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Linear Algebra and its Applications 375 (2003) 115–134

LINEAR ALGEBRA AND ITS APPLICATIONS

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Model reduction via truncation: an interpolation point of view ☆

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Received 7 March 2003; accepted 27 May 2003

Submitted by V. Mehrmann

Abstract

In this paper, we focus our attention on linear time invariant continuous time linear systems with one input and one output (SISO LTI systems). We consider the problem of constructing a reduced order system via truncation of the original system. Given a SISO strictly proper transfer function $T(s)$ of McMillan degree N and a strictly proper SISO transfer function $\hat{T}(s)$ of McMillan degree $n < N$, we prove that $\hat{T}(s)$ can always be constructed via truncation of the system $T(s)$. The proof is mainly based on interpolation theory, and more precisely on multipoint Padé interpolation. Moreover, new results about Krylov subspaces are developed.

Keywords: Model reduction; Rational interpolation; Modal approximation; Linear time invariant system; Truncation technique; Embedding

1. Introduction

All the matrices considered throughout this paper have coefficients in the field $\mathbb{C}$, or in the polynomial ring $\mathbb{C}[\lambda]$, in which case they are called λ-matrices and denoted

☆ A research fellowship from the Belgian National Fund for Scientific Research is gratefully acknowledged by the second author. This paper presents research supported by the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. This work was also supported by the National Science Foundation under grant no. CCR-9912415.

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0024-3795/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0024-3795(03)00648-7

$A(\lambda)$, $B(\lambda)$, ... We use A, B, ... exclusively for constant matrices. For two scalar polynomials $\alpha(\lambda)$ and $\beta(\lambda)$, the symbol $\alpha(\lambda)|\beta(\lambda)$ means that $\alpha(\lambda)$ divides $\beta(\lambda)$.

Let us consider the following standard state-space model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

with input $u(t) \in \mathbb{C}^m$, state $x(t) \in \mathbb{C}^N$, output $y(t) \in \mathbb{C}^p$, and with system matrices A, B, C, D that belong to respectively $\mathbb{C}^{N\times N}$, $\mathbb{C}^{N\times m}$, $\mathbb{C}^{p\times N}$ and $\mathbb{C}^{p\times m}$. Unless specified differently, we assume here that there is only one input and one output, i.e. $m = p = 1$. Without loss of generality, we assume that the system is controllable and observable since otherwise we can always find a smaller dimensional model that is controllable and observable, and that has exactly the same transfer function. A quadruple of matrices (A, B, C, D) is called a realization of the proper transfer function $T(s)$ if $T(s) = C(sI - A)^{-1}B + D$. A realization (A, B, C, D) of a transfer function $T(s)$ is minimal if and only if the pair (A, C) is observable and the pair (A, B) is controllable. The McMillan degree of $T(s)$ is the dimension N of the matrix A of any minimal realization of the proper transfer function $T(s) = C(sI - A)^{-1}B + D$.

When the system order N is too large for solving various control problems within a reasonable computing time, it is natural to consider approximating it by a reduced order system

$$\begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t), \end{cases} \tag{2}$$

driven with the same input $u(t) \in \mathbb{C}^m$, but having a different output $\hat{y}(t) \in \mathbb{C}^p$ and state $\hat{x}(t) \in \mathbb{C}^n$. The matrix $\hat{A}$ belongs to $\mathbb{C}^{n\times n}$. For the same reasons as above, we will assume that the realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of the reduced order model $\hat{T}(s)$ is minimal. The degree n of the reduced order system is also assumed to be much smaller than the degree N of the original system. The objective of the reduced order model is to reduce the dimension of the state-space (of dimension N) of the system to a lower dimension n in such a way that the “behavior" of the reduced order model is sufficiently close to that of the full order system. For a same input $u(t)$, we thus want $\hat{y}(t)$ to be close to $y(t)$. One shows that in the frequency domain, this is equivalent to imposing conditions on the frequency responses of both systems [1]: we want to find a reduced order model such that the transfer functions of both models, i.e.

$$\begin{aligned} T(s) &= C(sI_N - A)^{-1}B + D, \\ \hat{T}(s) &= \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}, \end{aligned}$$

are such that the error $\|T(\cdot) - \hat{T}(\cdot)\|$ is minimal for the H_∞ norm. A particular way of constructing a reduced order model is the following truncation technique.

Definition 1. The transfer function $\hat{T}(s) \doteq \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}$ of McMillan degree n, with m inputs and p outputs, is constructed via truncation of the transfer function $T(s) = C(sI_N - A)^{-1}B + D$ (with m inputs and p outputs) of McMillan degree N if and only if there exist projecting matrices $Z, V \in \mathbb{C}^{N\times n}$ such that $Z^{\mathrm{T}}V = I_n$ and

$$\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\} = \{Z^{\mathrm{T}}AV, Z^{\mathrm{T}}B, CV, D\}. \tag{3}$$

If we change the state-space coordinate basis of the system (1) by choosing

$$\tilde{x} = Sx,$$

with the matrix $S \in \mathbb{C}^{N\times N}$ invertible, the system (1) is equivalent to the system

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \\ y = \tilde{C}\tilde{x} + \tilde{D}u, \end{cases} \tag{4}$$

where

$$\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\} = \{SAS^{-1}, SB, CS^{-1}, D\}. \tag{5}$$

Because the matrix D does not depend on the dimension N of the state-space, it does not play any role in the model reduction framework. From now on, we therefore assume that $D = \hat{D} = 0$. A rational transfer function $T(s)$ is called strictly proper when $\lim_{s\to\infty} T(s) = 0$, i.e. when $D = 0$. A triple of matrices (A, B, C) is called a realization of the strictly proper transfer function $T(s)$ when $T(s) = C(sI - A)^{-1}B$.

It can be shown that the reduced-order system $\hat{T}(s)$ can be constructed via the truncation technique from $T(s)$ if and only if there exists a state-space coordinate basis in which the matrices of the original system $T(s)$ are (see for instance [2] for a proof)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{6}$$

and the matrices of the reduced-order model are taken to be

$$\hat{A} = A_{11}, \quad \hat{B} = B_1, \quad \hat{C} = C_1.$$

For instance, from Eqs. (3) and (5), we can choose the projecting matrices $Z^T \in \mathbb{C}^{n\times N}$ and $V \in \mathbb{C}^{N\times n}$ to be respectively the first n rows of S and the first n columns of S^{-1}, where S is the coordinate basis change that put the original system in the form (6).

