Simultaneous Diagonalization of Rectangular Complex Matrices*

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ABSTRACT

A number of necessary and sufficient conditions are given for the existence of unitary matrices $U$ and $V$, such that $UAV$ is a diagonal matrix for every matrix $A$ in some set $\Gamma$ of rectangular complex matrices. Two related questions are then considered. A necessary and sufficient condition for the existence of unitary matrices $U$ and $V$ such that $UAV$ is a real diagonal matrix for every $A$ in $\Gamma$ is obtained, and an improvement on a necessary and sufficient condition discovered by R. C. Thompson for the existence of real orthogonal matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix for every $A$ in $\Gamma$ is given.

It is well known that if $A$ is an $s \times t$ complex matrix then there exist unitary matrices $U$ and $V$ such that $UAV$ is a diagonal matrix. In this paper we consider the existence of unitary matrices $U$ and $V$ such that $UAV$ is a diagonal matrix for every matrix $A$ in some set $\Gamma$ of $s \times t$ complex matrices. One of our results shows that a necessary and sufficient condition for the existence of $U$ and $V$ obtained by N. A. Wiegmann [1] contains redundant requirements. Two variations on the basic problem are also considered. One of these is to impose the additional requirement that $UAV$ be real for every $A$ in $\Gamma$, and the other is to require that $U$ and $V$ be real orthogonal matrices.

Let $A$ be a complex matrix. Let $\overline{A}$, $A^T$, $A^*$, and $A^+$ be the conjugate, the transpose, the transposed conjugate, and the Moore-Penrose generalized inverse of $A$, respectively. Denote the rank of $A$ by $r(A)$.

*This research was supported by the National Science Foundation and the Research Committee of The University of Alabama in Huntsville.

Throughout this paper \( \Gamma \) will be a set of \( s \times t \) complex matrices. Since the properties that we consider hold for \( \Gamma \) if and only if they hold for every finite subset of nonzero matrices of \( \Gamma \), we assume that \( \Gamma \) is a finite set of nonzero matrices,

\[
\Gamma = \{ A_1, A_2, \ldots, A_n \}.
\]

As J. Williamson [2] has shown, if \( A \) and \( B \) are \( s \times t \) complex matrices then there exist unitary matrices \( U \) and \( V \) such that both \( UAV \) and \( UBV \) are diagonal matrices if and only if \( AB^* \) and \( B^*A \) are normal. Wiegmann [1] pointed out that this does not generalize. For example, if

\[
\Gamma_0 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\},
\]

then \( AB^* \) and \( B^*A \) are normal for all \( A, B \in \Gamma_0 \), but there exist no unitary matrices \( U \) and \( V \) such that \( UAV \) is a diagonal matrix for every \( A \in \Gamma_0 \). Wiegmann [1] showed that if \( s = t \), then there exist unitary matrices \( U \) and \( V \) such that \( UAV \) is a diagonal matrix for every \( A \in \Gamma \) if and only if \( AB^*C = CB^*A \) and both \( AB^* \) and \( B^*A \) are normal for all \( A, B, C \in \Gamma \). We shall see that the requirement that \( AB^* \) and \( B^*A \) be normal is redundant. Moreover, there is no reason to require that \( s = t \).

A lemma on simultaneous unitary diagonalization of square matrices will be useful in proving part of our principal theorem. If \( D \) is restricted to being the identity matrix, then this lemma is well known.

**Lemma 1.** Let \( D \) be a real diagonal matrix with all diagonal elements positive. If \( DA \) is normal and \( AD(BD)^* = (BD)^*AD \) for all \( A, B \in \Gamma \), then there exists a unitary matrix \( U \) such that \( UAU^* \) is a diagonal matrix for every \( A \in \Gamma \).

**Proof.** We may assume that

\[
D = \begin{bmatrix}
d_1 I_1 & 0 & \cdots & 0 \\
0 & d_2 I_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & d_m I_m
\end{bmatrix},
\]  

(1)
where $I_k$ is an identity matrix of order $s_k$ for $k = 1, \ldots, m$, and
\[ d_1 > d_2 > \cdots > d_m > 0. \]  
(2)

Let $A \in \Gamma$, and partition $A$ as $D$ is partitioned, $A = (A_{ik})$. The matrices $D$, $\tilde{D}A$, and $AD$ are normal. Therefore, as shown by Wiegmann [3, Theorem 2],
\[ D^*DA = ADD^*. \]

Since (2) holds, we see that
\[ j \neq k \Rightarrow A_{jk} = 0, \quad j, k = 1, \ldots, m. \]

