqslant s<t\leqslant 1, \quad \text{a.e.},$$

$$f_2(t)\, g_2(s)=1, \qquad \tfrac{1}{2}<s\leqslant t\leqslant 1, \quad \text{a.e.},$$

which is impossible. So $\mathrm{l}(k)\neq 1$, and hence $\mathrm{l}(k)=2$.

Next, one computes

$$\operatorname{rank}\left(\int_\gamma^1 F(t)^*\, F(t)\, dt\right)=\begin{cases}2 & \text{for } 0\leqslant\gamma<\tfrac{1}{2},\\ 1 & \text{for } \tfrac{1}{2}\leqslant\gamma<1,\end{cases}$$

$$\operatorname{rank}\left(\int_0^\gamma G(s)\, G(s)^*\, ds\right)=\begin{cases}1 & \text{for } 0<\gamma\leqslant\tfrac{1}{2},\\ 2 & \text{for } \tfrac{1}{2}<\gamma\leqslant 1.\end{cases}$$

So there is no γ, $0<\gamma<1$, for which the two operators

$$\int_\gamma^1 F(t)^*\, F(t)\, dt, \qquad \int_0^\gamma G(s)\, G(s)^*\, ds$$

are invertible. Nevertheless $\{F, G\}$ is a minimal lower separable representation of the kernel (4.5).

5. Uniform Lower Order and Uniform Minimality

Let $k(t, s)\colon Z \to Y$ be an operator-valued kernel on the square $a \leqslant t \leqslant b$ and $a \leqslant s \leqslant b$. For $[\alpha, \beta] \subset [a, b]$ we define $k_{\alpha\beta}$ to be the restriction of k to the square $\alpha \leqslant t \leqslant \beta$, $\alpha \leqslant s \leqslant \beta$. Thus in the notation of Section 3 we have $k_{\alpha\beta} = k_{[\alpha,\beta] \times [\alpha,\beta]}$. If the kernel k is lower separable, then the same is true for $k_{\alpha\beta}$. The corresponding lower orders are related as follows:

$$\mathrm{l}(k_{\alpha\beta}) \leqslant \mathrm{l}(k), \qquad a \leqslant \alpha < \beta \leqslant b. \tag{5.1}$$

We say that k has a *uniform lower order* if in (5.1) equality holds for each $[\alpha, \beta] \subset [a, b]$.

Not every lower separable kernel has a uniform lower order. For example, the finite rank kernel

$$h(t, s) = \chi_{(1/2, 1]}(t)\, \chi_{[0, 1/2]}(s), \qquad 0 \leqslant t \leqslant 1, \quad 0 \leqslant s \leqslant 1$$

has lower degree 1, but $\mathrm{l}(h_{0, 1/2}) = 0$, and thus h does not have a uniform lower order. On the other hand we shall see that k has a uniform lower order whenever k has a lower separable representation $\{F, G\}$ of which the components F and G are analytic functions.

Let $\{F, G\}$ be a lower separable representation of k. Denote by $F_{\alpha\beta}$ and $G_{\alpha\beta}$ the restrictions of F and G, respectively, to the interval $[\alpha, \beta]$. We say that $\{F, G\}$ is *uniformly minimal* if for each $[\alpha, \beta] \subset [a, b]$ the pair $\{F_{\alpha\beta}, G_{\alpha\beta}\}$ is a minimal lower separable representation of $k_{\alpha\beta}$. From the definitions it is clear that a lower separable kernel k has a uniformly minimal lower separable representation if and only if k has a uniform lower order, and in that case any minimal lower separable representation of k is uniformly minimal.

5.1. Theorem. *A lower separable representation $\{F, G\}$ of k is uniformly minimal if and only if for each $[\alpha, \beta] \subset [a, b]$ the following operators are invertible*:

$$\int_\alpha^\beta G(t)\, G(t)^*\, dt, \qquad \int_\alpha^\beta F(t)^*\, F(t)\, dt. \tag{5.2}$$

Proof. Assume $\{F, G\}$ is uniformly minimal. Then $\{F_{\alpha\beta}, G_{\alpha\beta}\}$ is a minimal lower separable representation of $k_{\alpha\beta}$. So by Proposition 4.1 the operators (5.2) are invertible.

