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Grothendieck rings of o-minimal expansions of ordered abelian groups

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Abstract

We will calculate completely the Grothendieck rings, in the sense of first order logic, of o-minimal expansions of ordered abelian groups by introducing the notion of the bounded Euler characteristic.

Keywords: Grothendieck rings; O-minimal structures; Bounded Euler characteristic

1. Introduction

The notion of the Grothendieck ring for a first-order structure was introduced by [1,2], independently. In [1], J. Krajíček and T. Scanlon clarified the relation between the triviality of this ring and the non-existence of non-trivial weak Euler characteristic maps. More precisely, they used weak Euler characteristics and Grothendieck rings to handle the following situations. For instance, for a finite model and when any one-to-one function is onto (PHP, pigeonhole principle), however, for an infinite model, this does not holds in general. In [2], J. Denef and F. Loeser showed that for T the theory of algebraically closed field containing a fixed field k, it coincides with the notion of the Grothendieck ring of algebraic varieties over k. They treated with the motivic integration which was introduced by M. Kontsevich.

For an arbitrary $\mathcal{L}$-structure $\mathcal{M}$, $K_0(\mathcal{M})$ and $K_0(M, \mathcal{L})$ denote the Grothendieck ring of the $\mathcal{L}$-structure $\mathcal{M}$.

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In [3–5], the Grothendieck rings of fields are calculated explicitly as follows:

(1) $K_0(R, \mathcal{L}_{\mathrm{or}}) = \mathbb{Z}$, where R is a real closed field and $\mathcal{L}_{\mathrm{or}}$ is the language $(<, +, -, \cdot, 0, 1)$.
(2) $K_0(\mathbb{Q}_p, \mathcal{L}_{\mathrm{ring}}) = 0$, where p is a prime number, $\mathbb{Q}_p$ is the p-adic number field and $\mathcal{L}_{\mathrm{ring}}$ is the language $(+, -, \cdot, 0, 1)$.
(3) $K_0(\mathbb{F}_p((t)), \mathcal{L}_{\mathrm{ring}}) = 0$, where p is a prime number and $\mathbb{F}_p((t))$ is the quotient field of the formal power series in the indeterminate t over the finite field $\mathbb{F}_p$.
(4) $K_0(F, \mathcal{L}_{\mathrm{ring}}) = 0$, where F denotes Laurent series fields $L((t_1))$, $L((t_1))((t_2))$, $L((t_1))((t_2))((t_3))$ and L is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q$. Here p is a prime number and q is a power of p.

In [1,2], it is shown that the Grothendieck ring $K_0(\mathbb{C}, \mathcal{L}_{\mathrm{ring}})$ is extremely big and complicated:

(5) There exists a ring embedding $\mathbb{Z}[X_j \mid j \in \mathfrak{c}] \hookrightarrow K_0(\mathbb{C}, \mathcal{L}_{\mathrm{ring}})$, where $\mathfrak{c}$ is the cardinality of continuum and X_j $(j \in \mathfrak{c})$ are indeterminates.

Although the Grothendieck rings of some structures have been calculated as above, many other Grothendieck rings are not known yet and the Grothendieck rings of o-minimal expansions of ordered abelian groups are known only a little. See [3] for the precise definition of an o-minimal structure.

In the present paper, we will calculate the Grothendieck rings of o-minimal expansions of ordered abelian groups completely, namely, we have the following theorem:

Theorem 1. *Let $\mathcal{G} = (G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group. Then $K_0(\mathcal{G})$ is isomorphic to either $\mathbb{Z}$ or the quotient ring $\mathbb{Z}[T]/(T^2 + T)$ as a ring, where $\mathbb{Z}[T]$ is a polynomial ring in an indeterminate T over $\mathbb{Z}$ and $(T^2 + T)$ is the ideal of $\mathbb{Z}[T]$ generated by $T^2 + T$.*

2. Grothendieck rings

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The notation $\mathrm{Def}^n(\mathcal{M})$ denotes the family of all definable subsets of M^n. We set $\mathrm{Def}(\mathcal{M}) := \bigcup_{n=0}^{\infty} \mathrm{Def}^n(\mathcal{M})$. Two definable sets $A, B \in \mathrm{Def}(\mathcal{M})$ are *definably isomorphic*, denoted by $A \cong B$, if there is a definable bijection $A \to B$.

Definition 2 *(Grothendieck ring).* The *Grothendieck group* of an $\mathcal{L}$-structure $\mathcal{M}$ is the abelian group $K_0(\mathcal{M})$ generated by symbols $[X]$, where $X \in \mathrm{Def}(\mathcal{M})$ with the relations $[X] = [Y]$ if X and Y are definably isomorphic, and $[U \cup V] = [U] + [V]$ where $U, V \in \mathrm{Def}^n(\mathcal{M})$ and $U \cap V = \emptyset$. The ring structure is defined by $[X][Y] = [X \times Y]$ where $X \times Y$ is the Cartesian product of definable sets. The ring $K_0(\mathcal{M})$ with this multiplication is called *Grothendieck ring* of the $\mathcal{L}$-structure $\mathcal{M}$.