Hence,
\[ A_j = \begin{bmatrix}
    C_{j1} & 0 & \cdots & 0 \\
    0 & C_{j2} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & C_{jm}
\end{bmatrix}, \quad j = 1, \ldots, n, \]  
(3)

where $C_{jk}$ has order $s_k$ for $k = 1, \ldots, m$. Let $k \in \{1, \ldots, m\}$. Since $A_j D (A_j D)^* = (A_j D)^* A_j D$,

\[ C_{jk} C_{jk}^* = C_{jk}^* C_{jk}, \quad i, j = 1, \ldots, n. \]

This implies that \( \{C_{jk}| j = 1, \ldots, n\} \) is a set of commuting normal matrices. Therefore, there exists a unitary matrix $U_k$ such that $U_k C_{jk} U_k^*$ is a diagonal matrix for $j = 1, \ldots, n$. If we let $U$ be the direct sum of $U_1, U_2, \ldots, U_m$, then $U$ is a unitary matrix such that $UAU^*$ is a diagonal matrix for every $A \in \Gamma$.

**Remark.** It appears that Theorem 3 of [3] could be used instead of Theorem 2 in our proof of Lemma 1. Unfortunately, Theorem 3 lacks a qualifying phrase. Not only does it imply Lemma 1, but it implies that every square complex matrix is normal. To see this, let $B$ be arbitrary and $A$ the zero matrix. Then $AB = 0 = BA$ is normal and a polar form of the normal matrix $A$ is $A = 0I$. It now follows from this theorem that $BI = B$ is normal. Theorem 3 of [3] would be valid if $A$ were required to be nonsingular.

**Theorem 1.** The following are equivalent.

(a) There exist unitary matrices $U$ and $V$ such that $UAV$ is a diagonal matrix for every $A \in \Gamma$. 

(b) \( AB^*C = CB^*A \) for all \( A, B, C \in \Gamma \).
(c) \( A^\dagger B C^* = C^* B A^\dagger \) for all \( A, B, C \in \Gamma \).
(d) \( AB^\dagger C = CB^\dagger A \), while both \( AB^\dagger \) and \( B^\dagger A \) are normal, for all \( A, B, C \in \Gamma \).
(e) \( B^*A \) is normal and \( AB^*B C^* = BC^*A B^* \) for all \( A, B, C \in \Gamma \).
(f) \( B^\dagger A \) is normal and \( AB^\dagger B C^* = BC^*A B^\dagger \) for all \( A, B, C \in \Gamma \).
(g) \( B^\dagger A \) is normal and \( AB^\dagger (CB^\dagger)^* = (CB^\dagger)^* AB^\dagger \) for all \( A, B, C \in \Gamma \).

Proof. We prove by induction on \( s + t \) that (c) \( \Rightarrow \) (b). Clearly this is true for \( s + t = 2 \). Assume (c), where \( s + t > 2 \). Let \( M \) be a matrix in \( \Gamma \) of minimal rank. If \( r(M) = s = t \), then each matrix in \( \Gamma \) is nonsingular,

\[
\Lambda (A^{-1}B C^*)\Lambda = \Lambda (C^* B A^{-1})\Lambda
\]

for all \( A, B, C \in \Gamma \), and (b) follows. Suppose that \( r(M) < (s + t)/2 \). There exist unitary matrices \( R \) and \( S \), such that

\[
RMS = \begin{bmatrix}
D & 0 \\
0 & 0
\end{bmatrix},
\]

where \( D \) is a nonsingular matrix of order \( r(M) \). Let \( A \in \Gamma \),

\[
RAS = \begin{bmatrix}
B & X \\
Y & C
\end{bmatrix},
\]

where \( B \) has the same order as \( D \). We have

\[
S^* M^\dagger M A^* R^* = (RMS)^\dagger RMS (RAS)^* = \begin{bmatrix}
B^* & Y^* \\
0 & 0
\end{bmatrix}
\]

\[
S^* A^* M M^\dagger R^* = (RAS)^* RMS (RMS)^\dagger = \begin{bmatrix}
B^* & 0 \\
X^* & 0
\end{bmatrix}
\]

Therefore, since \( M^\dagger M A^* = A^* M M^\dagger \),

\[
X = 0, \quad Y = 0.
\]
This implies that

$$RA_iS = \begin{bmatrix} B_i & 0 \\ 0 & C_i \end{bmatrix}, \quad i = 1, \ldots, n,$$

where $B_i$ has order $r(M)$. Since it holds for $\Gamma$, (c) holds for both $\{B_i| i = 1, \ldots, n\}$ and $\{C_i| i = 1, \ldots, n\}$. Hence, by the inductive assumption, (b) is satisfied for both of these sets, and it follows that (c) $\Rightarrow$ (b).