For the converse, assume that for each $[\alpha, \beta] \subset [a, b]$ the operators (5.2) are invertible. Take $\alpha < \gamma < \beta$. Then the operators

$$\int_\alpha^\gamma G(t)\, G(t)^*\, dt, \qquad \int_\gamma^\beta F(t)^*\, F(t)\, dt \tag{5.3}$$

are invertible. So we can apply Corollary 4.3 to show that $\{F_{\alpha\beta}, G_{\alpha\beta}\}$ is a minimal lower separable representation of $k_{\alpha\beta}$. Since $[\alpha, \beta] \subset [a, b]$ is arbitrary, it follows that $\{F, G\}$ is a uniformly minimal separable representation. ▮

5.2. COROLLARY. *Two uniformly minimal lower separable representations of the same kernel are similar. Equivalently, a kernel of uniform lower order is lower unique.*

Proof. Assume k has uniform lower order. Then k has a uniformly minimal lower separable representation, $\{F, G\}$ say. It follows that for F and G the operators (5.2) are invertible for each $[\alpha, \beta] \subset [a, b]$. In particular, for each $a < \gamma < b$ the operators (5.3) are invertible. But then we can apply Corollary 4.3 to show that k is lower unique. Further we see that up to similarity $\{F, G\}$ is the unique (uniformly) minimal lower separable representation of k. ▮

5.3. THEOREM. *If k has a lower separable representation $\{F, G\}$ such that F and G are analytic functions, then k has uniform lower order.*

Proof. Let $\{F, G\}$ be a lower separable representation of k, and assume that F and G are analytic on $[a, b]$. Put $h(t, s) = F(t)\, G(s)$, $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$. Then $F(t)\, G(s)$ is an analytic separable representation of the finite rank kernel h. We know that $F(t)\, G(s)$ is a dilation of an irreducible separable representation $F_0(t)\, G_0(s)$ (see Sect. 1). From the definition of a dilation it is clear that the operators $F_0(t)$ and $G_0(s)$ depend analytically on the variables t and s, respectively. Since h and k have the same lower triangular part, the pair $\{F_0, G_0\}$ is a lower separable representation of k. So, to prove the theorem, we may without loss of generality assume (see Theorem 1.1) that the operators

$$\int_a^b G(t)\, G(t)^*\, dt, \qquad \int_a^b F(t)^*\, F(t)\, dt \tag{5.4}$$

are invertible. Because of the analyticity of $F(\cdot)$ and $G(\cdot)$ the spaces

$$\operatorname{Im}\left(\int_\alpha^\beta G(t)\, G(t)^*\, dt\right), \qquad \operatorname{Ker}\left(\int_\alpha^\beta F(t)^*\, F(t)\, dt\right)$$

are independent of the particular choice of α, β as long as $a \leqslant \alpha < \beta \leqslant b$. Since the operators in (5.4) are invertible, we conclude that for each $a \leqslant \alpha < \beta \leqslant b$ the operators (5.2) are invertible. Now apply Theorem 5.1 to finish the proof. ▮

Results analogous to the ones proved in this section also hold for upper separable kernels and the upper order. We omit the details.

6. Two Open Questions

Let $k(t, s)\colon Z \to Y$, $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$, be an operator-valued kernel, and let $\{F, G\}$ be an arbitrary lower separable representation of k. The first question we want to pose asks for a constructive procedure to get from $\{F, G\}$ a minimal lower separable representation of k. The first step of such a procedure is clear and consists of the reduction described in the proof of Theorem 1.1. Let X be the internal space of the representation $\{F, G\}$. Introduce the following auxiliary operators:

$$\Gamma\colon\ L_2([a, b], Z) \to X, \qquad \Gamma\varphi = \int_a^b G(s)\,\varphi(s)\,ds,$$

