A formal fuzzy reasoning system and reasoning mechanism based
on propositional modal logic

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Abstract

We establish in this paper a fuzzy propositional modal logic, \(\mathcal{FPM}\), and the associated semantics, fuzzy Kripke semantics. We prove that \(\mathcal{FPM}\) is sound and complete. Furthermore, we set up a formalized reasoning mechanism based on \(\mathcal{FPM}\).

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1. Introduction

Modal logic [12] is now widely used as a formalism for knowledge representation in artificial intelligence and computer science [7,1,3]. Several systems with various kinds of modal operators have been constructed. The connection between the possible world semantics for S5, the modal epistemic logic, and the approximation space in rough sets theory [13] is now well-known, where the modalities \(\square\) and \(\Diamond\) in the fuzzy S5 can be considered as fuzzy analogs of lower and upper approximation operators from the Pawlak's theory of rough sets. The system S5 has been shown to be useful in the analysis of knowledge in various areas [6]. As fragments of the classical first-order logic, modal logics are limited in dealing with crisp assertions, as the associated possible world semantics is crisp. However, assertions encountered in the real world are not that precise, and cannot be treated by simply using yes–no questions. Fuzzy logic deals with the notion of vagueness and imprecision, and is now used in various research areas, such as interval mathematics [2], possibility theory [5], Rough Sets theory [14] and artificial neural networks. Hájek initiated the study of fuzzy modal logic in [10,9], and provided a complete axiomatization of the fuzzy S5 system, where the accessibility
relation corresponds to the universal relation. In [16,8], Godo and Rodríguez gave a complete axiomatic system for
the extension of Hájek’s logic, where another modality corresponds to a fuzzy similarity relation. In [19], Zhang
et al. established a formal reasoning system, which is based on fuzzy propositional modal logic (FPML). This paper
is devoted to a further study of the properties of fuzzy reasoning based on this system.

We organize the paper as follows. In Section 2, we introduce the system of FPML, and its associated semantics.
Section 3 is focused on a comparison between the consequence relation Σ ⊨ φ, based on the ordinary Kripke semantics,
and the corresponding relation Σ ≃ φ, based on the fuzzy Kripke semantics, where Σ is a set of fuzzy assertions and
φ is a fuzzy assertion. In Section 4, we establish a fuzzy reasoning system for deciding whether Σ ⊨ φ is true or
not. This is based on the notion of a fuzzy constraint. Based on the notion of satisfiability, we prove in Section 5 the
soundness and the completeness of the reasoning system FPML. In Section 6, we set up a reasoning tree mechanism
for FPML. Section 7 includes a short list of concluding remarks.

2. Fuzzy assertions in propositional modal logic

Modal logic is an extension of the classical logic by adding necessity operator □ and possibility operator ◦. If
PV = {p1, p2, . . .} is a set of propositional symbols in the propositional modal logic (PML), then a Kripke model
for the PML is a triple M = ⟨W, R, V⟩, where W is a set of possible worlds, R is a binary relation on W, (R is
called an accessibility relation), and V : W × PV → [0, 1] is a truth assignment evaluating the truth value of each
propositional symbol in each possible world. The function V can be easily extended to all well-formed formulas (wffs)
of PML inductively.

Let M = ⟨W, R, V⟩ be a Kripke model of PML and φ be a wff. Say that φ is satisfied by a possible world w of W
in M, denoted by M ⊨ wφ, if V(w, φ) = 1. Say that M is a model of φ, denoted by M ⊨ φ, if M ⊨ w φ for every w in
W. Let Σ be a set of formulas of PML. M is a model of Σ if M is a model of each formula in Σ. FPML is a natural
extension of PML. The notion of fuzzy assertions, as defined below, plays a crucial role in this extension.

Definition 1. (1) A fuzzy assertion based on PML is an ordered pair ⟨φ, n⟩, where φ is a wff of PML and n is a
real number in [0, 1].
(2) In a fuzzy assertion ⟨φ, n⟩, the number n is called the believable degree of φ.
(3) We call a fuzzy assertion ⟨φ, n⟩ atomic, if φ is a propositional symbol.

FPML is obtained by replacing wffs of PML with fuzzy assertions, and correspondingly, we call its semantics
fuzzy Kripke semantics. A fuzzy Kripke model for FPML is also a triple M = ⟨W, R, V⟩, where W is a set of
possible worlds, R is an accessibility relation on W, and V is a function from W × PV to the unit interval [0, 1] (not
just (0, 1)) such that for each p ∈ PV and n ∈ [0, 1], if at the possible world w, the believable degree of proposition
p is n, then V(w, p) = n. V is called a believable degree function. V can be extended to all wffs in PML as follows:
(2.1) V(w, ∼φ) = 1 − V(w, φ),
(2.2) V(w, φ ∧ ψ) = min[V(w, φ), V(w, ψ)],
(2.3) V(w, φ ∨ ψ) = max[V(w, φ), V(w, ψ)],
(2.4) V(w, φ → ψ) = max[1 − V(w, φ), V(w, ψ)],
(2.5) V(w, □φ) = inf{V(u, φ) : ⟨u, w⟩ ∈ R},
(2.6) V(w, ◦φ) = sup{V(u, φ) : ⟨u, w⟩ ∈ R}.

Definition 2. Let M = ⟨W, R, V⟩ be defined as above, w ∈ W be a possible world and ⟨φ, n⟩ be a fuzzy assertion
in FPML. Define Sat(w, ⟨φ, n⟩), which means that ⟨φ, n⟩ is satisfied at the possible world w ∈ M, inductively as
follows:
(1) Sat(w, ⟨p, n⟩) iff V(w, p) ≥ n for proposition symbols p;
(2) Sat(w, ⟨¬ψ, n⟩) iff V(w, ψ) ≤ 1 − n;
(3) Sat(w, ⟨ψ1 ∧ ψ2, n⟩) iff Sat(w, ⟨ψ1, n⟩) and Sat(w, ⟨ψ2, n⟩);
(4) Sat(w, ⟨ψ1 ∨ ψ2, n⟩) iff either Sat(w, ⟨ψ1, n⟩) or Sat(w, ⟨ψ2, n⟩);
(5) Sat(w, ⟨ψ1 → ψ2, n⟩) iff either V(w, ψ1) ≤ 1 − n or Sat(w, ⟨ψ2, n⟩);
(6) Sat(w, ⟨□ψ, n⟩) iff for all w′ with ⟨w′, w⟩ ∈ R, Sat(w′, ⟨ψ, n⟩);
(7) Sat(w, ⟨◦ψ, n⟩) iff there exists w′ such that ⟨w′, w⟩ ∈ R and Sat(w′, ⟨ψ, n⟩).
Given a fuzzy assertion \( \langle \varphi, n \rangle \) of \( \mathcal{FPML} \), say that \( \langle \varphi, n \rangle \) is satisfiable in \( \mathcal{M} \), denoted as \( \mathcal{M} \models_{w} \langle \varphi, n \rangle \), if there exists a \( w \in \mathcal{W} \) such that \( \text{Sat}(w, \langle \varphi, n \rangle) \). Furthermore, say that \( \langle \varphi, n \rangle \) is valid in \( \mathcal{M} \) or \( \mathcal{M} \) is a model of \( \langle \varphi, n \rangle \), denoted as \( \mathcal{M} \models \langle \varphi, n \rangle \), if for all possible worlds \( w \in \mathcal{W} \), \( \mathcal{M} \models_{w} \langle \varphi, n \rangle \).

