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Routing through a Network with Maximum Reliability

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1. INTRODUCTION

The problem of finding an optimal path through a network and in particular the shortest path in time has been studied by many people during the past 25 years. For a thorough review and references, see Dreyfus [1].

In this paper we discuss a stochastic version of this problem in which various probabilistic elements are introduced. One of the important applications of this problem is in communication networks, where reliability is a major requirement. Namely, what happens if one or more link(s) in the network fail for some reason. In Section 2, we discuss the failure problem and in Section 3 we give a method for finding the most reliable path in the network. A work similar to this was first done by Christofides [5]. In Section 4 we consider the same problem under resource constraint and in Section 5 we give a method to compute the second, third, etc., most reliable paths in the network. Pollack's algorithm [3] will be discussed in Section 6, and finally we have some discussion in the last section.

2. THE PROBLEM OF FAILURE

One standard method to obtain the shortest path through a stochastic network is to use expected value.

The basic disadvantage of the expected value model is its inapplicability to cases where we have one or more failures of link(s) in the network.

If we use an expected value model and we have a non-zero probability of failure of one link the expected time along that link is infinity. Thus we may end up with a disconnected network, while the network is actually connected. A simple example of this is shown in a network of 5 nodes below. We assume the probability of link (4, 5) failing is 0.2; then by using the expected value model we end up with a disconnected network where node 5 is

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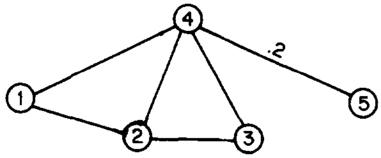


isolated. Hence this is a good motivation to search for another method which enables us to handle the failure problem in a network.

3. A Path of Maximum Probability of Getting Through the Network

Here instead of expected value we use probabilities. We assume we are given the probability distribution for time for each link in the network. These probability distributions include a probability that a link disappears. We are assuming independence.

Here instead of the time matrix or expected time matrix we have a distribution of times for each link. These probability distributions may have a finite probability that the time to go along a particular link is infinite; i.e., physically the link may not be there at all or there is a probability that the link has failed. Our objective is to obtain the maximum probability of going from node i to terminal node N for $i = 1, 2, \ldots, N-1$. In other words, we are looking for the most reliable path to go from i to N. Considering the assumptions above, we have the functional equation

$$p_i = \max_{j \in S(i)} (p_{ij} \cdot p_j) \qquad \text{for} \quad i = 1, 2, \ldots, N-1, \tag{1}$$

where p_i denotes the maximum probability of going from node i to N, and p_{ij} is the probability of going from i to j along the link connecting i and j. If we take the logarithm of both sides of Eq. (1) we have

$$\log p_i = \log(\max_{j \in S(i)} (p_{ij} \cdot p_j)). \tag{2}$$

Since both log and maximum are monotone functions we have

$$\log p_i = \max_{j \in S(i)} (\log(p_{ij} \cdot p_j)). \tag{3}$$

Since $p_{ij} \cdot p_j < 1$ then $\log p_{ij} \cdot p_j < 0$, and hence we have

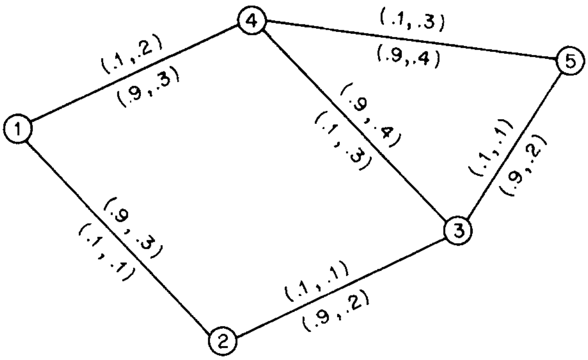
$$-\log p_i = \min_{j \in S(i)} (-\log p_{ij} - \log p_j). \tag{4}$$

Let $-\log p_i = q_i$ and $-\log p_j = q_j$; thus,

$$q_i = \min_{j \in S(i)} (-\log p_{ij} + q_j). \tag{5}$$

This is the same kind of equation as in the deterministic case or in the expected value model.

It is worthwhile to note that the most reliable path and the minimum expected time path are necessarily the same. Let us consider a simple example of a network with five nodes and six links as shown in the scheme below. For simplicity we assume for each link only two probabilities for the



time being, 0.1 and 0.9. A pair (p, t) on a link denotes that we can traverse a particular link with probability p in time t. t is a random variable for which we have the probability distribution function. We applied both methods, the expected value model and the most reliable path. We also consider only the paths from node 1 to node 5. The table below shows the computational results.

Expected time	Probability	Nodes (path)
0.68	0.81	1,4,5*
0.87	0.729	1,4,3,5
0.66	0.729	1,2,3,5*
1.25	0.648	1,2,3,4,5

This simple example shows that a path of maximum probability from node 1 to 5 is the path (1,4,5) with probability 0.81 and a path of minimum expected time is the path (1,4,3,5) with expected time 0.66. These two paths are shown in the table above by asterisks.

This simple result shows that a path of maximum reliability is not necessarily a path of minimum time. This result is important in many cases in transportation or communication networks when dealing with emergency situations, when our primary goal is to make sure a message reaches its destination.

How to handle emergency cases is discussed in a forthcoming paper.