$$\Lambda\colon\ X \to L_2([a, b], Y), \qquad (\Lambda x)(t) = F(t)\,x.$$

Put $X_1 = \operatorname{Ker} \Lambda$, define X_0 to be a direct complement of $\operatorname{Ker} \Lambda \cap \operatorname{Im} \Gamma$ in $\operatorname{Im} \Gamma$, and let X_2 be a direct complement of $X_1 + X_0$ in X. Then $X = X_1 \oplus X_0 \oplus X_2$ and relative to this decomposition $G(\cdot)$ and $F(\cdot)$ admit the following partitioning:

$$G(\cdot) = \begin{bmatrix} * \\ G_0(\cdot) \\ 0 \end{bmatrix}, \qquad F(\cdot) = [0 \quad F_0(\cdot) \quad *].$$

Thus $\{F_0, G_0\}$ is a lower separable representation of k and in general its order is less than the order of $\{F, G\}$. If F and G are analytic functions, then we are finished, because in that case the proof of Theorem 5.3 and the results of Section 1 imply that $\{F_0, G_0\}$ is a minimal lower separable representation. So for nonanalytic representations the question is what other steps does one have to take in order to get a minimal lower separable representation.

The second question is to analyse the set $\mathscr{E}(k)$ of all minimal rank extensions of the lower triangular part of k. It is clear that the set $\mathscr{E}(k)$ can be a singleton (the case when k is lower unique) or an infinite set. To see the latter, let f and g be as in (2.2) and (2.3), and for $0 < \gamma < 1$ put $g_\gamma(t) = \gamma + (1 - \gamma)\,g(t)$. Then for each $0 < \gamma < 1$ the kernel $h_\gamma(t, s) = f(t)\,g_\gamma(s)$ is a minimal rank extension of the lower triangular part of the kernel $k(t, s) = f(t)$, $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant 1$. Can it happen that the set $\mathscr{E}(k)$ is a finite set, but not a singleton?

7. Minimality for Systems with Separable Boundary Conditions

This section concerns time varying linear systems with separable boundary conditions (SB-systems) and their input–output operators. An SB-system has the following state space representation:

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t), & a \leqslant t \leqslant b, \\ y(t) &= C(t)\,x(t) + D(t)\,u(t), & a \leqslant t \leqslant b, \\ (I-P)\,x(a) &= 0, & PU(b)^{-1}\,x(b) = 0. \end{aligned} \tag{7.1}$$

The coefficients $A(t)\colon X \to X$, $B(t)\colon Z \to X$, $C(t)\colon X \to Y$, and $D(t)\colon Z \to Y$ are linear operators acting between finite dimensional inner product spaces, which (as functions of t) satisfy the following integrability conditions: $A(\cdot)$ is integrable, $B(\cdot)$ and $C(\cdot)$ are square integrable and $D(\cdot)$ is measurable and essentially bounded on the interval $[a, b]$. The symbol $U(\cdot)$ denotes the *fundamental operator* of the system, which by definition is the unique absolutely continuous solution of the operator differential equation:

$$\dot{U}(t) = A(t)\,U(t), \qquad a \leqslant t \leqslant b, \quad U(a) = I_X. \tag{7.2}$$

The operator P appearing in the boundary conditions acts on the *state space* X, and it is important here that P is assumed to be a projection. (The latter property is expressed by the word "separable" in the description of the boundary conditions.) If $P = 0$, then (7.1) is a classical causal system; for $P = I$ we have an anti-causal system. Various aspects of systems with boundary conditions (separable or nonseparable) have been studied in [1 – 3, 6–8, 11–13].