Assume (b). Then

$$B^*AA^*B = B^*BA^*A = A^*BB^*A,$$

$$AB^*BC^* = BB^*AC^* = BC^*AB^*$$

for all $A, B, C \in \Gamma$. Therefore, (b) $\Rightarrow$ (c).

Assume (d). Then

$$AB^tCB^t = CB^tAB^t$$

for all $A, B, C \in \Gamma$. Hence, for each $B \in \Gamma$, $\{AB^t| A \in \Gamma\}$ is a set of commuting normal matrices. Therefore,

$$AB^t(CB^t)^* = (CB^t)^*AB^t$$

for all $A, B, C \in \Gamma$, and it follows that (d) $\Rightarrow$ (g).

We prove by induction on $s + t$ that (g) $\Rightarrow$ (a). Clearly this is true for $s + t = 2$. Assume (g), where $s + t > 2$. Let $M \in \Gamma$. These exist unitary matrices $R$ and $S$ such that $RMS$ has the form of (4), where $D$ is a real diagonal matrix of order $r(M)$ with all diagonal elements positive. It follows from $M^tA_i$ and $A_iM^t$ being normal that $RA_iS$ has the form of (5), where $B_i$ has order $r(M)$ for $i = 1, \ldots, n$. Since $M^tA_i$ is normal, $D^{-1}B_i$ is normal for $i = 1, \ldots, n$. We have

$$RA_iM^t(A_iM^t)^*R^* = RA_iS(RMS)^t(RA_iS(RMS)^t)^*,$$

$$R(A_iM^t)^*A_iM^tR^* = (RA_iS(RMS)^t)^*RA_iS(RMS)^t.$$

Hence, since $A_iM^t(A_iM^t)^* = (A_iM^t)^*A_iM^t$,

$$B_iD^{-1}(B_iD^{-1})^* = (B_iD^{-1})^*B_iD^{-1}, \quad i, j = 1, \ldots, n.$$

Therefore, according to Lemma 1, there exists a unitary matrix $P$ such that $PB_iP^*$ is a diagonal matrix for $i = 1, \ldots, n$. Since (g) holds for $\Gamma$, it holds for
\{C_i \mid i = 1, \ldots, n\}. By the inductive assumption, there exist unitary matrices \(U_0\) and \(V_0\) such that \(U_0 C_i V_0\) is a diagonal matrix for \(i = 1, \ldots, n\). This implies the existence of unitary matrices \(U\) and \(V\) such that \(U A V\) is a diagonal matrix for every \(A \in \Gamma\). Therefore, (g) \(\Rightarrow\) (a).

Assume (f). Let \(M \in \Gamma\). There exist unitary matrices \(R\) and \(S\) such that \(R M S\) has the form of (4), where \(D\) is a real diagonal matrix of order \(r(M)\) with all diagonal elements positive. It follows that \(R A_i S\) has the form of (5), where \(B_i\) has order \(r(M)\). Since \(A_i M^\dagger MM^* = MM^* A_i M^\dagger\) and every diagonal element of \(D\) is positive, \(D\) and \(B_i\) commute. Hence, since \(A_i M^\dagger M A_i^* = MA_i^* A_i M^\dagger\), \(\{B_i \mid i = 1, \ldots, n\}\) is a set of commuting normal matrices. Therefore, an inductive argument shows that (f) \(\Rightarrow\) (a). Obviously (a) implies (c), (d), and (f). Since our proof that (g) \(\Rightarrow\) (a) can easily be altered to show that (c) \(\Rightarrow\) (a), the theorem follows.

Theorem 1 certainly does not give an exhaustive list of conditions equivalent to the existence of unitary matrices \(U\) and \(V\) such that \(U A V\) is a diagonal matrix for every \(A \in \Gamma\). For example, another such condition is that there exist unitary matrices \(U\) and \(V\) such that \(U A B^* U^*\) and \(V B^* A V^*\) are diagonal matrices for all \(A, B \in \Gamma\).