In [18], a comparative study of fuzzy sets and rough sets is given. At the end of this section, we take a look at the relation between fuzzy modal logics and rough set theory. Rough sets theory was introduced by Pawlak, and has been widely used in the areas of data mining and knowledge representation and reasoning [14,15]. The basic ingredients in rough sets theory are the lower and upper approximations. More precisely, let \((U, R)\) be an approximation space in the sense of Pawlak, where \( U \) is a non-empty universe and \( R \) is an equivalence relation on \( U \), then for any subset \( X \) of \( U \), the lower and upper approximations of \( X \) are defined as follows:

\[
X_{R} = \{ y \in U : [y]_{R} \subseteq X \}, \quad \overline{X}_{R} = \{ y \in U : [y]_{R} \cap X \neq \emptyset \}.
\]

Say that \( X \) is definable (or exact) if it is a union of the \( R \)-equivalence classes. \( X \) is rough if it is not exact. Now let \( \mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle \) be a fuzzy Kripke model, where \( \mathcal{R} \) is an equivalence relation on \( \mathcal{W} \). Define

\[
\| \langle \varphi, n \rangle \| = \{ w \in \mathcal{W} : w \models \langle \varphi, n \rangle \}.
\]

Then the following relation can be easily verified.

**Theorem 1.** For any formula \( \varphi \) and \( n \in [0, 1] \),

\[
\| (\Box \varphi, n) \| = \| \langle \varphi, n \rangle \| _{\mathcal{R}}, \quad \| (\Diamond \varphi, n) \| = \| \langle \varphi, n \rangle \| _{\mathcal{R}}^{\mathcal{R}}.
\]

### 3. Semantic properties of \( \mathcal{FPML} \)

In this section we study \( \mathcal{FPML} \) from semantic point of view. Our discussion will be focused on those semantic models \( \mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle \), where the relation \( \mathcal{R} \) is an equivalence relation on \( \mathcal{W} \).

#### 3.1. Fuzzy assertions about axioms in \( \mathcal{PML} \)

There are various types of \( \mathcal{PML} \) systems such as K-system, D-system, T-system, S4-system, S5-system, and so on. If \( \mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle \) is a model where \( \mathcal{R} \) is an equivalence relation on \( \mathcal{W} \), then \( \mathcal{M} \) can be viewed as a model of S5. As usual, we use \( \sim, \rightarrow \) and \( \Box \) as the basic connectives, and define \( \varphi \land \psi \equiv \sim (\varphi \rightarrow \sim \psi) \), \( \varphi \lor \psi \equiv (\sim \varphi \rightarrow \psi) \) and \( \Diamond \varphi \equiv \sim \Box \sim \varphi \).

S5 contains the following axioms and inference rules:

- **Axioms:**
  
  \[
  \begin{align*}
  \text{Ap}_1 \quad & \varphi \rightarrow (\psi \rightarrow \varphi), \\
  \text{Ap}_2 \quad & (\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)), \\
  \text{Ap}_3 \quad & (\sim \varphi \rightarrow (\sim \psi)) \rightarrow (\psi \rightarrow \varphi), \\
  \text{K} \quad & \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi), \\
  \text{T} \quad & \Box \varphi \rightarrow \varphi, \\
  \text{E} \quad & \Diamond \varphi \rightarrow \Box \Diamond \varphi.
  \end{align*}
  \]

- **Inference rules:**
  
  \[
  \begin{align*}
  \text{N} \quad & \text{if } \vdash \varphi \text{ then } \vdash \Box \varphi; \\
  \text{MP} \quad & \text{if } \vdash \varphi \rightarrow \psi \text{ and } \vdash \varphi \text{ then } \vdash \psi.
  \end{align*}
  \]

In classical \( \mathcal{PML} \)s, axioms are always valid in the associated models. That is, if \( \varphi \) is an axiom of some \( \mathcal{PML} \) system and \( \mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle \) is an associated model, then \( w \models \varphi \) for any \( w \in \mathcal{W} \). One natural question is whether a fuzzy assertion \( \langle \varphi, 1 \rangle \), where \( \varphi \) is an axiom, is valid in its associated fuzzy semantics, or for which \( n \in [0, 1] \) can the fuzzy assertion \( \langle \varphi, n \rangle \) be always valid in the associated fuzzy models? The following proposition gives the answer.
Proposition 2. Let $\mathcal{M} = \langle W, R, V \rangle$ be a model of $FPML$, where $R$ is an equivalence relation on $W$. Then,

1. $\mathcal{M} \models (\neg \neg \varphi, 1, 0.5)$;
2. $\mathcal{M} \models (\neg \neg \neg \varphi, 0.5)$;
3. $\mathcal{M} \models (\neg \neg \neg \neg \varphi, 0.5)$;
4. $\mathcal{M} \models (K, 0.5)$;
5. $\mathcal{M} \models (T, 0.5)$;
6. $\mathcal{M} \models (E, 0.5)$.

The following proposition shows that the necessity rule and modus ponens over fuzzy assertions are valid in $FPML$ under some conditions. In the necessity rule, the believable degree of $\Box \varphi$ depends on that of $\varphi$. In modus ponens, the believable degree of $\psi$ depends on the believable degrees of $\varphi \rightarrow \psi$ and $\varphi$.

Proposition 3. Let $\mathcal{M} = \langle W, R, V \rangle$ be any model of $FPML$, where $R$ is an accessibility relation on $W$. Then,

1. If $\mathcal{M} \models (\varphi, n)$ then $\mathcal{M} \models (\Box \varphi, n)$.
2. If $\mathcal{M} \models (\varphi \rightarrow \psi, n)$ and $\mathcal{M} \models (\varphi, m)$, where $n, m \in [0, 1]$ and $n > 1 - m$, then $\mathcal{M} \models (\psi, n)$.