It is known that several existing model reduction techniques, such as modal truncation [3], balanced truncation [1], ADI method [4], and multipoint Padé interpolation [5], use a truncation technique to construct the reduced order transfer function. The purpose of this paper is to show that *every* strictly proper SISO transfer function of McMillan degree n can be constructed by truncation of *any* strictly proper SISO transfer function of McMillan degree $N > n$. Moreover, we give an explicit way of constructing the projecting matrices Z and V. The proof is based on the multipoint Padé technique discussed in the SISO case in [5,6]. In the MIMO case, we refer to [7,8] for first results in this direction.

The contents of this paper are as follows. In Section 2, some general results are given about Krylov subspaces and embedding of matrices. In Section 3, we solve the problem of finding a reduced order model that interpolates the original model at some given frequencies, up to a given order. A first solution to this problem is already known as the multipoint Padé method (see [5,9]), but we think that our derivation of the solution gives new insights in the problem. This will permit us, in Section 4, to solve the problem of finding projectors Z and V such that a reduced order transfer function $\hat{T}(s)$ can be constructed via truncation of an original transfer function $T(s)$. Some concluding remarks are developed in Section 5.

2. Some general results

A necessary condition for a transfer function $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ to be obtained by truncation of $T(s) = C(sI - A)^{-1}B$ is that the matrix A is equivalent to a matrix having $\hat{A}$ as a submatrix. This problem of embedding of matrices has already been solved and the main results are recalled in Section 2.1. In Section 2.2 we prove some results about Krylov subspaces that will be useful in the sequel. We first need the following definition.

Definition 2. The invariant polynomials of a square matrix $A \in \mathbb{C}^{N\times N}$ are the polynomials $h_1(A), \ldots, h_N(A)$ appearing on the diagonal of the canonical Smith form of the polynomial matrix $sI_N - A$ and satisfying $h_1(A)|\cdots|h_N(A)$.

2.1. Embedding of polynomial matrices

We say that $A(\lambda)$ is λ-embeddable in $B(\lambda)$ whenever $B(\lambda)$ is equivalent to a λ-matrix having $A(\lambda)$ as a submatrix. The following theorem has been proved independently by [10,11].

Theorem 3. *Let the matrices $A \in \mathbb{C}^{N\times N}$ and $\hat{A} \in \mathbb{C}^{n\times n}$ with invariant polynomials (including trivial invariant polynomials)*

$$h_1(A)|\cdots|h_N(A), \quad h_1(\hat{A})|\cdots|h_n(\hat{A}),$$

with the following convention

$$h_{n+1}(\hat{A}) = h_{n+2}(\hat{A}) = \cdots = h_N(\hat{A}) = 0.$$

Then $\hat{A}$ is a principal submatrix of some similarity transform of A if and only if the two following relations hold:

$$\begin{array}{ccc} h_1(A) & | \quad h_1(\hat{A}) & | \; h_{1+2(N-n)}(A), \\ h_2(A) & | \quad h_2(\hat{A}) & | \; h_{2+2(N-n)}(A), \\ & \vdots & \\ h_{2n-N}(A) & | \; h_{2n-N}(\hat{A}) \; | & h_N(A), \end{array}$$

and

$$\text{degree}\,(h_1(A)\cdots h_N(A)) = N, \quad \text{degree}\left(h_1(\hat{A})\cdots h_n(\hat{A})\right) = n.$$

Let us consider a minimal transfer function $T(s) = C(sI - A)^{-1}B$ of McMillan degree N with m inputs and p outputs. The observability and controllability of $T(s)$ implies that $h_1(A) = \cdots = h_{N-i}(A) = 1$, where $i = \min(m, p)$.

Let (A, B, C) be a minimal realization of the strictly proper SISO transfer function $T(s)$ of McMillan degree N. From the minimality assumption of (A, B, C), only one invariant polynomial of $sI_N - A$ can be different from 1. Let $(\hat{A}, \hat{B}, \hat{C})$ be a minimal realization of the strictly proper SISO transfer function $\hat{T}(s)$ of McMillan degree $n < N$. For the same reason as above, $sI_n - \hat{A}$ must have only one invariant polynomial different from 1. A consequence of Theorem 3 is that there exists always a state-space realization of $T(s)$, say (A, B, C), such that $\hat{A}$ is a principal submatrix of A. Indeed, the conditions of Theorem 3 are trivially satisfied. It remains to prove in the SISO case that there exists a state-space realization (A, B, C) of $T(s)$ such that not only $\hat{A}$ is a principal submatrix of A, but also that $\hat{C}$ is a submatrix of C and $\hat{B}$ of B. This is what we prove in this paper. On the other hand, in the MIMO case, the conditions of Theorem 3 may not be satisfied. This more general case will not be treated here.

2.2. *Some facts about Krylov subspaces*

Most of the results of this section are very close to those developed in [12]. Let $T(s) = C(sI - A)^{-1}B$ be a continuous time, linear time invariant transfer function of McMillan degree N, with one input and one output.

Definition 4. For any matrix $X \in \mathbb{C}^{N\times k}$, the subspace $\mathrm{Im}(X)$ is defined to be the linear subspace spanned by the columns of X. In order to keep our notation consistent, we will use this definition also in the case X is a single vector. We define the Krylov subspace of order $k \in \mathbb{N}_0$, written $\mathscr{K}_k(A, B)$, as

$$\mathscr{K}_k(A, B) = \mathrm{Im}([B, AB, \ldots, A^{k-1}B]).$$

If $k \leqslant 0$, then we define

$$\mathscr{K}_k(A, B) = \{0\}.$$

Two well-known matrices of a SISO transfer function $T(s) = C(sI - A)^{-1}B$ with $A \in \mathbb{C}^{N\times N}$ are the controllability matrix $\mathrm{Contr}(A, B) \in \mathbb{C}^{N\times N}$ and the observability matrix $\mathrm{Obs}(A, C) \in \mathbb{C}^{N\times N}$ defined by

$$\mathrm{Contr}(A, B) \doteq [B, \ldots, A^{N-1}B], \quad \mathrm{Obs}(A, C) \doteq \begin{bmatrix} C \\ \vdots \\ CA^{N-1} \end{bmatrix}.$$

Lemma 5. *Consider an arbitrary pair of matrices* (A, B) *with* $A \in \mathbb{C}^{N\times N}$ *and* $B \in \mathbb{C}^{N\times 1}$. *Consider* N *polynomials of degree at most* $N-1$,

$$\phi_j(x) = \sum_{i=0}^{N-1} \alpha_{i,j} x^i, \quad 1 \leqslant j \leqslant N.$$

