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European Journal of Combinatorics 25 (2004) 1107–1116

European Journal of Combinatorics

www.elsevier.com/locate/ejc

On edge-transitive Cayley graphs of valency four

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Received 17 October 2000; accepted 24 July 2003

Available online 8 February 2004

Abstract

A characterization is given of a class of edge-transitive Cayley graphs, providing methods for constructing Cayley graphs with certain symmetry properties. Various new half-arc transitive graphs are constructed.

1. Introduction

Let G be a group, and let S be a subset of G such that $S = S^{-1} := \{s^{-1} \mid s \in S\}$ and S does not contain the identity of G. The *Cayley graph* $\mathsf{Cay}(G, S)$ of G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G,\ s \in S\}$.

Obviously, the Cayley graph $\mathsf{Cay}(G, S)$ has valency $|S|$, and it easily follows that $\mathsf{Cay}(G, S)$ is connected if and only if $\langle S\rangle = G$. Further, the set of all right multiplications of G (that is, $g : x \to xg$) form a subgroup of the automorphism group of $\mathsf{Cay}(G, S)$, acting regularly, on the vertex set G. Thus in particular $\mathsf{Cay}(G, S)$ is a vertex-transitive graph. However, there exist Cayley graphs which are not edge-transitive. The purpose of this paper is to study edge-transitive Cayley graphs of valency 4, associated with certain classes of insoluble groups.

For a positive integer s, an *s-arc* of a graph Γ is an $(s + 1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices such that α_{i-1}, α_i are adjacent $1 \le i \le s$ and $\alpha_{i-1} \ne \alpha_{i+1}$ for $1 \le i \le s - 1$. If $X \le \mathsf{Aut}\,\Gamma$ and X is transitive on $V\Gamma$ and on the set of s-arcs of Γ, then Γ is called a *(X, s)-arc-transitive* graph; while if in addition X is not transitive on the set of $(s + 1)$-arcs of Γ, then Γ is said to be *(X, s)-transitive*. In particular, a $(\mathsf{Aut}\,\Gamma, s)$-transitive graph

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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2003.07.008

Table 1
Exceptional candidates

s	G
$\frac{1}{2}$, 1	M_{12}, M_{22}, J_2, Suz A_{2^m-1} for $m \geq 3$ $\mathrm{PSL}_n(2^e)$, $\mathrm{PSU}_n(2^e)$ for $n \geq 4$; $\mathrm{PSp}_n(2^e)$ for $n \geq 6$ $\mathrm{E}_6(2^e)$, $\mathrm{E}_7(2^e)$, ${}^2\mathrm{E}_6(2^e)$, ${}^2\mathrm{G}_2(2^e)$
2	$\mathrm{PSL}_2(11)$, M_{11}, M_{23}, A_{11}
3	$\mathrm{PSL}_2(11)$, or A_{n-1} where $n = 2^r 3^2$ for $r \in \{2, 3, 4\}$
4, 7	$\mathrm{PSL}_4(2)$, $\mathrm{PSL}_5(2)$, $\mathrm{PSL}_3(9)$, $\mathrm{PSL}_3(27)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSL}_5(3)$, $\mathrm{PSL}_6(3)$, $\mathrm{PSU}_4(3)$ A_{n-1} where $n = 2^r 3^{s-1}$ for $r \in \{2, 3, 4\}$

is simply called an *s-transitive graph*. A vertex- and edge-transitive graph is said to be $1/2$-*transitive* if it is not 1-arc-transitive.

The transitivity of a Cayley graph Γ is defined by the full automorphism group $\mathsf{Aut}\,\Gamma$. The problem of determining what symmetry degree a Cayley graph has is therefore a hard problem since it is hard to determine the full automorphism group. Let