3.2. Relation between $\models$ and $\models$

Let $\Sigma$ be a set of wffs of $\mathcal{PML}$. A wff $\varphi$ is said to be a conclusion of $\Sigma$, denoted by $\Sigma \models \varphi$, if every model of $\Sigma$ is also a model of $\varphi$. In $FPML$, we call a set of fuzzy assertions a fuzzy knowledge base. Let $\tilde{\Sigma}$ be a fuzzy knowledge base and $(\varphi, n)$ be a fuzzy assertion. If every model of $\tilde{\Sigma}$ is also a model of $(\varphi, n)$, then we say that assertion $(\varphi, n)$ is a conclusion of $\tilde{\Sigma}$, and we denote it as $\tilde{\Sigma} \models (\varphi, n)$. If $\tilde{\Sigma}$ is a fuzzy knowledge base, then we define $\tilde{\Sigma} = \{\varphi : (\varphi, n) \in \tilde{\Sigma}\}$, and we call it the crisp knowledge base of $\tilde{\Sigma}$.

Theorem 4. Let $\tilde{\Sigma}$ be a fuzzy knowledge base and $\Sigma$ be the crisp knowledge base of $\tilde{\Sigma}$ defined as above. Then for any $\varphi$ and $n > 0$, $\tilde{\Sigma} \models (\varphi, n)$ implies that $\Sigma \models \varphi$.

The theorem can be easily proved by the fact that any model of $\Sigma$, $M$ say, is also a model of $\tilde{\Sigma}$. If $w \in M$ is a possible world, then we have $\mathcal{V}(w, \varphi) \geq n > 0$, i.e., $\mathcal{V}(w, \varphi) = 1$. In general, the converse of Theorem 4 may not be true. To see a closer relation between $\models$ and $\models$, we introduce the notion of normalized fuzzy knowledge bases as follows.

Definition 3. A fuzzy assertion $(\varphi, n)$ is normalized if $n > 0.5$. A fuzzy knowledge base is normalized if every fuzzy assertion in it is normalized.

Let $\mathcal{M} = \langle W, R, V \rangle$ be a fuzzy Kripke model and $\hat{w}$ be a possible world in $W$. The possible world in crisp semantics of $\hat{w}$, denoted as $\text{Cri}(\hat{w})$, is defined as follows:

$$
\text{Cri}(\hat{w})(p_i) = 1 \text{ if and only if } \hat{w}(p_i) > 0.5
$$

for every $p_i \in PV$, where $\hat{w}(\varphi)$ is an abbreviation for $\mathcal{V}(\hat{w}, \varphi)$.

Lemma 5. Let $\mathcal{M} = \langle W, R, V \rangle$ be a fuzzy Kripke model, $\hat{w} \in W$ and $\varphi$ be any formula of $\mathcal{PML}$. Then:

1. If $\hat{w}(\varphi) > 0.5$, then $\text{Cri}(\hat{w})(\varphi) = 1$, and there exists a crisp Kripke model $M = \langle W, R, V \rangle$ such that $\text{Cri}(\hat{w}) \in W$ and $M \models \varphi$;
2. If $\hat{w}(\varphi) < 0.5$ then $\text{Cri}(\hat{w})(\varphi) = 0$, and there exists a crisp Kripke model $M = \langle W, R, V \rangle$ such that $\text{Cri}(\hat{w}) \in W$ and $M \models \sim \varphi$.

Proof. The proof is by induction on the structure of $\varphi$. It is easy to verify in the basic step that $\varphi$ is a propositional symbol and also in the induction step that $\varphi$ is either $\sim \psi$ or $\psi_1 \rightarrow \psi_2$. Here we give a proof when $\varphi$ is $\Box \psi$.

If $\hat{w}(\varphi) > 0.5$, then $\hat{w}(\psi) > 0.5$. By induction hypothesis, $\text{Cri}(\hat{w})(\psi) = 1$ and there exists a model $M$ such that $\text{Cri}(\hat{w}) \in M$ and $M \models \psi$. By Proposition 3(1), $M$ is also a model of $\Box \psi$ and hence $\text{Cri}(\hat{w})(\Box \psi) = 1$. 

If $\tilde{w}(\varphi) < 0.5$, then there is a possible world $\tilde{w}_0 \in \mathcal{W}$ such that $(\tilde{w}_0, \tilde{w}) \in \mathcal{R}$ and $\tilde{w}_0(\psi) < 0.5$. By induction hypothesis, there is a model $M_0$ such that $\text{Cri}(\tilde{w}_0) \in M_0$ and $\text{Cri}(\tilde{w}_0)(\psi) = 0$ and $M_0 \models \sim \psi$. Let $M$ be the model obtained by putting $\text{Cri}(\tilde{w})$ into $M_0$ and extending $R_0$ to $R$ such that $[\text{Cri}(\tilde{w})]_R = [\text{Cri}(\tilde{w}_0)]_R$. It is easy to verify that $M$ is a model of $\sim \Box \psi$, and $\text{Cri}(\tilde{w})(\Box \psi) = 0$.

This completes the proof of the lemma. □

Note that in the proof, we only need to consider the connectives $\sim$, $\rightarrow$ and the modality $\Box$, since the others are definable. By Theorem 4 and Lemma 5, we have:

**Theorem 6.** If $\tilde{\Sigma}$ is finite and normalized, then there is an $n \geq 0.5$ such that $\tilde{\Sigma} \models (\varphi, n)$ iff $\Sigma \models \varphi$.

### 4. Fuzzy reasoning based on $\mathcal{FPML}$

A process of deciding whether $\tilde{\Sigma} \models \gamma$ holds or not is called a fuzzy reasoning procedure, which is based on $\mathcal{FPML}$. By definition, to decide whether $\tilde{\Sigma} \models \gamma$ holds, one has to verify that every model of $\tilde{\Sigma}$ is also a model of $\gamma$, which is not easy to deal with.

In this section, we develop a reasoning mechanism which enables us to perform the fuzzy reasoning efficiently. We first introduce the notion of fuzzy constraints and then establish a formal system of fuzzy reasoning based on $\mathcal{FPML}$, which is a “fuzzy” version of the “semantic tableaux” (see [11]), a well-known method used in classical modal logic. We will combine the constraint propagation method introduced in [17] with the semantic chart method given in [20], where the former one is usually proposed in the context of description logics [4], while the latter one is used to solve the decidability problem of modal propositional calculus [11]. Let $S$ be a set of fuzzy constraints. Since the formulas in the constraints of $S$ are usually compound formulas, it is difficult to find directly an interpretation $\mathcal{I}$ satisfying $S$. To get around of such a difficulty, we introduce the reasoning rules and the notion of educed sets, which will enable us to change complicated formulas to simpler ones.

#### 4.1. Basic definitions

The alphabet of our fuzzy reasoning system contains the symbols used in $\mathcal{PML}$, a set of possible world symbols $w_1, w_2, \ldots$, a set of relation symbols $\{<, \leq, >, \geq\}$ and a special symbol $R$.