Define the matrix $M \in \mathbb{R}^{N\times N}$ *such that*

$$M(i, j) = \alpha_{i-1,j}, \quad 1 \leqslant i, j \leqslant N.$$

If M *is invertible* (*i.e. if the polynomials are independent*), *then*

$$\mathrm{Im}\,([\phi_1(A)B, \ldots, \phi_N(A)B]) = \mathscr{K}_N(A, B).$$

Proof. Because the functions $\phi_j(x)$ are polynomial and of finite degree, they are analytic in a neighborhood of the spectrum of A and the functions $\phi_j(A)$ are well defined. The proof of the lemma now follows from the following equation

$$[\phi_1(A)B, \ldots, \phi_N(A)B] = [B, \ldots, A^{N-1}B]M,$$

and the fact that M is invertible. □

Remark 6. By Cayley–Hamilton and by considering the Jordan canonical form of the matrix $A \in \mathbb{C}^{N\times N}$, it is well known that any function $\phi(\cdot)$ analytic in a neighborhood of the spectrum of A, denoted by $\Lambda(A)$, can be written as a polynomial function of A of degree $N-1$. Hence, the polynomial form of the functions ϕ_i is quite general.

This leads us to the following definition.

Definition 7. Let A be a square matrix of dimension N, let $\phi(\cdot)$ be a function analytic in a neighborhood of the spectrum of A, the polynomial function of minimal degree, $r(\cdot)$ (obtained via Cayley–Hamilton), such that the matrices $r(A)$ and $\phi(A)$ are equal, is called the interpolating polynomial of $\phi(\cdot)$ with respect to the matrix $A \in \mathbb{C}^{N\times N}$.

Lemma 8. *Consider an arbitrary pair of matrices* (A, B) *with* $A \in \mathbb{C}^{N\times N}$ *and* $B \in \mathbb{C}^{N\times 1}$. *Let* $\phi(\cdot)$ *be any function such that the matrix* $\phi(A) \in \mathbb{C}^{N\times N}$ *is invertible. Then*

$$\phi(A)\mathscr{K}_N(A, B) = \mathscr{K}_N(A, B).$$

Proof. By Cayley–Hamilton,

$$\phi(A)\mathscr{K}_N(A,B) = r(A)\mathscr{K}_N(A,B) \subset \mathscr{K}_N(A,B),$$

where $r(A)$ is the interpolating polynomial of $\phi(A)$. By invertibility of $\phi(A)$,

$$\dim(\phi(A)\mathscr{K}_N(A,B)) = \dim(\mathscr{K}_N(A,B)).$$

Equality of the two subspaces follows. □

Definition 9. An *interpolation set I*

$$I = \{(s_1, m_1), \ldots, (s_r, m_r)\},$$

is defined as a set of couples (s_i, m_i) where the points $s_i \in \mathbb{C} \cup \infty$ are distinct and the indices $m_i \in \mathbb{N}_0$. The size of the interpolation set I, denoted by $s(I)$ is defined as

$$s(I) = \sum_{i=1}^{r} m_i.$$

An interpolation set I is called an *$T(s)$-admissible interpolation set* when no interpolation point s_i is a pole of $T(s)$. A *minimal $T(s)$-admissible interpolation set* is a $T(s)$-admissible interpolation set of size N, where N is the McMillan degree of $T(s)$.

Definition 10. A couple of $T(s)$-admissible interpolation sets (I_1, I_2), denoted by

$$I_1 = \{(z_1, \mu_1), \ldots, (z_{r_1}, \mu_{r_1})\}, \quad I_2 = \{(w_1, \nu_1), \ldots, (w_{r_2}, \nu_{r_2})\},$$

is called a *separation* of I if the set of points of I is the union of those of I_1 and I_2 and if their corresponding indices add up. By that, we mean that for each couple $(s_k, m_k) \in I$ belonging to I_1 and I_2 we have

$$z_i = w_j = s_k \Rightarrow \mu_i + \nu_j = m_k,$$

and for each couple $(s_k, m_k) \in I$ belonging to only one set I_1 or I_2, we have (e.g. for I_1)

$$z_i = s_k \Rightarrow \mu_i = m_k.$$

As a consequence, we have

$$s(I_1) + s(I_2) = s(I).$$

A separation (I_1, I_2) is called *symmetric* when $s(I_1) = s(I_2)$.

The quantities occurring in $\mathrm{Contr}(A, B)$ and $\mathrm{Obs}(A, C)$

$$\gamma_{A,B}(\infty, k) \doteq A^{k-1}B, \quad \delta_{A,C}(\infty, k) \doteq CA^{k-1} \tag{7}$$

can be seen as "moments" of $(sI - A)^{-1}B$ and $C(sI - A)^{-1}$ about infinity. Similarly, we define the moments about a finite expansion point $\lambda \in \mathbb{C}$

$$\gamma_{A,B}(\lambda, k) \doteq (\lambda I - A)^{-k}B, \quad \delta_{A,C}(\lambda, k) \doteq C(\lambda I - A)^{-k}. \tag{8}$$

Definition 11. Let I be a $T(s)$-admissible interpolation set. For any state-space realization (A, B, C) of $T(s)$, we define the generalized controllability matrix $\mathscr{C}_{A,B}$ to be

$$\mathscr{C}_{A,B}(I) \doteq [\gamma(s_1, 1), \gamma(s_1, 2), \ldots, \gamma(s_1, m_1), \gamma(s_2, 1), \ldots, \gamma(s_r, m_r)],$$

and generalized observability matrix to be

$$\mathscr{O}_{A,C}(I) \doteq \begin{bmatrix} \delta(s_1, 1) \\ \delta(s_1, 2) \\ \vdots \\ \delta(s_1, m_1) \\ \delta(s_2, 1) \\ \vdots \\ \delta(s_r, m_r) \end{bmatrix}. \tag{9}$$

Let us introduce a final notation. Let (A, B) be a pair of matrices with $A \in \mathbb{C}^{N\times N}$ and $B \in \mathbb{C}^{N\times 1}$. If $s_i \neq \infty$ is not an eigenvalue of A, then define the matrix $A_i \in \mathbb{C}^{N\times N}$ by

$$A_i = (s_i I - A)^{-1}, \quad B_i = (s_i I - A)^{-1} B.$$

If $s_i = \infty$, then define

$$A_i = A, \quad B_i = B.$$

The following lemma is a straightforward consequence of the partial fraction expansion of a rational matrix. It will prove to be useful in the sequel.