$$\mathsf{Aut}(G, S) = \{\sigma \in \mathsf{Aut}(G) \mid S^\sigma = S\}.$$

Then every element of $\mathsf{Aut}(G, S)$ induces an automorphism of $\mathsf{Cay}(G, S)$, and by Lemma 2.1, $\mathbf{N}_{\mathsf{Aut}\,\Gamma}(G) = G : \mathsf{Aut}(G, S)$, a *semi-direct product* of G by $\mathsf{Aut}(G, S)$. Thus, although it is difficult to determine the full automorphism group $\mathsf{Aut}\,\Gamma$, the subgroup $\mathbf{N}_{\mathsf{Aut}\,\Gamma}(G)$ may be described in terms of G so that the action of $\mathbf{N}_{\mathsf{Aut}\,\Gamma}(G)$ has a very nice property, that is, the action is determined by translation and conjugation. This property has played an important role in the study of Cayley graphs, see for example [4, 7, 11, 19, 23]. An extreme case occurs when $\mathbf{N}_{\mathsf{Aut}\,\Gamma}(G) = \mathsf{Aut}\,\Gamma$, that is, when G is normal in $\mathsf{Aut}\,\Gamma$. In this case, $\Gamma = \mathsf{Cay}(G, S)$ is called a *normal Cayley graph*. The following theorem is one of the main results of this paper. It gives a characterization of a class of edge-transitive Cayley graphs of finite non-Abelian simple groups in terms of normality.

Theorem 1.1. *Let G be a non-Abelian simple group, and let $\Gamma = \mathsf{Cay}(G, S)$ be a connected edge-transitive graph of valency* 4. *Then either*

(1) $\mathsf{Aut}\,\Gamma = G : \mathsf{Aut}(G, S)$, $\mathsf{Aut}(G, S) = \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2^2, \mathrm{D}_8, \mathrm{A}_4$, *or* S_4, *and further*

 (i) *Γ is $1/2$-transitive if and only if* $\mathsf{Aut}(G, S) = \mathbb{Z}_2$;
 (ii) *Γ is 1-transitive if and only if* $\mathsf{Aut}(G, S) = \mathbb{Z}_4, \mathbb{Z}_2^2$, *or* D_8;
 (iii) *Γ is 2-transitive if and only if* $\mathsf{Aut}(G, S) = \mathrm{A}_4$ *or* S_4; *or*

(2) $\mathsf{Aut}\,\Gamma \neq G : \mathsf{Aut}(G, S)$, *$\Gamma$ is s-transitive, and G is one of the groups given in Table 1:*

Theorem 1.1 indicates that 'most' edge-transitive Cayley graphs of simple groups of valency 4 are normal, and that for $s \geq 3$, s-transitive Cayley graphs of valency 4 are very rare. It is shown in [12] that 3-arc transitive Cayley graphs of any given valency are rare. We have been unable to determine which groups G given in Table 1 have a connected

s-transitive Cayley graph $\Gamma = \mathsf{Cay}(G, S)$ of valency 4 that is not normal. In [4], a similar characterization is given of cubic Cayley graphs of non-Abelian simple groups.

The argument used in the proof of Theorem 1.1 also works for 2 valent Cayley digraphs, extending a result of [11]. More precisely, we have the following corollary.

Corollary 1.2. *Let G be a non-Abelian simple group, and let $\Gamma = \mathsf{Cay}(G, S)$ be a connected digraph of valency* 2. *Then either $G \lhd \mathsf{Aut}\,\Gamma$, or G is one of the groups given in Table* 1.

The characterization of 4 valent Cayley graphs given in Theorem 1.1 provides methods for constructing graphs satisfying certain symmetry properties, for example, half-arc transitivity. Constructing and characterizing half-arc transitive graphs is a currently active topic in algebraic graph theory, see for example [1, 2, 13–18] for references.

Theorem 1.3. *Let G be a non-Abelian simple group which does not occur in Table* 1, *and assume further that $G = \langle x, g\rangle$ where g is an involution. Let $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$, and let $\Gamma = \mathsf{Cay}(G, S)$. Let $A = \mathsf{Aut}\,\Gamma$, and let α be a vertex of Γ. Then $A_\alpha \le \mathrm{D}_8$, and Γ is not* 2*-arc-transitive. Further, Γ is half-arc transitive if and only if there exists no involution in $\mathbf{C}_G(g)$ which interchanges x and x^{-1} by conjugation.*

With this result, we can easily construct various half-arc transitive graphs. Here we give an example to illustrate the method. Let $G = \mathrm{BM}$, the Baby–Monster simple group. By [20] and the Atlas [3], it is easily shown that there exists an involution g and an element x of order 47 such that $G = \langle x, g\rangle$ and x is not conjugate to x^{-1}. Then, letting $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$, $\mathsf{Cay}(G, S)$ is half-arc transitive. By the Atlas [3], several other sporadic simple groups have this property, and so give rise to half-arc transitive graphs. Generally, we have the following corollary.