**Definition 4.** An expression in the fuzzy reasoning system is called a fuzzy constraint if it is of form $(w : \varphi \text{ rel} n)$ or $(\langle w, w' \rangle : R \geq 1)$, where $\varphi \in \mathcal{PML}$, $n \in [0, 1]$ and rel is a relation in $\{<, \leq, >, \geq\}$. A fuzzy constraint $(w : \varphi \text{ rel} n)$ is called atomic if $\varphi$ is a propositional symbol.

**Definition 5.** An interpretation $\mathcal{I}$ of the system contains an interpretation domain $\mathcal{W}$ such that

1. any $w$ is interpreted as an element of $\mathcal{W}$, $w^\mathcal{I}$, which is a mapping from $PV$ to $[0, 1]$;
2. $R$ is interpreted as a binary relation on $\mathcal{W}$, $R^\mathcal{I}$.

An interpretation $\mathcal{I}$ satisfies a fuzzy constraint $(w : \varphi \text{ rel} n)$, or $\langle (w, w') : R \geq 1 \rangle$ if $w^\mathcal{I}(\varphi) \text{ rel} n$, or $\langle w^\mathcal{I}, w'^\mathcal{I} \rangle \in R^\mathcal{I}$, respectively. Say that an interpretation $\mathcal{I}$ satisfies a set of fuzzy constraints $S$ if $\mathcal{I}$ satisfies every fuzzy constraint of $S$. Say that a set of fuzzy constraints $S$ is satisfiable if there exists an interpretation $\mathcal{I}$ satisfying $S$.

**Definition 6.** The system contains the following reasoning rules:

- **Reasoning rules about $R$:**
  - $(R_\varnothing) \emptyset \implies (\langle w, w \rangle : R \geq 1)$;
  - $(R_{\varnothing}) (\langle w, w' \rangle : R \geq 1) \implies (\langle w', w \rangle : R \geq 1)$;
  - $(R_{\varnothing}) (\langle w, w' \rangle : R \geq 1), (\langle w', w'' \rangle : R \geq 1) \implies (\langle w, w'' \rangle : R \geq 1)$.

- Basic reasoning rules:
  - $(\sim_{\geq}) (w : \sim \varphi \geq n) \implies (w : \varphi \leq 1 - n)$;
  - $(\sim_{\leq}) (w : \sim \varphi \leq n) \implies (w : \varphi \geq 1 - n)$;
  - $(\rightarrow_{\geq}) (w : \varphi \rightarrow \psi \geq n) \implies (w : \varphi \leq 1 - n) \mid (w : \psi \geq n)$.
We have
\[(→≤) \langle w: φ → ψ ≤ n \rangle \implies \langle w: φ ≥ 1 - n \rangle \text{ and } \langle w: ψ ≤ n \rangle;\]
\[(□≥) \langle w: □φ ≥ n \rangle, \langle (w', w): R ≥ 1 \rangle \implies \langle w': φ ≥ n \rangle;\]
\[(□≤) \langle w: □φ ≤ n \rangle \implies \langle (w', w): R ≥ 1 \rangle \text{ and } \langle w': φ ≤ n \rangle,\]
where “≤” used in \((→≤)\), means that from \(\langle w: φ → ψ ≥ n \rangle\), there will be two possible conclusions, \(\langle w: φ ≤ 1 - n \rangle\) and \(\langle w: ψ ≥ n \rangle\), each of which will be considered separately during the reasoning procedure.

There are six more basic reasoning rules for \(<\) and \(>\), \((~>)\), \((→>)\), \((→<)\), \((□>)\) and \((□<)\), each of which can be obtained by replacing \(≤\), \(≥\) with \(<\), \(>\), respectively. From these basic reasoning rules, we can easily obtain the reasoning rules \((∧rel)\), \((∨rel)\), \((≡rel)\). For instance:
\[(∧≥) \langle w: φ ∧ ψ ≥ n \rangle \implies \langle w: φ ≥ n \rangle \text{ and } \langle w: ψ ≥ n \rangle;\]
\[(∨≥) \langle w: φ ∨ ψ ≥ n \rangle \implies \langle w: φ ≥ n \rangle \text{ or } \langle w: ψ ≥ n \rangle;\]
\[(≡≥) \langle w: φ ≥ n \rangle \implies \langle (w', w): R ≥ 1 \rangle \text{ and } \langle w': φ ≥ n \rangle.\]

During the reasoning process, some new fuzzy constraints, called conclusions, are deduced from the given fuzzy constraints, by using these reasoning rules.

4.2. Educed set of fuzzy constraints and satisfiability

Now we introduce the notion of educed sets.

**Definition 7.** A set of fuzzy constraints \(S'\) is **educed by** \(S\) (or an educed set of \(S\)) if \(S' \supseteq S\) and every constraint in \(S'\) is in \(S\) or can be deduced from some constraint of \(S\).

Obviously, if \(S'\) is an educed set of \(S\) and \(S''\) is an educed set of \(S'\), then \(S''\) is also an educed set of \(S\). We remark here that a given \(S\) may have several educed sets \(S'\). It depends on which fuzzy constraints in \(S\) are considered and which reasoning rules are applied. Now we give an example, illustrating how the reasoning procedure described above works.

**Example 1.** Decide whether \(\{⟨φ ∧ ψ, 0.7⟩, ⟨□ψ, 0.6⟩\} \approx ⟨φ ∧ ψ, 0.6⟩\) holds.

Let \(S^*_2 = \{⟨w: φ ≥ 0.7⟩, ⟨w: □ψ ≥ 0.6⟩\}\) and \(S = S^*_2 \cup \{⟨w: φ ∧ ψ < 0.6⟩\}\). The reasoning procedure is as follows:

1. \(\langle w: φ ≥ 0.7⟩\) Hypothesis
2. \(\langle w: □ψ ≥ 0.6⟩\) Hypothesis
3. \(\langle w: φ ∧ ψ < 0.6⟩\) Hypothesis
4. \(\{⟨w', w⟩: R ≥ 1\}, \{w': φ ≥ 0.7⟩\) \(\implies 1\langleφ ≥ 0.7⟩\)
5. \(\{w': ψ ≥ 0.6⟩\) \(\implies 2\{φ ≥ 0.7⟩\)
6. \(\{w': φ ∧ ψ < 0.6⟩\) \(\implies 3\{φ ≥ 0.7⟩\)
7. \(\{w': φ < 0.6⟩\) \(\implies 4\{ψ < 0.6⟩\)

We have \(S = \{(1), (2), (3)\}\) at the beginning, and then we have
\[
S_1 = S \cup \{(4)\},
S_2 = S_1 \cup \{(5)\},
S_3 = S_2 \cup \{(6)\}.
\]

There are two educed sets \(S'\) and \(S''\) of \(S_3\), where
\[
S' = S_3 \cup \{⟨w': φ < 0.6⟩\},
S'' = S_3 \cup \{⟨w': ψ < 0.6⟩\}.
\]
It is so because constraint (7) is obtained by applying \((\wedge, \prec)\) to the fuzzy constraint (6). Thus to decide whether \(\{(\Diamond \varphi, 0.7), (\Box \psi, 0.6)\} \models (\Diamond (\varphi \wedge \psi), 0.6)\), we need to consider the satisfiability of both educed sets \(S'\) and \(S''\) separately (see also [19]). The following propositions are clear.