Lemma 12. *Consider an arbitrary pair of matrices (A, B) with $A \in \mathbb{C}^{N\times N}$ and $B \in \mathbb{C}^{N\times 1}$. Let i and j be two non-negative integers such that $i + j \geqslant 1$.*

(1) *If $s_1 \neq \infty$, $s_2 \neq \infty$ and $s_1 \neq s_2$, then*

$$\operatorname{Im}(A_1^i A_2^j B) \subset \mathscr{K}_i(A_1, B_1) + \mathscr{K}_j(A_2, B_2). \tag{10}$$

(2) *If $s_1 \neq \infty$ and $s_2 = \infty$, then*

$$\operatorname{Im}(A_1^i A_2^j B) \subset \mathscr{K}_i(A_1, B_1) + \mathscr{K}_{j-i+1}(A_2, B_2). \tag{11}$$

(3) (a) *If $s_1 = s_2 \neq \infty$ then,*

$$\operatorname{Im}(A_1^i A_2^j B) \subset \mathscr{K}_{i+j}(A_1, B_1). \tag{12}$$

(b) *If $s_1 = s_2 = \infty$ then,*

$$\operatorname{Im}(A_1^i A_2^j B) \subset \mathscr{K}_{i+j+1}(A, B). \tag{13}$$

Proof. The third part of the lemma is obvious. Let us prove the two first parts.

First, we suppose that $s_1 \neq s_2$ and that s_1 and s_2 are both different from ∞. We obtain by partial fraction expansion the identity

$$(s_1 I - A)^{-1}(s_2 I - A)^{-1} = (s_1 I - A)^{-1}\frac{1}{s_2 - s_1} + (s_2 I - A)^{-1}\frac{1}{s_1 - s_2}. \tag{14}$$

By recursively applying this equation, we find that

$$\begin{aligned}(s_1 I - A)^{-i}(s_2 I - A)^{-j} &= \frac{1}{s_2 - s_1}(s_1 I - A)^{-i}(s_2 I - A)^{-j+1} \\ &\quad + \frac{1}{s_1 - s_2}(s_2 I - A)^{-i+1}(s_2 I - A)^{-j} &(15)\\ &= \sum_{k=1}^{i}\alpha_k (s_1 I - A)^{-k} + \sum_{l=1}^{j}\beta_l (s_2 I - A)^{-l}, &(16)\end{aligned}$$

where the last equation is obtained by recursively applying Eq. (15). The coefficients α_i and β_j are not explicitly given here. The important point is that they depend only on the points s_i and s_j, i.e. they are the same for any matrix A. Moreover, it is clear that the coefficients related to the highest moments of the partial fraction expansion of $(s_1 I - A)^{-i}(s_2 I - A)^{-j}$, i.e. α_i and β_j, are different from zero. Multiply both sides of Eq. (16) by B, and Eq. (10) is satisfied.

Secondly, suppose that $s_2 = \infty$. Then,

$$(s_1 I - A)^{-1}A = -I + s_1(s_1 I - A)^{-1}. \tag{17}$$

By recursively applying this equation to $(s_1 I - A)^{-i}A^j$ and following the same reasoning as before, we find that

$$\mathrm{Im}((s_1 I - A)^{-i}A^j B) \subset \mathscr{K}_i(A_1, B_1) \quad \text{if } i > j, \tag{18}$$

$$\mathrm{Im}((s_1 I - A)^{-i}A^j B) \subset \mathscr{K}_i(A_1, B_1) + \mathscr{K}_{j-i+1}(A, B) \quad \text{if } i \leqslant j. \tag{19}$$

Hence, Eq. (11) is satisfied. □

Another proof of the following lemma may be found in [13].

Lemma 13. *Let $T(s)$ be a strictly proper SISO LTI transfer function of McMillan degree N with a state-space realization $T(s) = C(sI - A)^{-1}B$. Let*

$$I = \{(s_1, m_1), \ldots, (s_r, m_r)\},$$

be a minimal $T(s)$-admissible interpolation set. Then

(1) $\mathrm{Im}\left(\mathscr{C}_{A,B}(I)\right) = \mathrm{Im}\,(\mathrm{Contr}(A, B))$.
(2) $\mathrm{Ker}\left(\mathscr{O}_{A,C}(I)\right) = \mathrm{Ker}\,(\mathrm{Obs}(A, C))$.

Proof. In the sequel, we drop the subscripts A, B, C. We prove only the first statement, the second one follows by transposition. For simplicity, we suppose that there is no point at infinity. This case can be treated similarly but with more tedious notation. The proof consists of showing that the condition of Lemma 5 is satisfied.

From the set I, define $\forall 1 \leqslant i \leqslant m_i$

$$\tilde{\gamma}(1, i) \doteq \gamma(s_1, i),$$

where $\gamma(\lambda, k)$ is defined in Eqs. (8) and (7).

Define $\forall 2 \leqslant i \leqslant r \; \forall 1 \leqslant k \leqslant m_i$,

$$\tilde{\gamma}(i, k) \doteq \left(\prod_{j=1}^{i-1} A_j^{m_j}\right) \gamma(s_i, k).$$

Define the matrix

$$\tilde{\mathscr{C}}(I) = [\tilde{\gamma}(1, 1), \tilde{\gamma}(1, 2), \ldots, \tilde{\gamma}(r, m_r)].$$

As a consequence of Lemma 12, we obtain

$$\operatorname{Im}\left(\tilde{\mathscr{C}}(I)\right) = \operatorname{Im}\left(\mathscr{C}(I)\right).$$

Now, we use Lemmas 5 and 8. The matrix

$$N = \prod_{i=1}^{r} (A_i)^{-m_i} \tilde{\mathscr{C}}(I),$$

satisfies the condition of Lemma 5 because every column is a polynomial function of A of a different order, with degree smaller than N. Hence, $\operatorname{Im}(N) = \operatorname{Im}(\operatorname{Contr}(A, B))$. By Lemma 8, $\operatorname{Im}(N) = \operatorname{Im}(\mathscr{C}(I))$. This concludes the proof. □

Lemma 14. *Consider an arbitrary pair of matrices (A, B) with $A \in \mathbb{C}^{N \times N}$ and $B \in \mathbb{C}^{N \times 1}$. Let $\mathscr{X}$ be a right invariant subspace of A. If*

$$\operatorname{Im}(A^i B) \subset \mathscr{K}_i(A, B) + \mathscr{X}, \tag{20}$$

then, $\forall k \in \mathbb{N}$,

$$\operatorname{Im}(A^{i+k} B) \subset \mathscr{K}_i(A, B) + \mathscr{X}.$$

Proof. Let us prove it for $k = 1$.

$$\begin{aligned}
\operatorname{Im}(A^{i+1} B) &= A \operatorname{Im}(A^i B) \\
&\subset A\mathscr{K}_i(A, B) + A\mathscr{X} \\
&\subset \mathscr{K}_i(A, B) + \operatorname{Im}(A^i B) + \mathscr{X} \\
&= \mathscr{K}_i(A, B) + \mathscr{X}.
\end{aligned}$$