Remark 3. By construction, the map $[\ ] : \mathrm{Def}(\mathcal{M}) \to K_0(\mathcal{M})$ satisfies the following universal mapping property.

Consider the map $\chi : \mathrm{Def}(\mathcal{M}) \to \mathbb{Z}$ with

(1) $\chi(U \cup V) = \chi(U) + \chi(V)$ for $U, V \in \mathrm{Def}^n(\mathcal{M})$ with $U \cap V = \emptyset$,

(2) $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ for $X, Y \in \mathrm{Def}(\mathcal{M})$,
(3) $\chi(Z) = \chi(Z')$ if $Z, Z' \in \mathrm{Def}(\mathcal{M})$, $Z \cong Z'$.

Then, there exists an unique ring homomorphism $\psi : K_0(\mathcal{M}) \to \mathbb{Z}$ such that $\psi \circ [\,] = \chi$.

Remark 4. The onto-pigeonhole principle *ontoPHP* is the statement that there is no set A, $a \in A$, and injective map f from A onto $A \setminus \{a\}$. By the construction of the Grothendieck ring of a structure $\mathcal{M}$, $K_0(\mathcal{M})$ is non-trivial if and only if $\mathcal{M} \models$ *ontoPHP*. See [1] for the details.

3. Grothendieck rings of o-minimal expansions of ordered abelian groups

We begin with the introduction of notations of an o-minimal structure $(G, <, \ldots)$.

For a definable set $X \subseteq G^m$, we put

$$\mathcal{C}(X) := \{ f : X \to G \mid f \text{ is definable and continuous} \},$$
$$\mathcal{C}_\infty(X) := \mathcal{C}(X) \cup \{-\infty, +\infty\},$$

where we regard $-\infty$ and $+\infty$ as constant functions on X. For $f \in \mathcal{C}(X)$, the graph of f is denoted by $\Gamma(f) \subseteq X \times G$. For $f, g \in \mathcal{C}_\infty(X)$, we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f, g)_X := \{ (x, r) \in X \times G \mid f(x) < r < g(x) \}.$$

We next show that the Grothendieck rings of o-minimal expansions of ordered abelian groups are of the simple form:

Lemma 5. *Let* $(G, <, +, 0, \ldots)$ *be an o-minimal expansion of an ordered abelian group. Then,*

$$K_0(G) = \mathbb{Z}\big[[C] \mid C \subseteq G \textit{ is a cell}\big].$$

Proof. Let $M \subseteq G^n$ be a definable set. By the cell decomposition theorem,

$$M = C_1 \cup \cdots \cup C_l,$$

where $C_1, \ldots, C_l$ are cells. Hence

$$[M] = [C_1] + \cdots + [C_l].$$

Therefore, it suffices to show that for every cell $C \subseteq G^n$, $[C] \in \mathbb{Z}[[C] \mid C \subseteq G$ is a cell]. We will prove this by induction on n. For simplicity we denote $\mathbb{Z}_{\mathrm{cell}} := \mathbb{Z}[[C] \mid C \subseteq G$ is a cell].

The claim obviously holds true in the case where $n = 1$. Assume that the claim is true for $n = k$, and we show that it holds for $n = k + 1$. Let $C \subseteq G^{k+1}$ be a cell.

If

$$C = \{ (x, t) \in A \times G \mid t = f(x) \},$$

where $A \in G^k$ is the image $\pi(C)$ of C under the projection $\pi : G^{k+1} \to G^k$ on the first k-coordinates and for some function $f \in \mathcal{C}(A)$. Hence there exist a definable bijection $C \cong A$. Because A is a cell, by the inductive assumption, $[C] = [A] \in \mathbb{Z}_{\text{cell}}$.

If

$$C = \{(x, t) \in A \times G \mid \alpha(x) < t < \beta(x)\},$$

where $A \in G^k$ is the image $\pi(C)$ of C under the projection $\pi : G^{k+1} \to G^k$ on the first k-coordinates and for some functions $\alpha, \beta \in \mathcal{C}_\infty(A)$.

Case 1. $\alpha = -\infty$, $\beta = +\infty$.

Then $C = A \times (-\infty, +\infty)$. Hence $[C] = [A] \cdot [(-\infty, +\infty)] \in \mathbb{Z}_{\text{cell}}$.

Case 2. $\alpha \in \mathcal{C}(A)$, $\beta = +\infty$.

Then we have a definable bijection,

$$\begin{aligned} A \times (0, +\infty) &\longrightarrow C, \\ (x, t) &\longmapsto (x, \alpha(x) + t). \end{aligned}$$

Hence, $[C] = [A] \cdot [(0, +\infty)] \in \mathbb{Z}_{\text{cell}}$.

Case 3. $\alpha = -\infty$, $\beta \in \mathcal{C}(A)$.

Then we have a definable bijection,

$$\begin{aligned} A \times (0, +\infty) &\longrightarrow C, \\ (x, t) &\longmapsto (x, \beta(x) - t). \end{aligned}$$

Hence, $[C] = [A] \cdot [(0, +\infty)] \in \mathbb{Z}_{\text{cell}}$.