Corollary 1.4. *Let G be a non-Abelian simple group which does not occur in Table* 1. *Assume that x is an element of G which is not conjugate in $\mathsf{Aut}(G)$ to x^{-1}. Let $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$, and let $\Gamma = \mathsf{Cay}(G, S)$. Let $A = \mathsf{Aut}\,\Gamma$, and let α be a vertex of Γ. If there exists an involution g such that $G = \langle x, g\rangle$, then $A_\alpha \cong \mathbb{Z}_2$, and Γ is a half-arc transitive graph.*

It is known that an element x of order 4 in a Suzuki simple group $\mathrm{Sz}(q)$ is not conjugate in $\mathsf{Aut}(\mathrm{Sz}(q))$ to x^{-1}, and that there exists an involution g such that $\langle x, g\rangle = \mathrm{Sz}(q)$, see [21]. Thus by Corollary 1.4, $\{x, x^{-1}, x^g, (x^{-1})^g\}$ gives rise to a half-arc transitive graph, and the family of Suzuki simple groups $\mathrm{Sz}(q)$ gives infinitely many half-arc transitive Cayley graphs of valency 4. Many other half-arc transitive Cayley graphs of $\mathrm{Sz}(q)$ of valency 4 may be constructed, and a characterization is given of such graphs, in the next theorem.

Theorem 1.5. *Let $G = \mathrm{Sz}(q)$, and let x be an element of G of order greater than* 2. *Then there exists an involution $g \in G$ such that $\Gamma = \mathsf{Cay}(G, S)$ is a half-arc transitive graph, where $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$. Conversely, all half-arc transitive Cayley graphs of G of valency* 4 *may be constructed in this way.*

We will also construct in Proposition 4.1 infinitely many half-arc transitive Cayley graphs from the alternating groups. It is quite likely that most simple groups have connected half-arc transitive Cayley graphs of valency 4. However, the next result shows that not all simple groups have half-arc transitive Cayley graphs of valency 4.

Theorem 1.6. *Let* $G = \mathrm{PSL}_2(2^e)$ *for some* $e \geq 2$. *Then* G *has no half-arc transitive Cayley graphs of valency* 4.

The rest of this paper is organized as follows. After giving some preliminary results in Section 2, we prove Theorem 1.1 in Section 3. Then in Section 4, we construct various half-arc transitive graphs, and in particular, prove Theorems 1.5 and 1.6.

2. Preliminary results

This section collects some results which will be used in the proof of our theorems. The first simple lemma is due to Godsil (1981).

Lemma 2.1 ([7, Lemma 2.1]). *For a Cayley graph* $\Gamma = \mathsf{Cay}(G, S)$, *we have that*

$$\mathbf{N}_{\mathsf{Aut}\,\Gamma}(G) = G : \mathsf{Aut}(G, S).$$

R. Guralnick [8] classified non-Abelian simple groups which contain subgroups of index a power of a prime. A consequence of the classification is the following lemma.

Lemma 2.2. *Let* X *be a non-Abelian simple group which has an insoluble subgroup* G *of index a power of* 2. *Then* $(G, X) = (\mathrm{A}_{2^n-1}, \mathrm{A}_{2^n})$ *for some* $n \geq 3$.

The next lemma gives an upper-bound of the order of the vertex-stabilizer X_α for a $(X, 2)$-arc-transitive graph of valency 4.

Lemma 2.3 (Gardiner [5, 6]). *Let* Γ *be a* $(X, 2)$*-arc-transitive graph of valency* 4. *Then* $|X_\alpha|$ *divides* $2^4 3^6$, *where* α *is a vertex of* Γ.

With the use of a computer, Wang [22] determined simple groups which have subgroups of index dividing $2^6 3^6$. More precisely, we have the following lemma.

Lemma 2.4 (Wang [22]). *Let* X *be a non-Abelian simple group which has a subgroup* G *of index dividing* $2^6 3^6$. *Then* X, G *and* $n := |X : G|$ *are given in the following table:*

T	*G*	*n*	*Remarks*
A_n	A_{n-1}	n	$n \mid 2^6 3^6$
M_{11}	$\mathrm{PSL}_2(11)$	12	
M_{12}	M_{11}	12	*Two classes*
M_{12}	$\mathrm{PSL}_2(11)$	12^2	
M_{24}	M_{23}	24	

Finally, let Γ be a graph such that $G \leq \mathsf{Aut}\,\Gamma$ is transitive on the vertex set $V\Gamma$, and let $N \lhd G$ be such that N is intransitive on $V\Gamma$. The *quotient graph* Γ_N induced by N is defined as the graph such that the set $\mathcal{B}$ of N-orbits in $V\Gamma$ is the vertex set of Γ_N and

$B, C \in \mathcal{B}$ are adjacent if and only if some vertex $u \in B$ is adjacent in Γ to some vertex $v \in C$.

3. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. The proof consists of a series of lemmas. Let $\Gamma = \mathsf{Cay}(G, S)$ be X-edge-transitive, where $G \leq X \leq \mathsf{Aut}\,\Gamma$. The first lemma gives information of the point-stabilizer X_α.

Lemma 3.1. *Let G be a finite group, and let $\Gamma = \mathsf{Cay}(G, S)$ be a connected X-edge-transitive graph of valency 4. Then for a vertex α, either X_α is a 2-group, or $|X_\alpha|$ divides $2^4 3^6$.*

Proof. Since Γ is X-edge-transitive, $X_\alpha^{\Gamma(\alpha)}$ is a half-arc transitive permutation group. Since $|\Gamma(\alpha)| = 4$, $X_\alpha^{\Gamma(\alpha)}$ is a $\{2, 3\}$-group, and Γ is $(X, 2)$-arc-transitive if and only if the order of $X_\alpha^{\Gamma(\alpha)}$ is divisible by 3. It follows since Γ is connected that X_α is a $\{2, 3\}$-group. Therefore, if Γ is not a $(X, 2)$-arc-transitive graph, then $X_\alpha^{\Gamma(\alpha)}$ is a 2-group and so is X_α; while if Γ is $(X, 2)$-arc-transitive, then by Lemma 2.3, X_α has order dividing $2^4 3^6$. $\square$

In the case where G is normal in X, we further have the following result.

Lemma 3.2. *Let G be a finite group, and let $\Gamma = \mathsf{Cay}(G, S)$ be a connected X-edge-transitive graph of valency 4. Assume that $G \lhd X$. Then $X_\alpha = \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2^2$, D_8, A_4, or S_4.*

Proof. Since $G \lhd X$ and Γ is connected, $X_\alpha \leq \mathsf{Aut}(G, S)$ and $\mathsf{Aut}(G, S)$ is faithful on S. Thus X_α is a half-arc transitive permutation group of degree 4. The conclusion of the lemma then follows. $\square$

A permutation group G on a set Ω is said to be *quasiprimitive* if each nontrivial normal subgroup of G is transitive on Ω. The next lemma treats the case where G is simple and X is quasiprimitive.

Lemma 3.3. *Let G be a non-Abelian simple group, and let $\Gamma = \mathsf{Cay}(G, S)$ be connected and X-edge-transitive of valency 4, where $G < X \leq \mathsf{Aut}\,\Gamma$. Assume that X is quasiprimitive on $V\Gamma$, and assume that G is not normal in X. Then G is isomorphic to one of the following groups: A_{2^n-1} with $n \geq 3$, $\mathrm{PSL}_2(11)$, M_{11}, M_{23}, or A_m with $m + 1$ dividing $2^4 3^6$.*

Proof. Let N be a minimal normal subgroup of X. Then $N = T_1 \times \cdots \times T_m \cong T^m$, where $m \geq 1$ and $T_i \cong T$ is simple. Since X is quasiprimitive on $V\Gamma$, N is transitive on $V\Gamma$. Thus, as $|V\Gamma| = |G|$ is not a power of a prime, T is non-Abelian. Since G is simple, $G \cap N = 1$ or G, and it then follows as G is non-Abelian simple and $|N_\alpha| \mid 2^5 3^6$ that $m = 1$ and $G \leq N = T$. If $G = T$, then G is normal in X, which is a contradiction. Assume that $G \neq T$. Then $|T : G|$ divides $|X_\alpha|$, and so either $|T : G|$ is a power of 2, or $|T : G|$ divides $2^4 3^6$. In the case where $|T : G|$ is a power of 2, by Lemma 2.2, $(G, T) = (\mathrm{A}_{2^n-1}, \mathrm{A}_{2^n})$ for some $n \geq 3$; while in the case where $|T : G|$ divides $2^4 3^6$, by Lemma 2.4, $(G, T) = (\mathrm{PSL}_2(11), \mathrm{M}_{11})$, $(\mathrm{M}_{11}, \mathrm{M}_{12})$, $(\mathrm{PSL}_2(11), \mathrm{M}_{12})$, $(\mathrm{M}_{23}, \mathrm{M}_{24})$, or $(\mathrm{A}_m, \mathrm{A}_{m+1})$ with $m+1$ dividing $2^4 3^6$. $\square$

Now we consider the case where X is not quasiprimitive on $V\Gamma$ by the next lemma.