An easy induction will complete the proof. □

Lemma 15. *Let $T(s)$ be a strictly proper SISO transfer function of McMillan degree N. Let $\mathscr{X}$ (resp. $\mathscr{Y}$) be a right (resp. left) invariant subspace of A of dimension K. Let the columns of the matrix $X \in \mathbb{C}^{N\times K}$ (resp. $Y \in \mathbb{C}^{K\times N}$) be a basis of $\mathscr{X}$ (resp. $\mathscr{Y}$). Let I be a $T(s)$-admissible interpolation set of size $N-K$, denoted again by*

$$I = \{(s_1, m_1), \ldots, (s_r, m_r)\}.$$

Let the triple (A, B, C) be a minimal realization of $T(s)$. Then

(1) $\operatorname{rank}\left(\begin{bmatrix} X & \mathscr{C}_{A,B}(I)\end{bmatrix}\right) = N.$

(2) $\operatorname{rank}\left(\begin{bmatrix} Y \\ \mathscr{O}_{A,C}(I)\end{bmatrix}\right) = N.$

Proof. Only the first part of the lemma will be proved, the second one follows by transposition. As a consequence of Lemma 13,

$$\operatorname{rank}\left(\mathscr{C}_{A,B}(I)\right) = N-K.$$

Indeed, the $T(s)$-admissible interpolation set I may be seen as a subset of a minimal $T(s)$-admissible interpolation set of $T(s)$. Hence, the columns of $\mathscr{C}_{A,B}(I)$ must be linearly independent. Let us consider the first column of $\mathscr{C}_{A,B}(I)$. The matrices A_i and B_i associated with the point s_i are defined as usual. Suppose that

$$\begin{aligned}&\dim\left(\mathscr{X} + \mathscr{K}_{m_1}(A_1, B_1) + \cdots + \mathscr{K}_{m_r}(A_r, B_r)\right)\\ &\quad = q < K + m_1 + \cdots + m_r = N.\end{aligned}$$

Then, necessarily, $\exists 1 \leqslant p \leqslant r$ and $0 \leqslant k_p \leqslant m_p - 1$ such that

$$\begin{aligned}\operatorname{Im}(A_p^{k_p} B_p) \subset\, & \mathscr{X} + \mathscr{K}_{m_1}(A_1, B_1) + \cdots + \mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1})\\ & + \mathscr{K}_{k_p}(A_p, B_p).\end{aligned} \tag{21}$$

Some care must be taken when $s_p = \infty$. Firstly, suppose that $s_p = \infty$ and $k_p = 0$. Then, multiply both sides of Eq. (21) by A. From Lemmas 12 and 14, and Eqs. (18) to (19), we obtain the following relations:

$$\begin{aligned}\operatorname{Im}(AB) &\subset A\mathscr{X} + A\mathscr{K}_{m_1}(A_1, B_1) + \cdots + A\mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1})\\ &\subset \mathscr{X} + \mathscr{K}_{m_1}(A_1, B_1) + \cdots + \mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}) + \operatorname{Im}(B)\\ &= \mathscr{X} + \mathscr{K}_{m_1}(A_1, B_1) + \cdots + \mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}),\end{aligned}$$

where the last equation comes from Eq. (21) with $s_p = \infty$ and $k_p = 0$. But this implies that

$$\dim\left(\mathscr{K}_N(A, B)\right) \leqslant q < N.$$

This contradicts the fact that the pair (A, B) is controllable.

If $s_p = \infty$ and $k_p > 0$, then

$$\begin{aligned}\mathrm{Im}(A^{k_p+1}B) &\subset A\mathscr{X} + A\mathscr{K}_{m_1}(A_1, B_1) + \cdots + A\mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}) \\ &\quad + A\mathscr{K}_{k_p}(A, B) \\ &\subset \mathscr{X} + \mathscr{K}_{m_1}(A_1, B_1) + \cdots + \mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}) \\ &\quad + \mathscr{K}_{k_p}(A_p, B_p) + \mathrm{Im}(A^{k_p}B) \\ &= \mathscr{X} + \mathscr{K}_{m_1}(A_1, B_1) + \cdots + \mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}) \\ &\quad + \mathscr{K}_{k_p}(A_p, B_p),\end{aligned}$$

and again, the transfer function $T(s)$ is not of McMillan degree N.

Suppose now that $\forall 1 \leqslant i \leqslant p$, $s_i \neq \infty$, and multiply again both sides of Eq. (21) by A_p. From Lemmas 12 and 14, we find that

$$\begin{aligned}\mathrm{Im}(A_p^{k_p+1}B_p) &\subset \mathscr{X} + \mathscr{K}_{m_1}(A_1, B_1) + \cdots + \mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}) \\ &\quad + \mathscr{K}_{k_p}(A_p, B_p).\end{aligned} \tag{22}$$

This implies that

$$\dim\left(\mathscr{K}_N(A_p, B_p)\right) \leqslant q < N.$$

But, from Lemma 15, $\dim\left(\mathscr{K}_N(A_p, B_p)\right) = N$. This is impossible.

Finally, suppose that $\exists 1 \leqslant i \leqslant p$ such that $s_i = \infty$. From our previous discussion, $i < p$. For simplicity, suppose that $s_1 = \infty$. In such a case, by following the same reasoning as before,

$$\begin{aligned}\mathrm{Im}(A_p^{k_p+1}B_p) &\subset A_p\mathscr{X} + \mathscr{K}_{m_1}(A, B) + A_p\mathscr{K}_{m_2}(A_2, B_2) + \cdots \\ &\quad + A_p\mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}) + \mathscr{K}_{k_p}(A_p, B_p) + \mathrm{Im}(A_p^{k_p}B_p) \\ &\subset \mathscr{X} + \mathscr{K}_{m_1}(A, B) + \mathscr{K}_{m_2}(A_2, B_2) + \cdots \\ &\quad + \mathscr{K}_{m_{p-1}}(A_{p-1}, B_{p-1}) + \mathscr{K}_{k_p}(A_p, B_p).\end{aligned}$$

This is again a contradiction with the controllability of the pair (A, B). $\square$

3. Model reduction via rational interpolation

If $T(s)$ and $\hat{T}(s)$ are both strictly proper transfer functions, necessarily,

$$T(\infty) = \hat{T}(\infty).$$

As explained in the introduction, in the model reduction framework, we generally suppose that the original transfer function and the reduced order transfer function are both strictly proper. This leads to the following definition.