Case 4. $\alpha, \beta \in \mathcal{C}(A)$.

Then,

$$C \cup \Gamma(\alpha) \cup D = \{(x, t) \in A \times G \mid t < \beta(x)\},$$

where $D = \{(x, t) \in A \times G \mid t < \alpha(x)\}$. Hence, by considering Case 3

$$[C] + [\Gamma(\alpha)] + [D] \in \mathbb{Z}_{\text{cell}}.$$

Because $[\Gamma(\alpha)], [D] \in \mathbb{Z}_{\text{cell}}$, thus $[C] \in \mathbb{Z}_{\text{cell}}$. □

Corollary 6. *Let* $(G, <, +, 0, \dots)$ *be an o-minimal expansion of an ordered abelian group. We set* $X := [(0, +\infty)]$. *Then the equation* $X^2 + X = 0$ *holds true, and*

$$K_0(G) = \{m + nX \mid m, n \in \mathbb{Z}\}.$$

Proof. First, we prove the following claim.

Claim 7.

(i) *For the interval* (a, b) *where* $a, b \in G$, $[(a, b)] = -1$,
(ii) $[(-\infty, +\infty)] = 2X + 1$,
(iii) $X^2 = -X$.

Proof. (i) Because $(a, b) \cong (0, b - a)$, we may assume $a = 0$ and show that $[(0, b)] = -1$. $(0, b) \cong (0, b/2) \cong (b/2, b)$ and $[(0, b)] = [(0, b/2)] + 1 + [(b/2, b)]$. Hence, $[(0, b)] = -1$.

(ii) $(-\infty, 0) \cong (0, +\infty)$ and $[(-\infty, +\infty)] = [(-\infty, 0)] + 1 + [(0, +\infty)]$ thus $[(-\infty, +\infty)] = 2X + 1$.

(iii) Let I be the interval $(0, +\infty)$ and $f : I \to I\,(x \mapsto x)$ be a function. Then, $I \times I = (0, f)_I \cup \Gamma(f) \cup (f, +\infty)_I$. We can construct the following definable bijections,

$$\begin{aligned} (0, f)_I &\longrightarrow (f, +\infty)_I \\ (x, y) &\longmapsto (y, x) \end{aligned} \quad \text{and} \quad \begin{aligned} I \times I &\longrightarrow (f, +\infty)_I \\ (x, y) &\longmapsto (x, x + y). \end{aligned}$$

Because $\Gamma(f) \cong I$,

$$\begin{aligned} [I \times I] &= \big[(0, f)_I\big] + \big[\Gamma(f)\big] + \big[(f, +\infty)_I\big] \\ &= [I \times I] + [I] + [I \times I]. \end{aligned}$$

We get $[I \times I] + [I] = 0$. Thus $X^2 + X = 0$. □

By Lemma 5, for each element $F \in K_0(G)$ there exist cells $C_1, \dots, C_n$ in G such that

$$F = \sum_{j_1, \dots, j_n} a_{j_1, \dots, j_n} [C_1]^{j_1} \cdots [C_n]^{j_n},$$

where $a_{j_1, \dots, j_n} \in \mathbb{Z}$. Each cell C_i $(i = 1, \dots, n)$ is a point or an interval and $(0, +\infty) \cong (a, +\infty) \cong (-\infty, b) \cong (-\infty, 0)$ where $a, b \in G$. Using the above claim, we obtain $F = m + nX$ for some $m, n \in \mathbb{Z}$. □

Next we will define a class of definable sets for every o-minimal expansion of an ordered abelian group and show its useful properties to calculate the Grothendieck rings of o-minimal expansions of ordered abelian groups.

Definition 8. Let $(G, <, +, 0, \dots)$ be an o-minimal expansion of an ordered abelian group. We call that a definable set $M \subseteq G^n$ is *bounded* if $M \subseteq [b, b']^n$ for some $b, b' \in G$, where $[b, b'] := \{t \in G \mid b \leqslant t \leqslant b'\}$.

Lemma 9. *Let* $(G, <, +, 0, \ldots)$ *be an o-minimal expansion of an ordered abelian group and* $M \subseteq G^n$ *be a bounded definable set with* $\dim M = 1$. *Then, there exists a definable bijection* $M \to D$ *for some bounded definable set* $D \subseteq G$.

Proof. Since $\dim M = 1$, by the cell decomposition theorem we get the following decomposition:

$$M = C_1 \cup \cdots \cup C_l \cup C_{l+1} \cup \cdots \cup C_m,$$

where $C_1, \ldots, C_m$ are cells, $\dim C_1 = 1, \ldots, \dim C_l = 1$ and $\dim C_{l+1} = 0, \ldots, \dim C_m = 0$.