Lemma 3.4. *Let G be a non-Abelian simple group, and let $\Gamma = \mathsf{Cay}(G, S)$ be connected. Assume that $G < X \le \mathsf{Aut}\,\Gamma$, and that X has a maximal normal subgroup N which is semiregular and intransitive on $V\Gamma$ and has order a power of 2. Then either $G \lhd X$, or G occurs in Table 1.*

Proof. Assume first that G centralizes N. Since N is an intransitive maximal normal subgroup, X/N is quasiprimitive on $V\Gamma_N$. It then follows that X/N is almost simple, that is, $T \le X/N \le \mathsf{Aut}(T)$ for some non-Abelian simple group T. Then $G \cong GN/N \le T$. Since $|X : G|$ is a power of 2, $|T : GN/N|$ is a power of 2. If $T \cong G$, it follows since G centralizes N that G is normal in X. Suppose that $T \ncong G$. Then T contains a simple group which is isomorphic to G and has index a power of 2. By Lemma 2.2, $G \cong \mathrm{A}_{2^n-1}$ and $T \cong \mathrm{A}_{2^n}$.

Assume now that G does not centralize N. Let $|G|_2$ denote the 2-part of $|G|$, that is the largest power of 2 dividing $|G|$. Since N is a semiregular 2-group, $|N| \le |G|_2$. Take a series of subgroups of N:

$$N = N_1 > N_2 > \cdots > N_k > N_{k+1} = 1,$$

such that $N_i \trianglelefteq X$ and N_i/N_{i+1} is elementary Abelian for every i. Assume $|N_i/N_{i+1}| = 2^{d_i}$. Consider the actions of G on N_i/N_{i+1} by conjugation. Since G does not centralize N, there exists some i such that the G-action on N_i/N_{i+1} non-trivially, see [9, X, Lemma 1.3]. It gives a d_i-dimensional projective 2-modular representation of G. Let d be the smallest dimension of nontrivial projective 2-modular representations of G.

Suppose that G is a sporadic simple group. By [10, Proposition 5.3.8], since $2^d \mid |G|_2$, we conclude that $G = \mathrm{M}_{12}$, M_{22}, J_2, or Suz. Suppose next that $G = \mathrm{A}_n$. It is clear that A_5 cannot act on $\mathbb{Z}_2^2$, and A_6 cannot act on $\mathbb{Z}_2^3$. By [10, Proposition 5.3.7], either $n \ge 9$ and $d = n-2$, or $(n, d) = (7, 4)$ or $(8, 4)$. It is known that $|\mathrm{A}_n|_2 \le 2^{n-3} < 2^d$, and hence $n \le 8$. For $(n, d) = (7, 4)$, 2^4 does not divide $|\mathrm{A}_7|_2 = 2^3$. Therefore, $G = \mathrm{A}_8 \cong \mathrm{PSL}_4(2)$. Suppose now that G is a simple group of Lie type of characteristic p. In the case where p is odd, it follows from [10, Theorem 5.3.9] that $2^d > |G|_2$, which is a contradiction. Thus $p = 2$. A lower bound for d is given in [10, Table 5.4.C]. Comparing the bound of d and the order of G leads to a proof of the lemma. $\square$

Next we treat the case where X is not quasiprimitive on $V\Gamma$ and Γ is not $(X, 2)$-arc-transitive.