C. Eckart and G. Young [4] showed that if \(A\) and \(B\) are \(s \times t\) complex matrices then there exist unitary matrices \(U\) and \(V\) such that both \(U A V\) and \(U B V\) are real diagonal matrices if and only if both \(A B^*\) and \(B^* A\) are hermitian. Wiegmann [1] generalized this by showing that there exist unitary matrices \(U\) and \(V\) such that \(U A V\) is a real diagonal matrix for every \(A \in \Gamma\) if and only if \(A B^*\) and \(B^* A\) are hermitian for all \(A, B \in \Gamma\). It is interesting that \(B^*\) can be replaced by \(B\).

**Theorem 2.** There exist unitary matrices \(U\) and \(V\) such that \(U A V\) is a real diagonal matrix for every \(A \in \Gamma\) if and only if \(A B^\dagger\) and \(B^\dagger A\) are hermitian for all \(A, B \in \Gamma\).

**Proof.** Suppose that \(A B^\dagger\) and \(B^\dagger A\) are hermitian for all \(A, B \in \Gamma\). We prove by induction on \(s + t\) that there exist unitary matrices \(U\) and \(V\) such that \(U A V\) is a real diagonal matrix for every \(A \in \Gamma\). This is clearly true for \(s + t = 2\). Let \(s + t > 2\), and let \(M\) be a matrix in \(\Gamma\) of minimal rank. If \(r(M) = s = t\), then \(\Gamma\) is a set of nonsingular matrices such that \(A B^{-1}\) and \(B^{-1} A\) are hermitian for all \(A, B \in \Gamma\). This implies that \(A^* B\) and \(B A^*\) are hermitian for all \(A, B \in \Gamma\). Therefore, as shown by Wiegmann [1], there exist unitary matrices \(U\) and \(V\) such that \(U A V\) is a diagonal matrix for every \(A \in \Gamma\). Suppose that \(r(M) < (s + t)/2\). Then there exist unitary matrices \(R\) and \(S\) such that \(R M S\) has the form of (4), where \(D\) is a nonsingular matrix of.
order $r(M)$. Since $AM^1$ and $M^1A$ are hermitian, we see that $RA_iS$ has the form of (5) for $i=1,\ldots,n$, where $B_i$ has order $r(M)$. If the inductive assumption is applied to $\{B_i|i=1,\ldots,n\}$ and $\{C_j|i=1,\ldots,n\}$, we see that there exist unitary matrices $U$ and $V$ such that $UAV$ is a diagonal matrix for every $A \in V$. The converse is trivial.

Let $m = \min\{s,t\}$. We say that $\Gamma$ has property $L$ for singular values if for all $A,B \in \Gamma$ there exist orderings $\alpha_1,\alpha_2,\ldots,\alpha_m$ and $\beta_1,\beta_2,\ldots,\beta_m$ of the singular values of $A$ and $B$, respectively, such that for all non-negative real numbers $x$ and $y$ the singular values of $xA+yB$ are $x\alpha_1+y\beta_1, x\alpha_2+y\beta_2,\ldots, x\alpha_m+y\beta_m$. It is not difficult to show that this definition is equivalent to R. C. Thompson's definition [5] of property $L$ for singular values whenever $\Gamma$ is finite. It follows from our proof of Theorem 2 that there exist unitary matrices $U$ and $V$ such that $UAV$ is a non-negative real diagonal matrix for every $A \in \Gamma$ if and only if $AB^1$ and $B^1A$ are positive semidefinite hermitian for all $A,B \in \Gamma$. This observation and Thompson's theorem [5] imply the following.

**Theorem 3.** The set $\Gamma$ has property $L$ for singular values if and only if $AB^1$ and $B^1A$ are positive semidefinite hermitian for all $A,B \in \Gamma$.

We now present a lemma that generalizes a well-known result on simultaneous orthogonal diagonalization of real symmetric matrices.

**Lemma 2.** If $A$ and $AB^*$ are symmetric for all $A,B \in \Gamma$, then there exists a real orthogonal matrix $P$ such that $PAP^T$ is a diagonal matrix for every $A \in \Gamma$.