Claim 10. *For all* $i = 1, \ldots, l$, *there exists a projection* $p_{n_i} : G^n \to G((x_1, \ldots, x_n) \mapsto x_{n_i})$ *for some* $1 \leqslant n_i \leqslant n$ *such that* $p_{n_i} \mid C_i : C_i \to p_{n_i}(C_i)$ *is definably bijective.*

Proof. We prove this claim by the induction on n. When $n = 1$, because each C_i is an interval or a point, the claim holds true. Under the assumption that the claim holds true for $n = k$, we show that the claim holds for $n = k + 1$. Let $p_1 : G^{k+1} \to G$ be the projection on the first coordinate.

Case 1. $\dim p_1(C_i) = 1$.

For the projections $\pi_q : G^{k+1} \to G^q$ $(q = 1, \ldots, k + 1)$ on the first q-coordinates, $\dim \pi_q(C_i) = 1$, because $\dim C_i \geqslant \dim \pi_q(C_i) \geqslant \dim p_1(C_i) = 1$. Hence, each cell $\pi_q(C_i)$ $(q = 2, \ldots, k + 1)$ is the graph of a definable function $f_q \in \mathcal{C}(\pi_{q-1}(C_i))$. By using $f_2, \ldots, f_k$, we inductively define functions $g_2, \ldots, g_{k+1} : p_1(C_i) \to G$ as follows: $g_2(x) := f_2(x)$ and we define g_{j+1} by $g_{j+1}(x) := f_{j+1}(x, g_2(x), \ldots, g_j(x))$ where $2 \leqslant j \leqslant k + 1$ and $x \in p_1(C_i)$. Then, for a definable function $g : p_1(C_i) \to G^k$ $(x \mapsto (g_2(x), \ldots, g_{k+1}(x)))$, $C_i = \Gamma(g)$. Thus we obtain a definable bijection $p_1 \mid C_i : C_i \to p_1(C_i)$.

Case 2. $\dim p_1(C_i) = 0$.

Since $\dim p_1(C_i) = 0$, there are a point $a_i \in G$ and a cell $D_i \subseteq G^k$ such that $C_i = \{a_i\} \times D_i$. By inductive assumption, there is a projection $p_{n_i} : G^k \to G$ such that $p_{n_i} \mid D_i$ is injective. Let τ be a projection such that $\tau : G^{k+1} \to G^k((x_1, \ldots, x_{k+1}) \mapsto (x_2, \ldots, x_{k+1}))$. Then, $p_{n_i+1} = p_{n_i} \circ \tau$ and $p_{n_i+1} \mid C_i : C_i \to p_{n_i}(C_i)$ is a definable bijection. □

By claim, each C_i $(i = 1, \ldots, l)$ is definably bijective to an interval of G and each C_i $(i = l + 1, \ldots, m)$ is a point set. Thus, we can define a definable bijection $M \to D$ for some bounded definable set $D \subseteq G$. □

Proposition 11. *Let* $\mathcal{G} = (G, <, +, 0, \ldots)$ *be an o-minimal expansion of an ordered abelian group,* $M \subseteq G^m$ *be a non-bounded definable set and* $N \subseteq G^n$ *be a bounded definable set. If* M *and* N *are definably isomorphic, then there exists a definable bijection* $(0, +\infty) \to D$ *for some bounded definable set* $D \subseteq G$.

Proof. Let $\pi_q : G^n \to G^q$ be the projection on the first q-coordinates. By the cell decomposition theorem,

$$M = C_1 \cup \cdots \cup C_m,$$

where $C_1, \dots, C_m$ are cells. Since M is a non-bounded definable set, we can choose a non-bounded cell C_i for some $1 \leqslant i \leqslant m$. Because C_i is non-bounded we may assume that $\pi_1(C_i)$ is a non-bounded interval I.

If $\pi_2(C_i) = \Gamma(f)$ for some $f \in \mathcal{C}(\pi_1(C_i))$, then we can define a definable injection $i_2 : I \to \pi_2(C_i)$ by $i_2(x) := (x, f(x))$.

If $\pi_2(C_i) = \{(x, y) \in I \times G \mid \alpha(x) < y < \beta(x)\}$ for some $\alpha, \beta \in \mathcal{C}_\infty(\pi_1(C_i))$, note that G is a vector space over $\mathbb{Q}$ [3, Chapter 1, Proposition 4.2], we can define a definable injection $i_2 : I \to \pi_2(C_i)$ by

$$i_2(x) := \begin{cases} (x, x) & \text{if } \alpha = -\infty,\ \beta = +\infty, \\ (x, \beta(x) - a) & \text{if } \alpha = -\infty,\ \beta \in \mathcal{C}(\pi_1(C_i)), \\ (x, \alpha(x) + a) & \text{if } \alpha \in \mathcal{C}(\pi_1(C_i)),\ \beta = +\infty, \\ (x, (\alpha(x) + \beta(x))/2) & \text{if } \alpha \in \mathcal{C}(\pi_1(C_i)),\ \beta \in \mathcal{C}(\pi_1(C_i)), \end{cases}$$

where a is a positive element of G.