Lemma 3.5. *Let G be a non-Abelian simple group, and let $\Gamma = \mathsf{Cay}(G, S)$ be connected and X-edge-transitive of valency 4, where $G < X \le \mathsf{Aut}\,\Gamma$. Assume that X is not quasiprimitive on $V\Gamma$ and that Γ is not $(X, 2)$-arc-transitive. Then either $G \lhd X$, or G occurs in Table 1.*

Proof. Let N be a maximal intransitive normal subgroup of X. It follows since G is simple that $G \cap N = 1$ and N is a 2-group. Suppose that N is not semiregular on $V\Gamma$. Then

$1 \neq N_\alpha \lhd X_\alpha$, and hence N_α acts on $\Gamma(\alpha)$ non-trivially. Thus N_α acts on $\Gamma(\alpha)$ half-arc transitively, and has orbits of size 2 or 4. It then follows that the valency of the quotient graph Γ_N is equal to 2 or 1, respectively, which is a contradiction since G is non-Abelian simple and $G \leq \mathsf{Aut}\,\Gamma_N$. Therefore, N is semiregular on $V\Gamma$. From Lemma 3.4, the lemma follows. $\square$

The last lemma of this section deals with the case where Γ is $(X, 2)$-arc-transitive.

Lemma 3.6. *Let G be a non-Abelian simple group, and let $\Gamma = \mathsf{Cay}(G, S)$ be connected $(X, 2)$-arc-transitive of valency 4, where $G < X \leq \mathsf{Aut}\,\Gamma$. Assume that X is not quasiprimitive on $V\Gamma$, and assume that G is not normal in X. Then either $G = \mathrm{PSL}_4(2)$, $\mathrm{PSL}_5(2), \mathrm{PSL}_3(9), \mathrm{PSL}_3(27), \mathrm{PSL}_4(3), \mathrm{PSL}_5(3), \mathrm{PSL}_6(3), \mathrm{PSU}_4(3), \mathrm{PSL}_2(11), \mathrm{M}_{11}, \mathrm{M}_{23}$; or $G \cong \mathrm{A}_m$ with $m + 1$ dividing $2^4 3^6$.*

Proof. Let N be a maximal intransitive normal subgroup of X. It follows since G is simple that $G \cap N = 1$, and by Lemma 2.3, N has order dividing $2^4 3^6$. Since $GN/N \cong G$ acts on $V\Gamma_N$, it follows that N is semiregular on $V\Gamma$.

Suppose that G does not centralize N. Let $|G|_r$ denote the r-part of $|G|$ where $r = 2$ or 3. Since N is semiregular on $V\Gamma$, $|N|$ divides $\min(|G|_2|G|_3, 2^4 3^6)$. Take a series of subgroups of N:

$$N = N_1 > N_2 > \cdots > N_s > N_{s+1} = 1,$$

such that $N_i \trianglelefteq X$ and N_i/N_{i+1} is an elementary Abelian r-group for every i, where $r = 2$ or 3. If G does not centralize N, then G is a subgroup of $\mathrm{GL}_5(2)$ or $\mathrm{GL}_6(3)$. All subgroups of $\mathrm{GL}_5(2)$ and $\mathrm{GL}_6(3)$ are known, refer to [10, Chapter 4]. It follows that $G = \mathrm{PSL}_4(2)$, $\mathrm{PSL}_5(2)$, $\mathrm{PSL}_3(9)$, $\mathrm{PSL}_3(27)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSL}_5(3)$, $\mathrm{PSL}_6(3)$, or $\mathrm{PSU}_4(3)$.

Suppose that G centralizes N. Since N is an intransitive maximal normal subgroup, X/N is quasiprimitive on $V\Gamma_N$. It then follows that X/N is an almost simple group, that is, $T \leq X/N \leq \mathsf{Aut}(T)$ with T simple. Then $G \cong GN/N \leq T$. Since $|X : G|$ divides $2^4 3^6$, $|T : GN/N|$ divides $2^4 3^6$. If $T \cong G$, it follows since G centralizes N that G is normal in X, which is a contradiction. Thus T contains a simple group which is isomorphic to G and has index dividing $2^4 3^6$. Hence, by Lemma 2.2, we have $G \cong \mathrm{PSL}_2(11)$, M_{11}, M_{23}, or A_m with $m + 1$ dividing $2^4 3^6$. $\square$

4. Constructing half-arc transitive graphs

Here we apply Theorem 1.1 in order to construct half-arc transitive graphs. We first prove Theorem 1.3.

Proof of Theorem 1.3. By the assumption, G is normal in $\mathsf{Aut}\,\Gamma$, and hence $\mathsf{Aut}\,\Gamma = G : \mathsf{Aut}(G, S)$. Since $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$ and $\mathsf{Aut}(G, S)$ acts on S by conjugation, if an element of $\mathsf{Aut}(G, S)$ fixes x then it must also fix x^{-1}. Hence $\mathsf{Aut}(G, S)$ is not 2-transitive on S, and Γ is not $(X, 2)$-arc-transitive.