In what follows $\theta = (A(t), B(t), C(t), D(t); P)_a^b$ stands for the SB-system (7.1). The input-output operator $T_\theta\colon L_2([a, b], Z) \to L_2([a, b], Y)$ of θ is the integral operator

$$(T_\theta \varphi)(t) = D(t)\,\varphi(t) + \int_a^b k(t, s)\,\varphi(s)\,ds, \qquad a \leqslant t \leqslant b,$$

of which the kernel k is given by

$$k(t, s) = \begin{cases} C(t)\,U(t)(I-P)\,U(s)^{-1}\,B(s), & a \leqslant s < t \leqslant b, \\ -C(t)\,U(t)\,PU(s)^{-1}\,B(s), & a \leqslant t < s \leqslant b. \end{cases} \tag{7.3}$$

The kernel k of the input-output operator is a *semiseparable* kernel, which means that k is both lower and upper separable (cf. [6, Sect. I.4]). Any integral operator with a semiseparable kernel is the input–output operator of an SB-system [6, Theorem I.4.1].

If $T = T_\theta$, then θ is called a *realization* of T. In this section we deal with the problem of minimal realization within the class of SB-systems. We say that θ is SB-*minimal* if θ is an SB-system and among all SB-systems with the same input–output operator as θ the dimension of the state space of θ is as small as possible.

To describe the SB-minimal systems we shall use the *order* of T_θ (notation: $\operatorname{ord}(T_\theta)$), which we define to be the sum of the lower order and the upper order of the kernel of T_θ. In other words $\operatorname{ord}(T_\theta) = \mathrm{l}(k) + \mathrm{u}(k)$, where k is the lower and upper separable kernel (7.3).

7.1. THEOREM. *Let θ be an* SB-*system, and let X be its state space. Then* $\operatorname{ord}(T_\theta) \leqslant \dim X$, *and θ is* SB-*minimal if and only if* $\operatorname{ord}(T_\theta) = \dim X$.

Proof. Let $\theta = (A(t), B(t), C(t), D(t); P)_a^b$, and let k be the kernel of its input–output operator. Furthermore, let $U(t)$ be the fundamental operator of θ. Write $U(t)^{-1}B(t)$ and $C(t)\,U(t)$ as block operator matrices relative to the decomposition $X = \operatorname{Ker} P \oplus \operatorname{Im} P$,

$$U(t)^{-1}B(t) = \begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix}, \qquad C(t)\,U(t) = [F_1(t) \quad -F_2(t)]. \tag{7.4}$$

Then

$$k(t, s) = \begin{cases} F_1(t)\,G_1(s), & a \leqslant s < t \leqslant b, \\ F_2(t)\,G_2(s), & a \leqslant t < s \leqslant b. \end{cases}$$

It follows that $\mathrm{l}(k) \leqslant \dim \operatorname{Ker} P$ and $\mathrm{u}(k) \leqslant \dim \operatorname{Im} P$, and thus $\operatorname{ord}(T_\theta) \leqslant \dim X$.

From the latter inequality we conclude that θ is SB-minimal whenever $\operatorname{ord}(T_\theta) = \dim X$. To prove the reverse implication, note that θ is SB-minimal if and only if $\{F_1, G_1\}$ is a minimal lower separable representation of k and $\{F_2, G_2\}$ is a minimal upper separable representation of k. Thus if θ is SB-minimal, then $\mathrm{l}(k) = \dim \operatorname{Ker} P$ and $\mathrm{u}(k) = \dim \operatorname{Im} P$, and hence $\operatorname{ord}(T_\theta) = \dim X$. ∎

SB-minimality may also be described in terms of the controllability and observability Gramians $\mathscr{C}(\theta)$ and $\mathscr{O}(\theta)$. Recall (see, e.g., [7, 11]) that

$$\mathscr{C}(\theta) = \int_a^b U(t)^{-1}\,B(t)\,B(t)^*(U(t)^{-1})^*\,dt,$$

$$\mathscr{O}(\theta) = \int_a^b U(t)^*\,C(t)^*\,C(t)\,U(t)\,dt.$$

Here $\theta = (A(t), B(t), C(t), D(t); P)$ and $U(t)$ is the fundamental operator of θ. To describe SB-minimality in terms of $\mathscr{C}(\theta)$ and $\mathcal{O}(\theta)$ we need the following finite rank integral operators:

$$(L_\theta \varphi)(t) = C(t)\, U(t)(I - P) \int_a^b U(s)^{-1}\, B(s)\, \varphi(s)\, ds,$$

$$M_\theta \varphi)(t) = -C(t)\, U(t)\, P \int_a^b U(s)^{-1}\, B(s)\, \varphi(s)\, ds.$$