**Proposition 7.** Let \(S\) be a set of fuzzy constraints. If \(S\) is satisfiable and \(\{w : \varphi \text{ rel } n\} \in S\), then \(S \cup \{\langle w : \varphi \text{ rel }^n 1 - n \rangle\}\) is also satisfiable, where \(\text{rel} \in \{\geq, \leq, >, <\}\) and \(\text{rel}^n\) is the converse of \(\text{rel}\). For example, if \(\text{rel} = \leq\), then \(\text{rel}^n = \geq\).

**Proposition 8.** Let \(S\) be a set of fuzzy constraints. If \(S\) is satisfiable and \(\langle w : \varphi \rightarrow \psi \leq n \rangle \in S\), then \(S \cup \{\langle w : \varphi \geq 1 - n \rangle, \langle w : \psi \leq n \rangle\}\) is also satisfiable. The proposition is also correct when \(\geq\) and \(\leq\) are replaced with \(>\) and \(<\), respectively.

**Proposition 9.** Let \(S\) be a set of fuzzy constraints. If \(S\) is satisfiable and \(\langle w : \varphi \rightarrow \psi \geq n \rangle \in S\), then at least one of \(S \cup \{\langle w : \varphi \leq 1 - n \rangle\}\) and \(S \cup \{\langle w : \psi \geq n \rangle\}\) is satisfiable. The proposition is also correct when \(\geq\) and \(\leq\) are replaced with \(>\) and \(<\), respectively.

**Proposition 10.** Let \(S\) be a set of fuzzy constraints. If \(S\) is satisfiable and \(\langle w : \Diamond \varphi \geq n \rangle \in S\) and \(\langle \langle w', w \rangle : R \geq 1 \rangle \in S\), then \(S \cup \{\langle w' : \varphi \geq n \rangle\}\) is also satisfiable. It is also correct when \(\geq\) is replaced with \(>\).

**Proposition 11.** If \(S\) is satisfiable and \(\langle w : \Box \varphi \leq n \rangle \in S\), then \(S \cup \{\langle \langle w', w \rangle : R \geq 1 \rangle, \langle w' : \varphi \leq n \rangle\}\) is also satisfiable, where \(w'\) is any possible world symbol not in \(S\). It is also correct when the relation symbol \(\leq\) is replaced with \(<\).

**Definition 8.** Say that two fuzzy constraints \(\xi, \zeta\) form a *conjugated pair*, if one of the following conditions holds:

1. \(\xi = \langle w : \varphi \geq n \rangle, \zeta = \langle w : \varphi \leq m \rangle\) and \(n > m\);
2. \(\xi = \langle w : \varphi \geq n \rangle, \zeta = \langle w : \varphi < m \rangle\) and \(n \geq m\);
3. \(\xi = \langle w : \varphi > n \rangle, \zeta = \langle w : \varphi \leq m \rangle\) and \(n \geq m\);
4. \(\xi = \langle w : \varphi > n \rangle, \zeta = \langle w : \varphi < m \rangle\) and \(n \geq m\).

**Definition 9.** A set of fuzzy constraints \(S\) contains a *clash*, if it contains a conjugated pair.

**Proposition 12.** If a set of fuzzy constraints \(S\) contains a clash, then \(S\) cannot be satisfied by any interpretation \(I\).

**Definition 10.** A fuzzy constraint \(\langle w : \varphi \text{ rel } n \rangle\) in \(S\) is *available* if one of the following is true:

1. \(\langle w : \varphi \text{ rel } n \rangle\) is of the form \(\langle w : \Diamond \psi \text{ rel } n \rangle\), where \(\text{rel} \in \{>, \geq\}\), and there is a \(w'\) such that \(\langle \langle w', w \rangle : R \geq 1 \rangle \in S\) and \(\langle w' : \varphi \text{ rel } n \rangle \notin S\).
2. \(\langle w : \varphi \text{ rel } n \rangle\) is not of the form \(\langle w : \Diamond \psi \text{ rel } n \rangle\), where \(\text{rel} \in \{>, \geq\}\), and \(\varphi\) is not a propositional symbol, and \(\langle w : \varphi \text{ rel } n \rangle\) has not been used by any reasoning rule to produce new constraints in the reasoning procedure.

**Definition 11.** Let \(S' \supseteq S\) be a set of fuzzy constraints educed by \(S\) in a reasoning procedure. Say that \(S'\) is *complete* with respect to \(S\) if every fuzzy constraint in \(S'\) is not available.

By Definitions 10 and 11, together with Propositions 7, 9 and 11, we can prove the following proposition by induction on the structure of \(\varphi\).

**Proposition 13.** Let \(S\) be a set of fuzzy constraints. If \(S\) is finite, then every educed set \(S'\) of \(S\) can be extended to a complete educed set of \(S\). Moreover, if \(S\) is satisfiable, then there exists a complete educed set \(S'\) of \(S\) such that \(S'\) is satisfiable.

Let \(S\) be a set of fuzzy constraints. If \(S' \supseteq S\) is a complete educed set with respect to \(S\), then each propositional symbol appearing in some fuzzy constraint of \(S\) also appears in some atomic fuzzy constraint of \(S'\). Thus, if \(S'\) contains
no clash, then we can define an interpretation $\mathcal{I}$ such that $w^\mathcal{I}(p) \rel n$ for every atomic fuzzy constraint $\langle w : p \rel n \rangle$ in $S'$. Obviously, $\mathcal{I}$ satisfies $S'$. So we have

**Proposition 14.** Let $S'$ be a complete educed set of $S$. If $S'$ contains no clash, then there exists an interpretation $\mathcal{I}$ satisfying $S'$.

The following theorem follows from Propositions 12–14.

**Theorem 15.** Let $S$ be a finite set of fuzzy constraints. Then $S$ is satisfiable if and only if there exists a set $S'$ such that $S'$ is complete with respect to $S$ and contains no clash.