Assume that $\mathsf{Aut}(G, S)$ is transitive on S. It then follows that $\mathsf{Aut}(G, S) = \mathbb{Z}_2^2$ or D_8. Thus there exists an involution $\tau \in \mathsf{Aut}(G, S)$ such that $\tau g = g\tau$, $x^\tau = x^{-1}$, and $(x^g)^\tau = (x^g)^{-1}$.

Assume now that $\tau \in \mathbf{C}_G(g)$ such that τ inverts x. Then $(x^g)^\tau = x^{g\tau} = x^{\tau g} = (x^\tau)^g = (x^{-1})^g$. So $\tau \in \mathsf{Aut}(G, S)$, $\mathsf{Aut}(G, S)$ is transitive on S, and Γ is arc-transitive. $\square$

Next we construct an infinite family of half-arc transitive Cayley graphs of the alternating groups.

Proposition 4.1. *Let $G = \mathrm{A}_n$, where n is odd and $n \neq 2^m - 1$. Then for any element $x \in G$ of order n, there exists an involution $g \in G$ such that $\mathsf{Cay}(G, S)$ is a connected half-arc transitive graph, where $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$.*

Proof. Let $n \geq 5$ be an odd integer such that $n \neq 2^m - 1$, and let $\Omega = \{1, 2, 3, \ldots, n - 1, n\}$. Let $x = (123\ldots n)$, and let $g = (12)(34)$. Then $\langle x, g\rangle = \mathrm{A}_n$. Let $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$. Assume that $\tau \in S_n$ is an involution such that $\langle x, \tau\rangle \cong \mathrm{D}_{2n}$. Then $\tau \in \tau_0^{\langle x\rangle}$, where $\tau_0 = (2, n)(3, n-1)\ldots(\frac{n+1}{2}, \frac{n+3}{2})$. It is easily shown that g centralizes no element of $\tau_0^{\langle x\rangle}$. By Theorem 1.3, $\mathsf{Cay}(G, S)$ is a half-arc transitive graph. $\square$

Now we use Theorem 1.3 to analyze edge-transitive Cayley graphs of Suzuki groups.

Proof of Theorem 1.5. Let $G = \mathrm{Sz}(q)$, where $q = 2^m$ for some odd integer $m \geq 3$. By [21], $|G| = q^2(q^2 + 1)(q - 1)$, and any two Sylow 2-subgroups of G have trivial intersection. Further, let M be a maximal subgroup of G. Then by [21], M is conjugate to one of the following subgroups:

$Q : \mathbb{Z}_{q-1}$ where Q is a Sylow 2-subgroup of G,
$\mathrm{D}_{2(q-1)}$,
$\mathbb{Z}_{q+r+1} : \mathbb{Z}_4$, and $\mathbb{Z}_{q-r+1} : \mathbb{Z}_4$, where $r = 2^{\frac{m+1}{2}}$;
$\mathrm{Sz}(2^l)$, where $l \mid m$ with m/l prime.

It follows that the order of an element of G divides 4, $q - 1$, $q + r + 1$, or $q - r + 1$. We complete the proof by a series of lemmas. The first lemma constructs half-arc transitive graphs based on elements of order 4.

Lemma 4.2. *Let $x \in G$ be of order 4. Then there exists an involution $g \in G$ such that $\mathsf{Cay}(G, S)$ is connected and half-arc transitive, where $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$.*

Proof. By [21], all cyclic subgroups of G of order 4 are conjugate, and hence for the element x, there exists an involution $g \in G$ such that $\langle x, g\rangle = G$. Further, x is not conjugate to x^{-1} in $\mathsf{Aut}(G)$. By Theorem 1.3, the lemma is true. $\square$

The next lemma constructs graphs based on elements of order dividing $q - 1$.