By continuing in the similarly way, we get a sequence of definable injections

$$I \xrightarrow{i_2} \pi_2(C_i) \xrightarrow{i_3} \cdots \xrightarrow{i_{n-1}} \pi_{n-1}(C_i) \xrightarrow{i_n} C_i.$$

Let $\iota : I \to C_i$ be the composition of these definable injections. Because $\dim f(\iota(I)) = 1$ by Lemma 9, there is a bounded definable set $D \subseteq G$ such that $f(\iota(I)) \cong D$. Thus we get a definable bijection between I and D. $\square$

It is easier to calculate the Grothendieck ring of the structure $\mathcal{G}$ in the case where a non-bounded definable set and a bounded definable set are definably isomorphic than in the other case. To treat the latter case, we rewrite the condition as follows:

Bounded Condition. Let $(G, <, +, 0, \dots)$ be an o-minimal expansion of an ordered abelian group, $M \subseteq G^m$ be a bounded definable set and $N \subseteq G^n$ be a definable set. If M and N are definably isomorphic, then N is bounded.

Example 12. Let $\mathcal{G} = (G, +, -, <, 0)$ be the ordered divisible abelian group. Then $\mathcal{G}$ satisfies Bounded Condition.

Proof. Suppose not. Then there are definable sets $X \subseteq G^m$, $Y \subseteq G^n$ such that X is non-bounded, Y is bounded and $X \cong Y$. By Proposition 11, there is a definable bijection $f : (0, +\infty) \to D$ for some bounded definable set $D \subseteq G$. Because $\mathcal{G}$ is o-minimal, we may assume that D is an interval (a, b) for some $a, b \in G$. By the monotonicity theorem [3, Chapter 3, Theorem 1.2], there are points $a_1 < \cdots < a_k$ in $(0, +\infty)$ such that on each subinterval (a_j, a_{j+1}) with $a_0 = 0$, $a_{k+1} = +\infty$, the function $f|(a_j, a_{j+1})$ is strictly monotone and continuous. Since $(g :=)\ f \mid (a_k, +\infty) : (a_k, +\infty) \to (a, b)$ is definable and the ordered divisible abelian group admits quantifier elimination [3, Chapter 1, Corollary 7.8], the definable function g is a polygonal line. By dividing suitably $(a_k, +\infty)$ again, we obtain points $a'_{k+1} < \cdots < a'_n$ in $(a_k, +\infty)$ with $a'_k = a_k$, $a'_{n+1} = +\infty$, and linear functions $g_{k,k+1} : (a'_k, a'_{k+1}) \to (a, b)$, $\dots$, $g_{n,n+1} : (a'_n, a'_{n+1}) \to (a, b)$ with $g_{k,k+1}, \dots, g_{n,n+1}$ are strictly monotone.

There exist $m, m' \in \mathbb{Z}$ such that $g_{n,n+1}(x) = mx + m'$, $m \neq 0$ where $x \in (a'_n, a'_{n+1})$. When $m > 0$ for $x_0 \in G$ with $(-m' + b)/m \ll x_0$, $g_{n,n+1}(x_0) > b$. This contradicts to the fact that the target space of $g_{n,n+1}$ is (a, b). We can also lead a contradiction when $m < 0$ in the same way. □

Example 13. Let $\mathcal{R} = (R, +, -, \cdot, <, 0, 1)$ be a real closed field. Then $\mathcal{R}$ does not satisfy Bounded Condition.

Proof. We can define a definable bijection $\phi : (0, 1) \to (1, +\infty)$ by $\phi(x) := x/(1 - x)$. □

4. Bounded Euler characteristic

We first recall the definition of the geometric Euler characteristic [3, Chapter 4].

Definition 14. Let $(G, <, \ldots)$ be an o-minimal structure and S be a definable subset of G^m. There exists a finite partition $\mathcal{P}$ of S into cells $\mathcal{P} = \{C_1, \ldots, C_l\}$ by the cell decomposition theorem. Then we define the *geometric Euler characteristic* of the definable set S:

$$\chi_g(S) := \sum_{C \in \mathcal{P}} (-1)^{\dim C}.$$

This definition is seem to depend on the partition $\mathcal{P}$ of S. However, the definition does not depend on the choice of finite partitions. Moreover, it is known that χ_g is invariant under definable bijections and satisfies the properties (1), (2) and (3) in Remark 3. See [3, Chapter 4] for the details.

Lemma 15. *Let $\mathcal{G} = (G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group. Consider the ring homomorphism $i : \mathbb{Z} \to K_0(G)$ given by $i(1) =$ [one point]. Then i is injective.*

Proof. Consider the geometric Euler characteristic $\chi_g : \mathrm{Def}(\mathcal{G}) \to \mathbb{Z}$. By Remark 3 there exists a ring homomorphism $\psi_g : K_0(\mathcal{G}) \to \mathbb{Z}$ such that $\psi_g \circ [\ ] = \chi_g$. Fix $n \in \ker(i)$. We may assume that $n \geqslant 0$. By the definition of χ_g,

$$n = \chi_g(n \text{ points}) = \psi_g \circ i(n) = 0.$$

We have shown that i is injective. □

By Lemma 15, we may consider naturally that $\mathbb{Z}$ is a subring of $K_0(\mathcal{G})$ for each o-minimal expansion of an ordered abelian group $\mathcal{G}$.