5. **Soundness and completeness based on satisfiability**

The soundness and completeness of our reasoning procedure will be based on satisfiability. In this section, we reduce the fuzzy reasoning problem to the satisfiability of some fuzzy constraints set. More precisely, to decide whether $\Sigma \models \langle \phi, n \rangle$ holds, we let

$$S_\Sigma = \{ \langle w : \psi \geq n \psi \rangle : \langle \psi, n, \psi \rangle \in \Sigma \}$$

and will prove that $\Sigma \models \langle \phi, n \rangle$ iff $S_\Sigma \cup \{ \langle w : \phi < n \rangle \}$ is not satisfiable. To prove it, we need the following lemma.

**Lemma 16.** Let $\mathcal{I}$ be an interpretation and suppose that $\mathcal{I}$ satisfies a fuzzy constraint $\langle w : \phi \rel n \rangle$, then there exists a model $M = \langle W, \mathcal{R}, \mathcal{V} \rangle$ such that $w^M \in W$ and for each $w \in W$, $\mathcal{V}(w, \phi) \rel n$.

**Proof.** We prove the lemma by induction on the structure of $\phi$.

1. $\phi$ is a proposition symbol. We define $W = \{ w^\mathcal{I} \}$, $\mathcal{R} = \{ w^\mathcal{I}, w^\mathcal{I} \}$ and put $\mathcal{V}(w^\mathcal{I}, p) = w^\mathcal{I}(p)$ for every $p \in PV$. Obviously, the model $M = \langle W, \mathcal{R}, \mathcal{V} \rangle$ is what we want.

2. $\phi$ is $\sim \psi$. Since $\mathcal{I}$ satisfies $\langle w : \phi \rel n \rangle$, $\mathcal{I}$ also satisfies $\langle w : \psi \rel n^* 1 - n \rangle$, where $\rel n^*$ is the converse of $\rel n$. By the induction hypothesis, there is a model $M$ such that $w^M \in W$ and for every $w \in W$, $\mathcal{V}(w, \psi) \rel n^* 1 - n$. Note that $\mathcal{V}(w, \psi) \rel n^* 1 - n$ iff $\mathcal{V}(w, \sim \psi) \rel n$, $M$ is also a model we need.

3. $\phi$ is $\psi_1 \rightarrow \psi_2$. There are two cases, according to the choice of $\rel$ in $\{ >, \geq \}$ or in $\{ <, < \}$. If $\rel$ is in $\{ >, \geq \}$, then either $\langle w : \psi_1 \rel n^* 1 - n \rangle$ or $\langle w : \psi_2 \rel n \rangle$ is satisfiable by $\mathcal{I}$. By the induction hypothesis, the model obtained according to either $\langle w : \psi_1 \rel n^* 1 - n \rangle$ or $\langle w : \psi_2 \rel n \rangle$ is what we need. If $\rel$ is in $\{ <, \leq \}$, then both $\langle w : \psi_1 \rel n^* 1 - n \rangle$ and $\langle w : \psi_2 \rel n \rangle$ are satisfiable by $\mathcal{I}$. Thus, by the induction hypothesis, we have two models, $M_1$, $M_2$, obtained by the facts that both $\langle w : \psi_1 \rel n^* 1 - n \rangle$ and $\langle w : \psi_2 \rel n \rangle$ are satisfiable by $\mathcal{I}$. Since $w^\mathcal{I}$ is in both $M_1$ and $M_2$, $W_1 \cap W_2 \neq \emptyset$. Let $W = W_1 \cup W_2$ and $\mathcal{R} = \mathcal{R}^I \cup \mathcal{V} = \{ w_1, w_2 \in \mathcal{R}^I : w_1, w_2 \in \mathcal{W} \}$. Then the model $M = \langle W, \mathcal{R}, \mathcal{V} \rangle$ is what we need.

4. $\phi$ is $\neg \psi$. Suppose that $\langle w : \neg \psi \rel n \rangle$ is satisfiable by $\mathcal{I}$. If $\rel$ is in $\{ >, \geq \}$, then $\langle w : \psi \rel n \rangle$ is also satisfiable by $\mathcal{I}$, thus the model exists. If $\rel$ is in $\{ <, \leq \}$, then there exists a symbol $w_1$ such that $\langle w^\mathcal{I}_1, w^\mathcal{I}_2 \rangle \in \mathcal{R}^I$ and $\langle w_1 : \psi \rel n \rangle$ is satisfiable by $\mathcal{I}$. By the induction hypothesis, there exists a model $M_1$ such that $w^\mathcal{I}_1 \in W_1$ and $\mathcal{V}(w, \psi) \rel n$ for any $w \in W_1$. Let $W = W_1 \cup \{ w^\mathcal{I}_1 \}$, $\mathcal{R} = \mathcal{R}^I \cup \mathcal{V}$. It is easy to verify that $M$ is the model we need.

**Corollary 17.** Let $S = \{ \langle w : \phi_i \rel_i n_i \rangle : 1 \leq i \leq m \}$ be a set of fuzzy constraints. If $S$ is satisfiable, then there exists a model $M$ such that the interpretation of $w$ is in $W$ and $\mathcal{V}(w, \phi_i) \rel_i n_i$ for each $w \in W$ and each $\langle w : \phi_i \rel_i n_i \rangle \in S$.

Now we prove the main result.

**Theorem 18.** If $\Sigma$ is finite, then $\Sigma \models \langle \phi, n \rangle$ if and only if $S_\Sigma \cup \{ \langle w : \phi < n \rangle \}$ is not satisfiable.

**Proof.** If $S_\Sigma \cup \{ \langle w : \phi < n \rangle \}$ is satisfiable by some $\mathcal{I}$, then by Corollary 17, there exists a model $M$ such that $w^\mathcal{I} \in M$. Obviously, $M$ is a model of $\Sigma$, but not a model of $\langle \phi, n \rangle$. Because for all $w \in M$, $\mathcal{V}(w, \psi) \geq n_\psi$ for any $\langle \psi, n_\psi \rangle \in \Sigma$ and $w^\mathcal{I}(\phi) < n$, thus $\Sigma \not\models \langle \phi, n \rangle$. 

Conversely, if \( \tilde{\Sigma} \not\models (\varphi, n) \), then there exists a model \( \mathcal{M} = (W, R, \mathcal{V}) \), and a possible world \( w \in W \) such that \( V(w, \psi) \geq m \) for any \((\psi, m) \in \tilde{\Sigma} \) and \( V(w, \varphi) < n \). Let \( \mathcal{I} \) be an interpretation such that \( w^2 = w \). Then \( S \tilde{\Sigma} \cup \{ (w : \varphi < n) \} \) is satisfied by the interpretation \( \mathcal{I} \). \( \square \)

6. Reasoning on trees

In this section, we establish a reasoning mechanism, which can be used to decide whether \( \tilde{\Sigma} \models (\varphi, n) \) holds, where \( \tilde{\Sigma} \) is a finite set of fuzzy assertions. A reasoning procedure will proceed on a reasoning tree.