Both L_θ and M_θ act from $L_2([a, b], Z)$ into $L_2([a, b], Y)$. Since the kernel of L_θ is a finite rank extension of k_ℓ and the kernel M_θ is a finite rank extension of k_u (where k is the kernel (7.3)) it is clear that

$$\operatorname{ord}(T_\theta) \leqslant \operatorname{rank} L_\theta + \operatorname{rank} M_\theta.$$

7.2. THEOREM. *The* SB-*system* θ *is* SB-*minimal if and only if the following two conditions hold true*:

(i) $\operatorname{Im}[\mathscr{C}(\theta)\, P\mathscr{C}(\theta)] = X$, $\operatorname{Ker}\begin{pmatrix}\mathcal{O}(\theta)\\ \mathcal{O}(\theta)P\end{pmatrix} = (0)$;

(ii) $\operatorname{ord}(T_\theta) = \operatorname{rank} L_\theta + \operatorname{rank} M_\theta$.

Here X *is the state space of* θ *and* P *is the projection appearing in the boundary conditions.*

Proof. We use the notation introduced in the first paragraph of the proof of Theorem 7.1. In particular, we assume that $U(t)^{-1} B(t)$ and $C(t)\, U(t)$ are partitioned as in (7.4). We already know that θ is SB-minimal if and only if the pair $\{F_1, G_1\}$ is a minimal lower separable representation of k and $\{F_2, G_2\}$ is a minimal upper separable representation of k, where, as before, k is the kernel of the input–output operator of θ. Next, note that $h_1(t, s) = F_1(t)\, G_1(s)$ is the kernel of L_θ and $h_2(t, s) = F_2(t)\, G_2(s)$ is the kernel of M_θ. It follows that condition (ii) is equivalent to the requirement that h_1 is a minimal rank extension of k_ℓ and h_2 is a minimal rank extension of k_u. On the other hand, condition (i) is equivalent to the statement that $h_1(t, s) = F_1(t)\, G_1(s)$ and $h_2(t, s) = F_2(t)\, G_2(s)$ are irreducible separable representations. These three observations, together with Proposition 4.1 (and its analogue for upper separability), prove the theorem. ∎

By similarity an SB-system can be transformed into an SB-system of which the main coefficient is identically zero and an SB-system with the latter property is a parallel connection of a causal and an anti-causal system. In this way the minimality theorems for SB-systems can be derived from the minimality theorems for causal and anti-causal systems. This method of reduction is used implicitly in the proofs of Theorems 7.1 and 7.2. Note, however, that also for the causal case Theorem 7.2 seems to be new.

In the terminology of [7], condition (i) in Theorem 7.2 means that θ is 2-controllable and 2-observable.

The SB-system $\theta=(A(t), B(t), C(t), D(t); P)_a^b$ will be called *uniformly* SB-*minimal* if for each subinterval $[\alpha, \beta] \subset [a, b]$ the system

$$\theta_{\alpha\beta}=(A(t), B(t), C(t), D(t); U(\alpha)\, PU(\alpha)^{-1})_\alpha^\beta \tag{7.6}$$

is SB-minimal. Note that $U(\alpha)\, PU(\alpha)^{-1}$ is a projection. Hence $\theta_{\alpha\beta}$ is an SB-system. A simple computation shows that the kernel $k_{\alpha\beta}$ of the input–output operator of $\theta_{\alpha\beta}$ is precisely the restriction of the kernel k of the input–output operator of θ to the square $[\alpha, \beta] \times [\alpha, \beta]$. This allows us to use the results of Section 5 to prove