Definition 16. Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group, $C \subseteq G^n$ be a cell and $p_k : G^n \to G^k$ be the projection on the first k-coordinates. A cell C is called *exceptional* if there exist $k \in \mathbb{N}$ and a cell $A \subseteq G^{k-1}$ with $p_k(C) = A \times G$. A non-exceptional cell C is called *bad* if there exist $k \in \mathbb{N}$ and a cell $A \subseteq G^{k-1}$ with

$$p_k(C) = \big\{(x, t) \in A \times G \mid t < f(x)\big\} \quad \text{or} \quad \big\{(x, t) \in A \times G \mid f(x) < t\big\},$$

where $f : A \to G$ is a definable function. A *good* cell C is a cell which is not neither exceptional nor bad.

Lemma 17. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group, $X \subseteq G^n$ be a definable set, $\mathcal{F}$ be a finite partition of X into cells any one of whose cell is not exceptional. We put*

$$\chi_b(X) := \begin{cases} \sum_{C\in\mathcal{F},\ C:\text{good}} (-1)^{\dim C}, & \text{if } \mathcal{F} \text{ includes a good cell,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\chi_b(X)$ does not depend on the choice of the finite partition $\mathcal{F}$.

Proof. We can take such a finite partition $\mathcal{F} = \{C\}$ of X by applying the cell decomposition theorem to definable sets X, $\{(x_1, \ldots, x_n) \in G^n \mid x_i > 0\}$, $\{(x_1, \ldots, x_n) \in G^n \mid x_i = 0\}$, and $\{(x_1, \ldots, x_n) \in G^n \mid x_i < 0\}$ $(i = 1, \ldots, n)$. We set

$$\chi_b^{\mathcal{F}}(X) := \sum_{C\in\mathcal{F},\ C:\text{good}} (-1)^{\dim(C)}.$$

Let $\mathcal{G} = \{D\}$ be another partition. Our purpose of this proof is to show $\chi_b^{\mathcal{G}}(X) = \chi_b^{\mathcal{F}}(X)$. Let $\mathcal{H}$ be a finer partition than $\mathcal{F}$ and $\mathcal{G}$. If $\chi_b^{\mathcal{F}}(X) = \chi_b^{\mathcal{H}}(X)$ and $\chi_b^{\mathcal{G}}(X) = \chi_b^{\mathcal{H}}(X)$, then $\chi_b^{\mathcal{G}}(X) = \chi_b^{\mathcal{F}}(X)$. Hence we may assume that $\mathcal{G}$ is a finer partition than $\mathcal{F}$. We prove $\chi_b^{\mathcal{G}}(X) = \chi_b^{\mathcal{F}}(X)$ by the induction on n. Remark that

$$\chi_b^{\mathcal{F}}(X) = \chi_g\Bigg(\bigcup_{C\in\mathcal{F},\ C:\text{good}} C\Bigg) = \sum_{C\in\mathcal{F},\ C:\text{good}} (-1)^{\dim(C)}.$$

We have only to show that, for any bad cell C of $\mathcal{F}$,

$$\sum_{D\in\mathcal{G},\ D\subseteq C,\ D:\text{good}} (-1)^{\dim(D)} = 0.$$

We fix $C \in \mathcal{F}$ and set

$$E := \bigcup_{D\in\mathcal{G},\ D\subseteq C,\ D:\text{good}} D.$$

Remark that

$$\sum_{D\in\mathcal{G},\ D\subseteq C,\ D:\text{good}} (-1)^{\dim(D)} = \chi_g(E).$$

When $n = 1$, $E = (a, b]$, $E = [a, b)$ or $E = \emptyset$ for some $a, b \in G$. Hence $\chi_g(E) = 0$.

We consider the case where $n > 1$. Let p be the projection on the first $(n-1)$-coordinates. Then $p(C)$ is a non-exceptional cell. Let $\mathcal{G}' = \{D'\}$ be the family of all good cells of the form: $D' = p(D)$ for some $D \in \mathcal{G}$. Set $F := \bigcup_{D'\in\mathcal{G}'} D'$. Consider two cases.

- First consider the case where C is of the form:

$$\big\{(x, t) \in p(C) \times G \mid t = f(x)\big\} \quad \text{or} \quad \big\{(x, t) \in p(C) \times G \mid f(x) < t < g(x)\big\},$$

where $f, g : p(C) \to G$ are definable functions. Remark that $\chi_g(F) = 0$ by the inductive hypothesis. Since $E = \{(x,t) \in F \times G \mid t = f(x)\}$ or $E = \{(x,t) \in F \times G \mid f(x) < t < g(x)\}$, $\chi_g(E) = 0$.