6.1. Reasoning trees

Let \( S = S_\tilde{\Sigma} \cup \{ (w : \varphi < n) \} \), where \( S_\tilde{\Sigma} = \{ (w : \psi \geq m) : (\psi, m) \in \tilde{\Sigma} \} \). The reasoning procedure will begin with a fuzzy constraint set \( S \). Some new possible world symbols may be introduced during the reasoning procedure.

**Definition 12.** Say that a possible world symbol \( w' \) is introduced by \( w \) if there is a constraint \( (w : \square \varphi \leq n) \) and \( (\square \leq) \) is applied, or there is a constraint \( (w : \square \varphi < n) \) and \( (\square <) \) is applied.

Let \( S \) be a set of fuzzy constraints and \( A \) be the set of those constraints which can be deduced from the constraints in \( S \) by applying the reasoning rules. The reasoning tree of \( S \), denoted as \( T_S \), is a subset of \( A^<\alpha \) such that each element of \( T_S \) is a finite sequence of constraints \( o \xi_1 \xi_2 \ldots \xi_k \), where \( o \) is a special letter denoting the root of the tree and \( \xi_i \in A \) for \( 1 \leq i \leq k \). An element of \( T_S \) is also called a path on the tree. We use Greek letters \( \alpha, \beta, \gamma \ldots \) to denote the paths on \( T_S \). If \( \alpha = o \xi_1 \ldots \xi_k \), then \( \alpha(0) = o \) and \( \alpha(i) = \xi_i \) for every \( 1 \leq i \leq k \). Let \( |\alpha| \) denote the length of \( \alpha \). Let \( \underline{\alpha} \) be the empty string and \( \underline{\alpha}' \) be the concatenation of \( \alpha \) followed by \( \beta \). \( \underline{\alpha}' \xi \), where \( \xi \) is a constraint, denotes that \( \xi \) is a direct successor of \( \alpha \).

We need an ordering among the constraints to establish the reasoning order. For any fuzzy constraints \( \xi_1 \) and \( \xi_j \), we say that \( \xi_1 \) is less than \( \xi_j \), denoted by \( \xi_1 < \xi_j \), if there is a path \( \alpha \) on \( T_S \) such that \( \alpha(k_1) = \xi_1 \) and \( \alpha(k_j) = \xi_j \) and \( k_i < k_j \). Note that any constraint in the form \( (w : \psi \geq m) \) produces two possible conclusions, \( (w : \varphi < n) \) and \( (w : \psi > n) \). If \( \gamma \) is a path on \( T_S \) and the direct successors of \( \gamma \) are \( (w : \varphi \leq 1 - n) \) and \( (w : \psi \geq n) \), then we set \( (w : \psi \geq n) \) on the left of \( (w : \psi \geq n) \). This will give an order on the paths through \( T_S \).

**Definition 13.** Let \( \alpha, \beta \) be two paths on \( T_S \). We say that \( \alpha \) is on the left of \( \beta \), denoted by \( \alpha <_L \beta \), if there exists a path \( \gamma \) with direct successors \( (w : \varphi \leq 1 - n) \) and \( (w : \psi \geq n) \) such that \( \gamma'(w : \varphi \leq 1 - n) \subseteq \alpha \) and \( \gamma'(w : \psi \geq n) \subseteq \beta \).

**Definition 14.** For any path \( \alpha \) on \( T_S \), let \( S_\alpha \) be the set of constraints such that \( S_\alpha = S \cup \{ \xi : \alpha(i) = \xi \text{ for some } 1 \leq i \leq |\alpha| \} \). Say that \( S_\alpha \) is complete if no reasoning rules can be applied to any constraint of it, i.e., any constraint in \( S_\alpha \) is not available. An \( \alpha \) on \( T_S \) is said to be terminating if either \( S_\alpha \) is complete or \( S_\alpha \) contains a clash.

A reasoning procedure is a refutation procedure. We will make an attempt to find a path \( \delta \) on \( T_S \) such that \( S_\delta \) is complete and has no clash in it. A reasoning procedure proceeds step by step on a reasoning tree, which grows correspondingly. At each stage \( s \), we will use \( \delta_s \) as an approximation of \( \delta \), and \( T^s_\delta \) as a portion of \( T_\delta \) formed by this stage.

We will use parameter \( \text{Col} \) to denote the result of our reasoning. If \( \tilde{\Sigma} \models (\varphi, n) \), then the value of \( \text{Col} \) is success. Otherwise, the value of \( \text{Col} \) is false.

For any path \( \alpha \), let \( W_\alpha \) be the set of all possible world symbols introduced during the reasoning along \( \alpha \). At stage \( s + 1, T^{s+1}_\delta = T^s_\delta \cup \{ \xi \} \) expresses the extension of \( T^s_\delta \) such that \( \xi \) is a new fuzzy constraint growing along the path \( \delta_s \) on \( T^s_\delta \). If \( T^{s+1}_\delta = T^s_\delta \cup \{ \xi_1, \xi_2 \} \) then \( T^{s+1}_\delta \) is an extension of \( T^s_\delta \) such that \( \xi_1 \) and \( \xi_2 \) are new fuzzy constraints growing along the path \( \delta_s \) on \( T^s_\delta \) such that \( \xi_1 \) is on the left of \( \xi_2 \).

6.2. Description of the reasoning procedure

**Stage** \( s = 0 \): Also an initialization stage at which we define

1. \( \delta_0 \) such that \( \delta_0(0) = o \) and \( S_{\delta_0} = S \cup \{ (w_1, w): R \geq 1 \} \), and
Stage S+1: Having $\delta_s$, $T^s$, $W_{\delta_s}$ and $Col = success$.

(1) If $S_{\delta_s}$ has a clash, then define $\delta_{s+1}$ as the left most on $T^s$ such that $\delta_s < L \delta_{s+1}, T^{s+1} = T^s$, $W_{\delta_{s+1}} = W_{\delta_s}$, and $S_{\delta_{s+1}} = S_{\delta_s}$. Goto the next stage. In this case, if no such $\delta_{s+1}$ exists, then stop the procedure.

(2) If $S_{\delta_s}$ is complete and has no a clash, then let $Col = false$ and stop the procedure.

(3) If $S_{\delta_s}$ has no clash and is not complete, then choose $x = \mu; (\gamma \leq \delta_s$ and $\gamma(\|\|)$ is available fuzzy constraint) and define parameters as follows:

(3.1) If $\alpha(z)$ is of the form $(w_i : \neg \psi \geq m)$, then define $\delta_{s+1} = \delta_s(w_i : \psi \leq 1 - m)$; $T^{s+1} = T^s \cup \{(w_i : \psi \leq 1 - m)\}$; $S_{\delta_{s+1}} = S_{\delta_s} \cup \{(w_i : \psi \leq 1 - m)\}$; $W_{\delta_{s+1}} = W_{\delta_s}$. Goto the next stage.