7.3. THEOREM. *The* SB-*system* $\theta=(A(t), B(t), C(t), D(t); P)_a^b$ *is uniformly* SB-*minimal if and only if for each* $a \leqslant \alpha < \beta \leqslant b$ *the following rank conditions are satisfied*:

$$\operatorname{rank}\left(\int_\alpha^\beta (I-P)\, U(t)^{-1}\, B(t)\, B(t)^*(U(t)^{-1})^*(I-P^*)\, dt\right)$$

$$= \operatorname{rank}\left(\int_\alpha^\beta (I-P^*)\, U(t)^*\, C(t)^*\, C(t)\, U(t)(I-P)\, dt\right) = \operatorname{rank}(I-P),$$

$$\operatorname{rank}\left(\int_\alpha^\beta PU(t)^{-1}\, B(t)\, B(t)^*(U(t)^{-1})^*\, P^*\, dt\right)$$

$$= \operatorname{rank}\left(\int_\alpha^\beta P^* U(t)^*\, C(t)^*\, C(t)\, U(t)\, P\, dt\right) = \operatorname{rank} P.$$

Here $U(t)$ *is the fundamental operator of* θ.

Proof. Let k be the kernel of the input–output operator of θ. As in (7.4) write $U(t)^{-1}\, B(t)$ and $C(t)\, U(t)$ as block operator matrices relative to the decomposition $X = \operatorname{Ker} P \oplus \operatorname{Im} P$. Then the system θ is uniformly SB-minimal if and only if the pair $\{F_1, G_1\}$ is a uniformly minimal lower separable representation of k and $\{F_2, G_2\}$ is a uniformly minimal upper separable representation of k. But then one can apply Theorem 5.1 (and its analogue for upper separable representations) to get the desired rank conditions. ∎

In the case when the system θ is causal (i.e., $P=0$) the rank conditions of Theorem 7.3 mean that the system θ is controllable and observable on each subinterval of $[a, b]$, which is just the classical criterium for uniform minimality (see [5]).

7.4. COROLLARY. *Let $\theta = (A(t), B(t), C(t), D(t); P)_a^b$ be an* SB-*system of which the coefficients $A(t)$, $B(t)$, and $C(t)$ are analytic on $a \leqslant t \leqslant b$ and let X be the state space of θ. Then θ is* SB-*minimal if and only if*

$$\operatorname{Im}[\mathscr{C}(\theta)\, P\mathscr{C}(\theta)] = X, \qquad \operatorname{Ker}\begin{pmatrix} \mathcal{O}(\theta) \\ \mathcal{O}(\theta)\, P \end{pmatrix} = (0), \tag{7.7}$$

and in that case θ is uniformly SB-*minimal.*

Proof. If θ is SB-minimal, then, because of Theorem 7.2, the identities (7.7) hold true. So we have to prove the reverse implication. Since P is a projection, (7.7) is equivalent to the statement that the following rank conditions hold true:

$$\operatorname{rank}\left(\int_a^b (I-P)\, U(t)^{-1}\, B(t)\, B(t)^*(U(t)^{-1})^*(I-P^*)\, dt\right)$$

$$= \operatorname{rank}\left(\int_a^b (I-P^*)\, U(t)^*\, C(t)^*\, C(t)\, U(t)(I-P)\, dt\right) = \operatorname{rank}(I-P),$$

$$\operatorname{rank}\left(\int_a^b PU(t)^{-1}\, B(t)\, B(t)^*(U(t)^{-1})^*\, P^*\, dt\right)$$

$$= \operatorname{rank}\left(\int_a^b P^*U(t)^*\, C(t)^*\, C(t)\, U(t)\, P\, dt\right) = \operatorname{rank} P.$$

Here $U(t)$ is the fundamental operator of θ. The fact that $A(t)$ is analytic on $a \leqslant t \leqslant b$ implies that $U(t)$ is also analytic on $a \leqslant t \leqslant b$ (see [4, Sect. VI.1]). It follows that the integrands of the four integrals above depend analytically on the variable t. We conclude that the above rank conditions remain valid if $[a, b]$ is replaced by any subinterval $[\alpha, \beta] \subset [a, b]$. But then we can apply Theorem 7.3 to show that for analytic systems (7.7) implies that θ is uniformly SB-minimal. ∎

By using a system similarity and the operation of dilation (see [7, 8] for the definition of these notions) an SB-system with analytic coefficients can be reduced to an SB-minimal system with analytic coefficients. Furthermore, two SB-minimal systems with analytic coefficients which are realization of the same input–output operator are similar. To prove the latter statement one first observes that for an SB-system θ with analytic coefficients the input–output operator determines the sequence of weighting patterns (see [7, 8]) of θ and next one applies the similarity theorem for systems with boundary conditions [8, Theorem 2.2].