Lemma 4.3. *Let $x \in G$ be such that $o(x) > 1$ divides $q-1$. Then there exists an involution $g \in G$ such that $\mathsf{Cay}(G, S)$ is half-arc transitive, where $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$. Further, if $o(x)$ is divisible by a primitive prime divisor of $2^m - 1$, then $\mathsf{Cay}(G, S)$ is connected.*

Proof. From the the list of maximal subgroups of G, it follows that $\mathbf{N}_G(\langle x\rangle) \cong \mathrm{D}_{2(q-1)}$. Thus there are $q - 1$ involutions τ_i $(i = 1, 2, \ldots, q - 1)$ of G which inverts x, and x normalizes two Sylow 2-subgroups of G. It is known that G has q^2+1 Sylow 2-subgroups. Thus there are Sylow 2-subgroups P of G such that $\tau_i \notin P$ and x does not normalizes P. Let g be an involution of P. Then g centralizes no τ_i. Thus $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$ gives rise to a half-arc transitive Cayley graph of G. If, further, $o(x)$ is divisible by a primitive prime divisor of $2^m - 1$, then $\langle x, g\rangle = G$, and so $\mathsf{Cay}(G, S)$ is connected. □

The last lemma deals with elements of order dividing $q \pm r + 1$.

Lemma 4.4. *Let $x \in G\backslash\{1\}$ be such that $o(x) \mid q \pm r + 1$. Then there exists an involution $g \in G$ such that $\mathsf{Cay}(G, S)$ is half-arc transitive, where $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$. Further, if $o(x) \nmid 2^{2l} + 1$ for any proper factor l of m, then $\mathsf{Cay}(G, S)$ is connected.*

Proof. From the list of maximal subgroups of G, it follows that $\mathbf{N}_G(\langle x\rangle) \cong \mathbb{Z}_{q\pm r+1} : \mathbb{Z}_4$. Thus there are $q \pm r + 11$ involutions τ_i ($i = 1, 2, \ldots, q \pm r + 1$) of G which inverts x, and x normalizes four Sylow 2-subgroups of G. It is known that G has $q^2 + 1$ Sylow 2-subgroups. Thus there are Sylow 2-subgroups P of G such that $\tau_i \notin P$ and x does not normalizes P. Let g be an involution of P. Then g centralizes no τ_i. Thus $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$ gives rise to a half-arc transitive Cayley graph of G. If, further, $o(x) \nmid 2^{2l} + 1$ for any proper factor l of m, then $\langle x, g\rangle = G$, and hence $\mathsf{Cay}(G, S)$ is connected. □

We end the section by proving Theorem 1.6.

Proof of Theorem 1.6. It is known that $G = \mathrm{PSL}_2(2^e)$ has the following properties:

(i) a Sylow 2-subgroup of G is isomorphic to $\mathbb{Z}_2^e$, G has $2^e + 1$ Sylow 2-subgroups P_i for $i \in \{1, 2, \ldots, 2^e + 1\}$, and any two Sylow 2-subgroups are disjoint;
(ii) an element of G is either an involution, or of order dividing $2^e - 1$ or $2^e + 1$.
(iii) G has maximal subgroups isomorphic to $\mathrm{D}_{2(2^e-1)}$ or $\mathrm{D}_{2(2^e+1)}$.

Let $\Gamma = \mathsf{Cay}(G, S)$ be a half-arc transitive graph. Then by Theorem 1.3, $S = \{x, x^{-1}, x^g, (x^{-1})^g\}$ such that $\langle x, g\rangle = G$, g is an involution, and x has order greater than 2. Thus $o(x) \mid 2^e - 1$, or $o(x) \mid 2^e + 1$.

Assume first that $o(x) \mid 2^e - 1$. Then there are $2^e - 1$ involutions τ_i in G inverting x for $i \in \{1, 2, \ldots, 2^e - 1\}$, and x normalizes two 2-Sylow subgroups P_{2^e}, P_{2^e+1} say. If $g \in P_i$ for some $i \leq 2^e - 1$, then $g\tau_i = \tau_i g$, which is a contradiction to Theorem 1.3; while if $g \in P_{2^e} \cup P_{2^e+1}$, then $\langle x, g\rangle < G$, which is a contradiction.

Assume now that $o(x) \mid 2^e + 1$. Then there are $2^e + 1$ involutions τ_i inverting x, where $i \in \{1, 2, \ldots, 2^e + 1\}$. Thus, if $g \in P_i$ then $g\tau_i = \tau_i g$, which is a contradiction. □

Acknowledgements

The authors are grateful to the referee for his constructive suggestions. The first author is supported by SRF for ROCS, SEM, China. The second author is supported by a QEII Research Fellowship and an ARC Discovery Project Grant from Australian Research Council. The third author is supported by Natural Science Foundation of China and the Research Fund for the Doctoral Program of Higher Education.

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