(3.2) If $\alpha(z)$ is of the form $(w_i : \neg \psi < m)$, then define $\delta_{s+1} = \delta_s(w_i : \psi \leq 1 - m)$; $T^{s+1} = T^s \cup \{(w_i : \psi \leq 1 - m)\}$; $S_{\delta_{s+1}} = S_{\delta_s} \cup \{(w_i : \psi \leq 1 - m)\}$; $W_{\delta_{s+1}} = W_{\delta_s}$. Goto the next stage.

(3.3) If $\alpha(z)$ is of the form $(w_i : \psi \geq 1 \rightarrow \psi_2 \geq m)$, then define $\delta_{s+1} = \delta_s(w_i : \psi_1 \leq 1 - m)$; $T^{s+1} = T^s \cup \{(w_i : \psi_1 \leq 1 - m), (w_i : \psi_2 \geq m)\}$; $S_{\delta_{s+1}} = S_{\delta_s} \cup \{(w_i : \psi_1 \leq 1 - m)\}$; $W_{\delta_{s+1}} = W_{\delta_s}$. Goto the next stage.

(3.4) If $\alpha(z)$ is of the form $(w_i : \psi \geq 1 \rightarrow \psi_2 \leq m)$, then define $\delta_{s+1} = \delta_s(w_i : \psi_1 \leq 1 - m)$; $T^{s+1} = T^s \cup \{(w_i : \psi_1 \leq 1 - m), (w_i : \psi_2 \leq m)\}$; $S_{\delta_{s+1}} = S_{\delta_s} \cup \{(w_i : \psi_1 \leq 1 - m)\}$; $W_{\delta_{s+1}} = W_{\delta_s}$. Goto the next stage.

(3.5) If $\alpha(z)$ is of the form $(w_i : \Box \psi \leq m)$, then define $\delta_{s+1} = \delta_s((w_j, w_i : r \geq 1) \rightarrow (w_j : \psi \leq m))$, (where $j \geq 1$ is the least integer such that $w_j \not\in W_{\delta_s}$); $T^{s+1} = (T^s \cup \{(w_j, w_i : r \geq 1)\}) \cup \{(w_j : \psi \leq m)\}$; $S_{\delta_{s+1}} = S_{\delta_s} \cup \{(w_j, w_i : r \geq 1), (w_j : \psi \leq m)\}$; $W_{\delta_{s+1}} = W_{\delta_s} \cup \{w_j\}$. Goto the next stage.

(3.6) If $\alpha(z)$ is of the form $(w_i : \Box \psi \geq m)$, then define $\delta_{s+1} = \delta_s(w_j : \psi \geq m)$, (where $j \geq 1$ is the least integer such that $w_j \in W_{\delta_s}$ and $(w_j : \psi \geq m) \not\in S_{\delta_s}$); $T^{s+1} = T^s \cup \{(w_j : \psi \geq m)\}$; $S_{\delta_{s+1}} = S_{\delta_s} \cup \{(w_j : \psi \geq m)\}$; $W_{\delta_{s+1}} = W_{\delta_s} \cup \{w_j\}$. Goto the next stage.

The procedure contains six additional clauses for the cases with $<$ and $>$, which can be formulated like the clauses (3.1)–(3.6) replacing $<$ with $<$ and similarly for $>$.

This ends the description of the reasoning procedure.

6.3. Verification of the reasoning procedure

Our reasoning procedure is used to decide whether $\Sigma \vdash \langle \rho, n \rangle$ holds, where $\Sigma$ is a set of fuzzy assertions and $\langle \rho, n \rangle$ is a fuzzy assertion. Let $S = S_{\Sigma} \cup \{\langle w : \rho < n \rangle\}$, where $S_{\Sigma} = \{\langle w : \psi \geq m \rangle : \langle \psi, m \rangle \in \Sigma\}$. By Theorem 18, we have
\[ \Sigma \models (\varphi, n) \] if and only if \( S \) is not satisfiable, and we only need to decide whether \( S \) is satisfiable. To do this, we need to check every educed set of \( S \) and see whether there is one which is satisfiable. During the reasoning procedure, each educed set of \( S \) will be produced along a path and its satisfiability will be verified at the terminating paths.

The following theorem follows from Proposition 13 and Theorems 15 and 18 immediately, which shows that the reasoning procedure is correct.

**Theorem 19.** Let \( \widetilde{\Sigma} \) be a set of fuzzy assertions and \( (\varphi, n) \) be a fuzzy assertion. If \( \widetilde{\Sigma} \) is finite then the reasoning procedure will stop in finitely many steps. Moreover, after the reasoning procedure, we have:

\[ \widetilde{\Sigma} \models (\varphi, n) \text{ if } \text{Col} = \text{success} \quad \text{and} \quad \widetilde{\Sigma} \not\models (\varphi, n) \text{ if } \text{Col} = \text{false}. \]

The reasoning tree is binary, and each node on the tree corresponds to a step in the deduction. Assume that one step of a deduction is a unit of the time complexity. Then, the time complexity of the reasoning can be counted as the number of nodes on a reasoning tree. Given a set \( S \) of fuzzy constraints, let \( T \) be the reasoning tree for \( S \). Let \( |S| = m, k = \max\{I_{\varphi} : (w : \varphi \ rel \ n) \in S\} \), where \( I_{\varphi} \) is the length of \( \varphi \). Then the height of \( T \) is at most \( mk \). Notice that during the reasoning procedure, branches in \( T \) are caused only by the fuzzy constraints in \( S \) in the form \((w : \varphi \rightarrow \psi \ rel \ n)\), where \( rel \in \{\geq, >\} \), and thus if \( t \) is the number of symbol \( \rightarrow \) in \( S \), then \( t \leq mk \), and as a consequence, \( T \) has at most \( t \) many branching nodes, and hence at most \( 2(t + 1) \) paths. Therefore, there are at most \( 2mk(t + 1) \) nodes on \( T \). The time complexity of the reasoning procedure is \( O(m^2k^2) \).

**7. Conclusions and further work**

In this paper, we introduce a fuzzy propositional modal logic, \( \mathcal{FPML} \), and investigate its semantics, fuzzy Kripke semantics. We take a close look at the relation between the reasoning \( \widetilde{\Sigma} \models (\varphi, n) \) and the satisfiability of fuzzy constraint sets. A formal reasoning system is introduced to decide whether \( \widetilde{\Sigma} \models (\varphi, n) \) holds or not. This paper gives a formalized description of the reasoning procedure and makes the reasoning procedure more applicable. It offers not only a reasoning mechanism but also a possibility that the reasoning procedure could be realized on a computer. The work in this paper is based on propositional logic, and our further work is to study the extension of our formal system on the base of first-order logic.

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**References**