Definition 16. Let $T(s)$ be a strictly proper SISO transfer function of McMillan degree N. Let $\hat{T}(s)$ be a strictly proper SISO transfer function of McMillan degree n. We are given one $T(s)$-admissible interpolation set I of size $2n$, denoted by

$$I = \{(s_1, m_1), \ldots, (s_r, m_r)\}.$$

We say that $T(s)$ interpolates $\hat{T}(s)$ at I when the following conditions are satisfied:

(1) $\forall 1 \leqslant i \leqslant r$ such that $s_i \neq \infty$,

$$T(s) - \hat{T}(s) = \mathrm{O}(s - s_i)^{m_i}. \tag{23}$$

(2) If ∞ is not a point of I, then

$$\lim_{s\to\infty} \Big(T(s) - \hat{T}(s)\Big) = 0. \tag{24}$$

(3) If ∞ is a point of I, say $s_k = \infty$, then

$$\lim_{s\to\infty} \Big(T(s) - \hat{T}(s)\Big) s^{m_k} = 0. \tag{25}$$

Let us consider a minimal realization (A, B, C) of $T(s)$ and a minimal realization $(\hat{A}, \hat{B}, \hat{C})$ of the reduced order transfer function $\hat{T}(s)$. Writing Eq. (23) is equivalent to imposing the m_i first coefficients of the Taylor expansions of $\hat{T}(s)$ and $T(s)$ about s_i to be equal, i.e. $\forall 1 \leqslant k \leqslant m_i$,

$$\hat{C}(s_i I_n - \hat{A})^{-k}\hat{B} = C(s_i I_N - A)^{-k} B.$$

Eq. (24) is automatically satisfied when the transfer functions $T(s)$ and $\hat{T}(s)$ are both strictly proper. Eq. (25) is equivalent to imposing the m_k first Markov parameters of both transfer functions to be equal, i.e. $\forall 0 \leqslant i \leqslant m_k - 1$,

$$\hat{C}\hat{A}^i\hat{B} = CA^iB.$$

Hence, an interpolation set of size $2n$ corresponds to $2n + 1$ interpolation conditions, one of them being trivially satisfied for any couple of strictly proper transfer functions. Generically, the solution of minimal McMillan degree of (23) is unique and of degree n. For the cases where this does not hold, we refer to [14]. For a more complete treatment of the interpolation problem of rational matrix functions, we refer to [15] and references therein.

In this paper, we are given a strictly proper SISO transfer function of McMillan degree N and a transfer function of McMillan degree $n < N$. The objective consists of finding the projecting matrices such that the transfer function of smallest McMillan degree can be constructed via truncation of the other transfer function. In trying to solve this problem, it turns out that all the interpolation problems we will have to consider will admit only one transfer function of minimal McMillan degree, and that this McMillan degree will be the half of the size of the interpolation set. Hence, there is no need to consider particular cases here. This will become clear in the proof of Theorem 20. From now on, we suppose therefore that there is only one solution of McMillan degree n of the interpolation conditions given in Definition 16, with $s(I) = 2n$. We call this solution $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$.

Lemma 17. *Let $T(s) = C(sI_N - A)^{-1}B$ be any strictly proper SISO transfer function and let I be a $T(s)$-admissible interpolation set. Let $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ be any strictly proper SISO transfer function. Then $\hat{T}(s)$ interpolates $T(s)$ at I if and only if either of the following two equivalent conditions hold*:

$$C\mathscr{C}_{A,B}(I) = \hat{C}\mathscr{C}_{\hat{A},\hat{B}}(I), \quad \mathscr{O}_{A,C}(I)B = \mathscr{O}_{\hat{C},\hat{A}}(I)\hat{B}. \tag{26}$$

Proof. It is simply another way to write down the interpolation conditions of Definition 16. □

Lemma 18. *Let $T(s) = C(sI_N - A)^{-1}B$ be a strictly proper SISO transfer function. Let I be a $T(s)$-admissible interpolation set and (I_1, I_2) be a symmetric separation of I. If the strictly proper SISO transfer function $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ interpolates $T(s)$ at I, then*

$$\mathscr{O}_{A,C}\mathscr{C}_{A,B} = \mathscr{O}_{\hat{C},\hat{A}}\mathscr{C}_{\hat{A},\hat{B}}. \tag{27}$$

Proof. Define A_i and B_i as usual, and consider one element of the matrix equality (27). We have to prove that

$$CA_i^{k_1}A_j^{k_2}B = \hat{C}\hat{A}_i^{k_1}\hat{A}_j^{k_2}\hat{B}. \tag{28}$$

From Lemma 12, we can rewrite this equation by partial fraction expansion as a linear combination of Eq. (26). This completes the proof of the lemma. □

Lemma 19. *Let $T(s) = C(sI_N - A)^{-1}B$ be a strictly proper SISO transfer function of McMillan degree N. Let I be a $T(s)$-admissible interpolation set of size $2n$ and (I_1, I_2) be a separation of I. Suppose that $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ is a strictly proper SISO transfer function of McMillan degree n, which interpolates $T(s)$ at I. Then*

$$\mathscr{O}_{A,C}A\mathscr{C}_{A,B} = \mathscr{O}_{\hat{C},\hat{A}}\hat{A}\mathscr{C}_{\hat{A},\hat{B}}. \tag{29}$$

Proof. Define A_i and B_i as usual, and consider again one element of the matrix equality (29). We have to prove that

$$CA_i^{k_1}AA_j^{k_2}B = \hat{C}\hat{A}_i^{k_1}\hat{A}\hat{A}_j^{k_2}\hat{B}. \tag{30}$$

The idea is that using partial fraction expansion it is possible to rewrite Eq. (30) as a linear combination of Eqs. (26) and (27).

The point at infinity requires more care. We show it for instance when $A_i = A$. From Definition 11, this implies that one of the points of I_1, say $s_{1,1}$ is equal to ∞. Then, $\forall u$, $1 \leqslant u \leqslant m_{1,1}$,

$$CA^{u-1}B = \hat{C}\hat{A}^{u-1}\hat{B}.$$

If $A_j = A$, then the point ∞ is also a point of I_2, say $s_{2,1} = \infty$. Then, $\forall v$, $1 \leqslant v \leqslant m_{2,1}$

$$CA^{v-1}B = \hat{C}\hat{A}^{v-1}\hat{B}.$$

Clearly, the point ∞ must be a point of I, say $s_1 = \infty$. Because (I_1, I_2) is a separation of I, $m_{1,1} + m_{2,1} = m_1$, and $\forall w$, $1 \leqslant w \leqslant m_1$,

$$CA^{w-1}B = \hat{C}\hat{A}^{w-1}\hat{B}. \tag{31}$$

Now, $k_1 + 1 + k_2 \leqslant m_{1,1} + m_{2,1} - 1 = m_1 - 1$, and equality (30) follows from Eq. (31). This concludes the proof for the case $A_i = A_j = A$. Suppose now that $A_j = (s_j I - A)^{-1}$ and $A_i = A$. Then, $\forall v$, $1 \leqslant v \leqslant m_{2,j}$,

$$CA_j^v B = \hat{C}\hat{A}_j^v \hat{B}.$$

From partial fraction expansion, it follows then that

$$CA^{k_1} A A_j^{k_2} B = -CA^{k_1} A_j^{k_2-1} B + s_j C A^{k_1} A_j^{k_2} B$$

Now, Eq. (30) follows from Lemmas 18 and 17. This completes the proof for $A_i = A$. The case $A_i \neq A$ is easier. □

This leads us to the main result of this section.