**Proof.** We induct on the number $n$ of matrices in $\Gamma$. Let

$$A_n = G + iH,$$

where $G$ and $H$ are real. Since $A_n$ is symmetric, both $G$ and $H$ are symmetric. Therefore, since $A_nA_n^*$ is symmetric, $G$ and $H$ commute. Hence, there exists a real orthogonal matrix $Q$ such that both $QGQ^T$ and $QHQ^T$ are diagonal matrices. This implies that $QA_nQ^T = D$ is a diagonal matrix. Suppose that $n > 1$. We may assume that $D$ has the form of (1), where $I_k$ is an identity matrix of order $s_k$ for $k=1,\ldots,m$, and $d_1,d_2,\ldots,d_m$ are distinct complex numbers. Since both $A_i$ and $A_iA_i^*$ are symmetric, $QA_iQ^T$ must have the form of (3) for $i=1,\ldots,n-1$, where $C_{ik}$ has order $s_k$ for $k=1,\ldots,m$. Let $k \in \{1,\ldots,m\}$. Since both $A_i$ and $A_iA_i^*$ are symmetric, both $C_{ik}$ and $C_{ik}^*$ are symmetric for $i,j=1,\ldots,n-1$. By the inductive assumption, there exists a
real orthogonal matrix $P_k$ such that $P_k C_{ik} P_k^T$ is a diagonal matrix for $i = 1, \ldots, n - 1$. If

$$P = \begin{bmatrix}
P_1 & 0 & \cdots & 0 \\
0 & P_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P_m
\end{bmatrix}$$

then $P$ is a real orthogonal matrix and $PAP^T$ is a diagonal matrix for every $A \in \Gamma$.

M. H. Pearl [6] proved that if $A$ is an $s \times t$ complex matrix then there exist real orthogonal matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix if and only if $AA^*$ and $A^*A$ are real. Thompson [7] generalized this by showing that there exist real orthogonal matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix for every $A \in \Gamma$ if and only if $AB^*, B^*A, AB^T$, and $B^TA$ are symmetric for all $A, B \in \Gamma$. The requirement that $AB^T$ and $B^TA$ be symmetric is redundant.

**Theorem 4.** There exist real orthogonal matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix for every $A \in \Gamma$ if and only if $AB^*$ and $B^*A$ are symmetric for all $A, B \in \Gamma$.

**Proof.** Let $AB^*$ and $B^*A$ be symmetric for all $A, B \in \Gamma$. We prove by induction on $s + t$ that there exist real orthogonal matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix for every $A \in \Gamma$. Clearly this is true for $s + t = 2$. Let $M \in \Gamma$, where $s + t > 2$. Both $M^*M$ and $MM^*$ are symmetric. Hence, as Pearl [6] and Thompson [7] have shown, there exist real orthogonal matrices $R$ and $S$ such that $RMS$ has the form of (4), where $D$ is a nonsingular diagonal matrix of order $r(M)$. Since $A_iM^*$ and $M^*A_i$ are symmetric matrices, it follows that $RA_iS$ has the form of (5), where $B_i$ has order $r(M)$ and

$$B_i\overline{D} = \overline{D}B_i^T, \quad B_i^T\overline{D} = \overline{D}B_i, \quad i = 1, \ldots, n.$$  \hspace{1cm} (6)

From (6), we have

$$B_i\overline{D}^2 = \overline{D}^2B_i, \quad i = 1, \ldots, n.$$ \hspace{1cm} (7)
Since we could multiply any of the rows of the real orthogonal matrix $R$ by $-1$ and still have a real orthogonal matrix, we may assume that if $d_i$ and $d_j$ are diagonal elements of $D$ then

$$d_i^2 = d_j^2 \Rightarrow d_i = d_j.$$ 

Therefore, (7) implies that

$$B_i \overline{D} = \overline{D} B_i, \quad i = 1, \ldots, n. \quad (8)$$

Since $\overline{D}$ is nonsingular, (6) and (8) imply that $B_i$ is symmetric for $i = 1, \ldots, n$. Moreover, since $A_i A_j^*$ is symmetric, $B_i B_j^*$ is symmetric for $i, j = 1, \ldots, n$. Hence, by Lemma 2, there exists a real orthogonal matrix $P_0$ such that $P_0 B_i P_0^T$ is a diagonal matrix for $i = 1, \ldots, n$. Since $A_i A_i^*$ and $A_i^* A_i$ are symmetric, $C_i C_i^*$ and $C_i^* C_i$ are symmetric for $i, j = 1, \ldots, n$. Therefore, by the inductive assumption, there exist real orthogonal matrices $U$ and $V$ such that $U C_i V$ is a diagonal matrix for $i = 1, \ldots, n$. If

$$P = \begin{bmatrix} P_0 & 0 \\ 0 & U \end{bmatrix} R, \quad Q = \begin{bmatrix} P_0^T & 0 \\ 0 & V \end{bmatrix},$$

then $P$ and $Q$ are real orthogonal matrices while $PAQ$ is a diagonal matrix for every $A \in \Gamma$. This completes the induction. The converse is trivial.

REFERENCES


Received September 30, 1972, revised June 5, 1973.