For SB-systems the questions of Section 6 are also relevant. In fact in terms of SB-systems the first question in Section 6 asks for a constructive procedure to get from an SB-system θ an SB-minimal system with the same input–output operator as θ. For SB-systems with analytic coefficients such a procedure is described in the previous paragraph, but for general SB-systems the question is open. SB-minimal realizations of the same input–output operator do not have to be similar, and for systems the second question in Section 6 asks to analyse the various nonsimilar SB-minimal realizations.

In general, it is important in which class of systems the problem of minimal realization is considered. For example, for an SB-system with analytic coefficients SB-minimality is the same as minimality in the class of SB-systems with analytic coefficients, but this statement does not remain true if one replaces analytic by time invariant. To see this, consider the system:

$$\Delta \begin{cases} \dot{x}_1 = u, \qquad \dot{x}_2 = x_1, \qquad \dot{x}_3 = 0, \quad 0 \leqslant t \leqslant 1, \\ y = x_3, \qquad 0 \leqslant t \leqslant 1, \\ x_1(0) = x_2(0) = 0, \qquad x_2(1) - x_3(1) = 0. \end{cases}$$

The system Δ is time invariant and has separable boundary conditions. In fact, the boundary conditions of Δ can be written in the form $(I-P)\,x(0) = 0$ and $PU(1)^{-1}x(1) = 0$ by setting

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \qquad U(t) = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The input–output operator of Δ is the integral operator

$$(T_\Delta \varphi)(t) = \int_0^1 (1-s)\,\varphi(s)\,ds, \qquad 0 \leqslant t \leqslant 1,$$

which is also realized by the following SB-system:

$$\Delta_0 \begin{cases} \dot{x}_1(t) = (1-t)\,u(t), \qquad \dot{x}_2(t) = (1-t)\,u(t), \quad 0 \leqslant t \leqslant 1, \\ y(t) = -x_1(t) + x_2(t), \qquad 0 \leqslant t \leqslant 1, \\ x_2(0) = 0, \qquad x_1(1) = 0. \end{cases}$$

Note, however, that the coefficients of Δ_0 are not time invariant, but depend analytically on t. It is not difficult to check that T_Δ cannot be realized by a time invariant SB-system of which the state space dimension is strictly less than three. Thus Δ is a minimal realization of T_Δ in the class of time invariant SB-systems, but it is not minimal in the class of

SB-systems with analytic coefficients. Thus minimality for time invariant SB-systems is not the same as minimality for analytic SB-systems.

The problem of minimal realization also changes if one allows for non-separable boundary conditions. To illustrate this consider the system

$$\Sigma \begin{cases} \dot{x}_1 = u, \qquad \dot{x}_2 = -u, \quad 0 \leqslant t \leqslant 1, \\ y = 2x_1 + x_2, \qquad 0 \leqslant t \leqslant 1, \\ x_1(0) = 0, \qquad x_2(1) = 0. \end{cases}$$

The system Σ has separable boundary conditions and one can apply Corollary 7.4 to show that Σ is SB-minimal. The input-output operator of Σ is also realized by the system:

$$\Sigma_0 \begin{cases} \dot{x} = u, \qquad 0 \leqslant t \leqslant 1, \\ y = x, \qquad 0 \leqslant t \leqslant 1, \\ 2x(0) - x(1) = 0. \end{cases}$$

Note that Σ_0 has no separable boundary conditions and the state space dimension of Σ_0 is strictly less than the dimension of the state space of Σ. Thus SB-minimality is not the same as minimality in the class of systems with arbitrary well-posed boundary conditions.

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