Theorem 20. *Let $T(s) = C(sI_N - A)^{-1}B$ be a strictly proper SISO transfer function of McMillan degree N. Let I be a $T(s)$-admissible interpolation set of size $2n$ and let (I_1, I_2) be a symmetric separation of I. Suppose that $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ is a strictly proper SISO transfer function of McMillan degree n, which interpolates $T(s)$ at I. Then $\hat{T}(s)$ can be obtained by truncation of $T(s)$ with*

$$Z^{\mathrm{T}} = \mathscr{O}_{\hat{C},\hat{A}}(I_1)^{-1}\mathscr{O}_{C,A}(I_1), \quad V = \mathscr{C}_{A,B}(I_2)\mathscr{C}_{\hat{A},\hat{B}}(I_2)^{-1}. \tag{32}$$

Moreover, $\hat{T}(s)$ is the unique transfer function of minimal Mc Millan degree n that interpolates $T(s)$ at I.

Proof. I_1 and I_2 is a separation of a minimal $\hat{T}(s)$-admissible interpolation set. From Lemma 13, the matrices $\mathscr{O}_{\hat{C},\hat{A}}(I_1)$ and $\mathscr{C}_{\hat{A},\hat{B}}(I_2)$ are invertible. From Lemmas 17 to 19, it is easy to check that conditions (3) of Definition 1 are satisfied with Z and V defined in Eq. (32). Uniqueness immediately follows. □

In other words, if the SISO strictly proper transfer function $\hat{T}(s)$ of McMillan degree n interpolates the SISO strictly proper transfer function $T(s)$ of degree $N > n$ at an interpolation set of size larger than $2n$, then $\hat{T}(s)$ can be obtained from $T(s)$ by a projection technique. Now, we have to consider the case when the interpolation set between $T(s)$ and $\hat{T}(s)$ is of size less than $2n$ (i.e. when the McMillan degree of $T(s) - \hat{T}(s)$ is less than or equal to $2n$). This special case where the McMillan degree drops will be treated in the next section.

4. Model reduction via truncation

The following lemma is well known in the literature (see for instance [16] and references therein for a proof).

Lemma 21. *Let the pair of matrices $A \in \mathbb{C}^{N\times N}$ and $C \in \mathbb{C}^{1\times N}$ be observable. Let $\mathscr{X}$ be a right invariant subspace of A of dimension K and let the matrix $X \in \mathbb{C}^{N\times K}$*

be full rank with $\mathscr{X} = \mathrm{Im}(X)$. *Define the matrices* $\tilde{A} \in \mathbb{C}^{K\times K}$ *and* $\tilde{C} \in \mathbb{C}^{1\times K}$ *by the following equations*:

$$AX = X\tilde{A}, \quad CX = \tilde{C}.$$

Then, the pair $(\tilde{C}, \tilde{A})$ *is observable and*

$$\Lambda(\tilde{A}) \subset \Lambda(A). \tag{33}$$

Remark 22. Since changing the basis X to XS results in a transformed pair $(S^{-1}AS, S^{-1}C)$, it is always possible to choose the basis X of the invariant subspace $\mathscr{X}$ such that the pair $(\tilde{A}, \tilde{C})$ is in observable canonical form (see for instance [1]).

Theorem 23. *Choose* $T(s) = C(sI_N - A)^{-1}B$, *an arbitrary strictly proper SISO transfer function of McMillan degree* N. *Choose* $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$, *an arbitrary strictly proper SISO transfer function of McMillan degree* $n < N$. *Then* $\hat{T}(s)$ *can be constructed via truncation of* $T(s)$.

Proof. By a recursive argument, it is not difficult to see that this theorem is true for every $N > n$ if and only if it is true for $N = n + 1$. We therefore prove it for $N = n + 1$ only. The proof is constructive: we construct Z and V such that the conditions of Definition 1 are satisfied. Define

$$T(s) \doteq \frac{n(s)}{d(s)}, \quad \hat{T}(s) \doteq \frac{\hat{n}(s)}{\hat{d}(s)},$$

where $d(s)$ and $\hat{d}(s)$ are monic polynomials of degree $n + 1$ and n, and where degree $(n(s)) < n + 1$ and degree $\left(\hat{n}(s)\right) < n$. Because the McMillan degree of $T(s)$ is $n + 1$ and that of $\hat{T}(s)$ is n, the polynomials $n(s)$ and $d(s)$ are coprime, and $\hat{n}(s)$ and $\hat{d}(s)$ are coprime as well. Define the error transfer function $E(s)$ to be

$$E(s) \doteq T(s) - \hat{T}(s) = \frac{n(s)\hat{d}(s) - \hat{n}(s)d(s)}{d(s)\hat{d}(s)} \doteq \frac{n_E(s)}{d_E(s)},$$

with

$$\begin{aligned} K &= \text{degree}\left(\gcd\left(d(s), \hat{d}(s)\right)\right) \\ &\text{degree}\,(d_E(s)) = 2n + 1 - K \\ &\text{degree}\,(n_E(s)) < 2n + 1 - K. \end{aligned}$$

We can write

$$\gcd\left(d(s), \hat{d}(s)\right) = (s - \beta_1)^{\nu_1} \cdots (s - \beta_p)^{\nu_p}, \qquad \sum_{i=1}^{p} \nu_i = K.$$