- Consider the other case, then there exist definable functions $f < g$ on $D' \in \mathcal{G}'$ such that

$$E \cap p^{-1}(D') = \{(x,t) \in D' \times G \mid f(x) < t \leqslant g(x)\},$$
$$E \cap p^{-1}(D') = \{(x,t) \in D' \times G \mid f(x) \leqslant t < g(x)\} \quad \text{or}$$
$$E \cap p^{-1}(D') = \emptyset.$$

In each case, $\chi_g(E \cap p^{-1}(D')) = 0$. Since $E = \bigcup_{D' \in \mathcal{F}'}(E \cap p^{-1}(D'))$, $\chi_g(E) = 0$. □

Lemma 18. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group. Let X and Y be definable sets. Then $\chi_b(X \cup Y) + \chi_b(X \cap Y) = \chi_b(X) + \chi_b(Y)$.*

Proof. This lemma follows from the definition of χ_b obviously. □

Proposition 19. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group, $X \subseteq G^{m+n}$ be a definable subset, $\mathcal{D}$ be a decomposition of G^{m+n} partitioning X and $\pi : G^{m+n} \to G^m$ be the projection on the first m-coordinates. Assume that all cells are not exceptional. Given a cell $A \in \pi(\mathcal{D})$ there is a constant e_A with $\chi_b(X \cap p^{-1}(a)) = e_A$ and $\chi_b(X \cap p^{-1}(A)) = \chi_b(A)e_A$.*

Proof. Fix $A \in \pi(\mathcal{D})$. For each cell C of $\mathcal{D}$, $C \cap \pi^{-1}(a) = \emptyset$ if $\pi(C) \neq A$ and $a \in A$. If $\pi(C) = A$, $C \cap \pi^{-1}(a)$ is a cell and its dimension does not depend on the choice of $a \in A$. Moreover, if $C \cap p^{-1}(a)$ is good for some $a \in A$, the same statement holds true for all $a \in A$. Set $e_A = \chi_b(X \cap \pi^{-1}(a))$ for some $a \in A$. Then e_A satisfies the requirement of the first statement of this lemma. It is also obvious that $\chi_b(X \cap p^{-1}(A)) = \chi_b(A)e_A$ by the definition of χ_b. □

Corollary 20. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group and $X \subseteq G^m$ and $Y \subseteq G^n$ be definable sets. Then $\chi_b(X \times Y) = \chi_b(X) \cdot \chi_b(Y)$.*

Proof. This corollary follows from Proposition 19. □

Lemma 21. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group. Moreover, assume that $\mathcal{G}$ satisfies Bounded Condition. Then a cell C is good if and only if C is bounded.*

Proof. It is obvious that a cell which is not good is not bounded. Hence we have only to show that a good cell $C \subseteq G^n$ is bounded. We prove it by the induction on n. When $n = 1$, it is obvious. Consider the case when $n > 1$. Let $p : G^n \to G^{n-1}$ be the projection on the first $(n-1)$-coordinates. The cell $p(C)$ is bounded by the inductive hypothesis. Let $d \in G$ such that $p(C) \subseteq [-d, d]^{n-1}$. Remark that C is of the form:

$$\{(x,t) \in p(C) \times G \mid t = f(x)\} \quad \text{or} \quad \{(x,t) \in p(C) \times G \mid f(x) < t < g(x)\},$$

where f and g are definable functions on $p(C)$. There exists positive $d' \in G$ such that $-d' < f(x) < d'$ and $-d' < g(x) < d'$ for all $x \in p(C)$. Set $d'' := \max\{d, d'\}$. Then $C \subseteq [-d'', d'']^n$, namely, C is bounded. □

Lemma 22. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group satisfying Bounded Condition. Let $X \subseteq G^m$ be a definable set and σ be a permutation of $\{1, \ldots, m\}$. We define a definable function $\Psi_\sigma : G^m \to G^m$ by $\Psi_\sigma(x_1, \ldots, x_m) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$. Then $\chi_b(X) = \chi_b(\Psi_\sigma(X))$.*

Proof. Since the symmetric group on $\{1, \ldots, m\}$ is generated by the transpositions $(i, i+1)$, we may assume that $\sigma = (i, i+1)$. By [3, Chapter 4, Proposition 2.13], there exists a cell decomposition $\mathcal{D}$ such that any cell is not exceptional and $\Psi_\sigma(C)$ are also cells for all cells $C \in \mathcal{D}$. Since a cell is good if and only if it is bounded by Lemma 21, $\Psi_\sigma(C)$ is good if and only if so is C. Hence, $\chi_b(X) = \chi_b(\Psi_\sigma(X))$ by the definition of χ_b. □

We are now ready to state the invariance of χ_b under bijections definable in an o-minimal expansion of an ordered abelian group which satisfies Bounded Condition.

Proposition 23. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group satisfying Bounded Condition. Let $X \subseteq G^m$ be a definable set and $f : X \to G^n$ be an injective definable map. Then $\chi_b(X) = \chi_b(f(X))$.*

Proof. Consider the graph $\Gamma(f) \subseteq G^{m+n}$ and the definable set $\Gamma'(f) = \{(f(x), x) \in G^n \times X\}$. By Proposition 19, $\chi_b(X) = \chi_b(\Gamma(f))$ and $\chi_b(f(X)) = \chi_b(\Gamma'(f))$. Because $\chi_b(\Gamma(f)) = \chi_b(\Gamma'(f))$ by Lemma 22. We obtain the conclusion. □

Definition 24. Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group satisfying Bounded Condition. For all definable sets $X \subseteq G^n$, we call $\chi_b(X)$ the *bounded Euler characteristic* of X.