Without loss of generality we can also write

$$d(s)=(s-\beta_1)^{n_1}\cdots(s-\beta_q)^{n_q}$$
$$\hat{d}(s)=(s-\hat{\beta}_1)^{\hat{n}_1}\cdots(s-\hat{\beta}_q)^{\hat{n}_{\hat{q}}};$$

where $\forall 1 \leqslant i \leqslant p$,

$$\hat{\beta}_i=\beta_i, \quad \min(n_i,\hat{n}_i)=\nu_i.$$

Clearly, $E(s)$ has $2n-K+1$ zeros, with at least one zero at ∞. Those zeros are the points where $\hat{T}(s)$ interpolates $T(s)$. More precisely, we can write

$$n_E(s)=\kappa(s-\alpha_1)^{\sigma_1}\cdots(s-\alpha_z)^{\sigma_z},$$
$$\sum_{i=1}^{z}\sigma_i=2n+1-K-\sigma_{z+1},$$

where $\sigma_{z+1}\in\mathbb{N}_0$ is the multiplicity of the zero at ∞ of $E(s)$. Indeed, it is not difficult to check from our definitions that

$$\lim_{s\to\infty} T(s)s^{\sigma_{z+1}}=\kappa,$$

where $\kappa\in\mathbb{C}$ is the gain of the transfer function $E(s)$. Moreover, $T(s)$ and $\hat{T}(s)$ have K poles in common. If $K=0$, then $\hat{T}(s)$ can be constructed by truncation of $T(s)$ via rational interpolation (see Theorem 20). We now suppose that $K>0$.

Clearly, $K\leqslant n$ and from Lemma 21, it is always possible to find a full rank matrix $X_1\in\mathbb{C}^{(n+1)\times K}$ such that the following relations hold:

$$AX_1=X_1\tilde{A}, \quad CX_1=\tilde{C},$$

where

$$\tilde{C}=[1,0,\ldots,0], \quad \tilde{A}=\begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0\\ -a_2 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ -a_{K-1} & 0 & 0 & \ldots & 1\\ -a_K & 0 & 0 & \ldots & 0 \end{bmatrix}, \tag{34}$$

where

$$s^K+a_1s^{K-1}+\cdots+a_K=(s-\beta_1)^{\nu_1}\cdots(s-\beta_p)^{\nu_p}.$$

This is indeed the observer canonical form associated to the common spectrum of $T(s)$ and $\hat{T}(s)$. Similarly, $\exists\hat{X}_1\in\mathbb{C}^{n\times K}$ such that the following relations hold:

$$\hat{A}\hat{X}_1=\hat{X}_1\tilde{A}, \quad \hat{C}\hat{X}_1=\tilde{C}.$$

Now, let us focus our attention to the $2n-K+1$ interpolation conditions. If $\sigma_{z+1}=1$, we define the interpolation set I to be

$$I\doteq\{(\alpha_1,\sigma_1),\ldots,(\alpha_z,\sigma_z)\}.$$

Otherwise, $\sigma_{z+1} > 1$, and we then define I to be

$$I \doteq \{(\alpha_1, \sigma_1), \dots, (\alpha_z, \sigma_z), (\infty, \sigma_{z+1} - 1)\}.$$

Clearly, I is a $T(s)$-admissible set of size $2n - K$. We separate this set into two $T(s)$-admissible sets. The first one, I_1, is of size n and the second one, I_2, is of size $n - K$. Define

$$Y \doteq \mathscr{O}_{A,C}(I_1), \quad \hat{Y} \doteq \mathscr{O}_{\hat{A},\hat{C}}(I_1), \quad X_2 \doteq \mathscr{C}_{A,B}(I_2), \quad \hat{X}_2 \doteq \mathscr{C}_{\hat{A},\hat{B}}(I_2).$$

From Lemma 15, the matrices $\hat{Y}$ and

$$\hat{X} \doteq [\hat{X}_1, \hat{X}_2]$$

are invertible. Finally, define

$$X = [X_1, X_2].$$

Now, we check that

$$CX = \hat{C}\hat{X}, \quad YB = \hat{Y}\hat{B}, \quad YX = \hat{Y}\hat{X}, \quad YAX = \hat{Y}\hat{A}\hat{X}. \tag{35}$$

To verify the first part of Eq. (35),

$$CX = [CX_1, CX_2] = [\hat{C}\hat{X}_1, \hat{C}\hat{X}_2] = \hat{C}\hat{X},$$

where the last equation follows from the construction of X_1 and $\hat{X}_1$ and Lemma 17. The second part of Eq. (35) follows from Lemma 17. Finally,

$$YX = [YX_1, YX_2].$$

Let $\phi(A)$ be a polynomial function of A, then

$$C\phi(A)X_1 = CX_1\phi(\tilde{A}) = \hat{C}\hat{X}_1\phi(\tilde{A}) = \hat{C}\phi(\hat{A})\hat{X}_1,$$

where the matrix $\tilde{A} \in \mathbb{C}^{K\times K}$ is defined in (34). Hence, $YX_1 = \hat{Y}\hat{X}_1$ and the third part of Eq. (35) follows from Lemma 18. For the same reasons, $YAX_1 = \hat{Y}\hat{A}\hat{X}_1$, and the fourth part of Eq. (35) follows from Lemma 19. Take then

$$V \doteq X\hat{X}^{-1}, \quad Z^T \doteq \hat{Y}^{-1}Y.$$

With such a choice of the projectors, Eq. (3) of Definition 1 are satisfied. Hence, $\hat{T}(s)$ can be constructed from truncation of $T(s)$. $\square$

5. Concluding remarks

Generically, two SISO transfer functions $T(s)$ and $\hat{T}(s)$, of order $n+1$ and n respectively, do not have common poles. Hence, almost every strictly proper SISO transfer function of McMillan degree n can be obtained from a strictly proper SISO transfer function of McMillan degree $N > n$ via *multipoint Padé* interpolation. This implies that a reduced order transfer function constructed via multipoint Padé may yield an error of arbitrarily large norm. As a consequence, the interpolation points must be chosen with care when trying to construct a reduced order transfer function via multipoint Padé.

There are many open questions. Given a strictly proper SISO transfer function $T(s)$ of McMillan degree N, and a strictly proper SISO transfer function $\hat{T}(s)$ of McMillan degree $n < N$, we have constructed one set of projecting matrices Z and V such that $\hat{T}(s)$ can be obtained from truncation of $T(s)$. The solution set for the matrices V and Z is certainly much larger, but is not known yet. For instance, when there are more than $2n$ interpolation points we can choose any subset of $2n$ zeros to construct a pair of projectors V and Z.

A more practical question about *multipoint Padé* approximation is how to find interpolation conditions that ensure to have a global error bound between the original and the reduced order transfer functions? For instance, is it possible to find an easy characterization of the interpolation points between a transfer function and a reduced order system obtained by *balanced truncation* or *optimal Hankel norm approximation* technique? How to choose interpolation points such that the reduced order transfer function is stable, is also not yet answered.

As already pointed out in Section 2, the MIMO case is more complicated and will be treated in a subsequent paper.

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