Remark 25. The following theorem ensures that our definition of χ_b coincides with the notion of the bounded Euler characteristic in [6].

Theorem 26. *Let $(G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group and $X \in G^n$ be a definable set. Let $d : X \to [0, \infty)$ be a definable function such that $d^{-1}(t)$ is bounded for any $t \geqslant 0$. Set $X_d(t) := \{x \in X \mid d(x) \leqslant t\}$ for any $t \in G$. Then there exists $\mu \in G$ with $\chi_g(X_d(t)) = \chi_b(X)$ for $t \geqslant \mu$.*

Proof. Consider the definable set $\Gamma'(d) := \{(t, x) \in G \times X \mid d(x) = t\}$. Let p be the projection of $\Gamma'(d)$ to the first factor. Apply the cell decomposition theorem to $\Gamma'(d)$. Let $\Gamma'(d) = C_1 \cup \cdots \cup C_k$ be the cell decomposition. We may assume that $C_1, \ldots, C_j$ are bounded and $C_{j+1}, \ldots, C_k$ are not bounded.

Since the fibres of d are bounded, the cell C_i is bounded if and only if $p(C_i)$ is bounded. Hence there exists $\mu \in G$ such that $C_i \cap p^{-1}(\{t \in G \mid t > \mu\}) = \emptyset$ for all $i = 1, \ldots, j$ and $C_i \cap p^{-1}(\{t \in G \mid t > \mu\}) \neq \emptyset$ for all $i = j+1, \ldots, k$. It is easy to see that the definable sets $C_i \cap p^{-1}(\{s \in G \mid s > t\})$ are cells of dimension $\dim C_i$ for all $i = j+1, \ldots, k$. Hence we omit the proof of this fact.

Fix $t \geqslant \mu$. Then

$$\begin{aligned}\chi_g\big(X_d(t)\big) &= \chi_g(X) - \chi_g\big(\{x \in X \mid d(x) > t\}\big) \\ &= \chi_g(X) - \sum_{i=j+1}^{k} (-1)^{\dim(C_i \cap p^{-1}(\{s \in G \mid s > t\}))} \\ &= \chi_g(X) - \sum_{i=j+1}^{k} (-1)^{\dim(C_i)} \quad \text{(by the above fact)} \\ &= \sum_{i=1}^{j} (-1)^{\dim(C_i)} \\ &= \chi_b\big(\Gamma'(d)\big) \quad \text{(by the definition)} \\ &= \chi_b(X) \quad \text{(by Proposition 23)}. \qquad \square\end{aligned}$$

5. Proof of Theorem 1

We are now ready to prove Theorem 1.

Proof.

Case 1. There exists a definable bijection between a non-bounded definable set and a bounded definable set.

Then by Proposition 11, we can take a definable bijection $(0, +\infty) \cong D$ for some bounded definable set $D \subseteq G$. Because $[(0, +\infty)] = [D] \in \mathbb{Z}$, the ring homomorphism $i : \mathbb{Z} \to K_0(\mathcal{G})$ given by $i(1) = [\text{one point}]$ is surjective. By Lemma 15, i is injective. Therefore $K_0(\mathcal{G})$ is isomorphic to $\mathbb{Z}$ as a ring.

Case 2. There exist no definable bijections of non-bounded definable sets into bounded definable sets.

Then, because $\mathcal{G}$ satisfies Bounded Condition, we can define the bounded Euler characteristic χ_b. By Corollary 6, the following ring homomorphism is surjective:

$$\begin{aligned}\phi : \mathbb{Z}[T]/\big(T^2 + T\big) &\longrightarrow K_0(\mathcal{G}), \\ 1 &\longmapsto [\text{one point}], \\ T &\longmapsto X,\end{aligned}$$

where $X = [(0, +\infty)]$.

We show that this ring homomorphism is injective. Fix $m + nX \in \ker(\phi)$ where $m, n \in \mathbb{Z}$. Considering the universal mapping property of $(K_0(\mathcal{G}), [\ ])$ for the geometric Euler characteristic χ_g, there exists an unique ring homomorphism $\psi_g : K_0(\mathcal{G}) \to \mathbb{Z}$ such that $\psi_g \circ [\ ] = \chi_g$. By the definition of ψ_g,

$$\psi_g(m + nX) = m + n\psi_g(X) = m + n\chi_g((0, +\infty)) = m - n.$$

Thus we get $m = n$. Similarly for the bounded Euler characteristic χ_b there exists an unique ring homomorphism $\psi_b : K_0(\mathcal{G}) \to \mathbb{Z}$ such that $\psi_b \circ [\ ] = \chi_b$. By the definition of ψ_b,

$$\psi_b(m + nX) = m + n\psi_b(X) = m + n\chi_b((0, +\infty)) = m.$$

Thus we get $m = n = 0$. We have shown ϕ is injective. $\square$

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