4. Resource Constraints

Different aspects of finding an optimal path when there are some constraints on the resources available to us have been discussed elsewhere. Here we consider the case where we have a limited amount of resource, c. The method developed in the previous section will be used, and we show how we can use the same analytic method as that in the deterministic case [6].

Let us assume that the velocity is constant and normalized to be unity. We also assume that the way the resource are used up is proportional to distance traveled.

Let us define $p_i(c)$ as the most reliable path from node 1 to N given c amount of resources. We have the following functional equation;

$$p_i(c) = \max_{j \in S(i)} (p_{ij} \cdot p_j(c - Kd_{ij})), \qquad Kd_{ij} \leqslant c \quad \text{for } c \geqslant 0,\ i = 1, 2, \ldots, N-1. \tag{6}$$

Let us take the logarithm of both sides:

$$\log p_i(c) = \log[\max_{j \in S(i)} (p_{ij} \cdot p_j(c - Kd_{ij})], \qquad Kd_{ij} \leqslant c.$$

By the same reasoning as that in the previous section we can write

$$\log p_i(c) = \max_{j \in S(i)} [\log(p_{ij} \cdot p_j(c - Kd_{ij})], \qquad Kd_{ij} \leqslant c,$$

which is equivalent to

$$-\log p_i(c) = \min_{j \in S(i)} [-\log p_{ij} \cdot -\log p_j(c - Kd_{ij})], \qquad Kd_{ij} \leqslant c.$$

Let $-\log p_i(c) = r_i(c)$; thus we have

$$r_i(c) = \min_{j \in S(i)} [-\log p_{ij} + r_j(c - Kd_{ij})], \qquad Kd_{ij} \leqslant c \quad \text{for } i = 1, 2, \ldots, N-1. \tag{7}$$

This is the same as the equations in the deterministic case. Ways of solving this type of equation are discussed in [6].

5. The 2nd, 3rd,...,Kth Most Reliable Path Through the Network

It is sometimes important to know the second and third, etc., most reliable path of getting through a network. If for any reason the most reliable path is not available then alternate routes are desirable. The unavailability of the best path can happen in case of one or more failure of the link(s) in a network. This problem has many important applications in road traffic, telephone, communication networks, etc.

This problem has been considered in the deterministic case by many people; see [1] for many references to these algorithms. Dreyfus [1] has modified the Bellman–Kalaba [4] and Hoffman–Pavley [2] procedure and discusses the comparison among existing procedures.

Now let us consider this problem in its stochastic version. We assume again that the probability of going from node i to j is p_{ij} and these probabilities are independent as was assumed in Section 2.

Let

$p_i =$ the maximum probability of going from node i to N.

Then, we have $p_i = \max_{j \neq i}(p_{ij} \cdot p_j)$. Also let

$q_j =$ the probability that the second best path is available (second most reliable path).

Then, we have

$$q_i = \max \begin{cases} \max\limits_{j \neq i}{}_2\, p_{ij} \cdot p_j, & i = 1, 2, \ldots, N-1, \\ p_{iK} \cdot p_{qK}, \end{cases} \tag{8}$$
$$q_N = 1$$

The term $\max_2(p_{ij} \cdot p_j)$ determines the value of the second best path starting at node i and deviating from the most reliable path at that node i. In Eq. (8) K is the node after i on the most reliable path from i.

Bellman and Kalaba [4] recommended solution of Eq. (8) by an iterative procedure. If the average node has n outgoing links and L is the average number of iterations until convergence of the iterative method, considering that we have N nodes the method requires on the average MNL additions and comparisons.

To determine the third most reliable path from all nodes to N assume r_i represents the probability of the third most reliable path from i. Then,

$$r_i = \max \begin{cases} p_{iK} \cdot r_K, \\ \max_2(p_{ij} \cdot p_j), & i = 1, \ldots, N-1, \end{cases}$$
$$r_N = 1$$

if a single node K follows i along both the most and the second most reliable path. But if K is the node following i on the most reliable path and m is the node following i on the second most reliable path, then we have

$$r_i = \max \begin{bmatrix} p_{iK} \cdot q_K \\ p_{im} \cdot p_m \\ \max_{j \neq i} {}_3[p_{ij} \cdot p_j] \end{bmatrix}. \qquad (9)$$

$$r_N = 1$$

In general, to obtain the Kth most reliable path we can use the method proposed by Dreyfus [1] in the deterministic case.

6. Pollack's Algorithm

Pollack [3] proposed a solution for obtaining the Kth best routes when K is small; that is, the second or third best route.

This method is good for people who do not know much mathematics and is practical in some applications. Here we apply his method to compute the second and third most reliable paths.

Given the most reliable path from i to N, the probability of each link is set in turn to one. The most reliable path problem is then solved for each such case. The maximum probability of all these cases is then the second most reliable path. The reason for this is that the second most reliable path must differ in at least one link from the most reliable path. This will be insured by setting the probability of one link at a time to one. If there are L links in the most reliable path, then in order to obtain the second most reliable path we should solve the most reliable path problem L times. Also if the second most reliable path has M links which are entirely different from the most reliable path then to compute the third most reliable path we have to solve the most reliable path problem ML times. The number ML will not usually occur because there are some common links between the most and the second most reliable paths.

The advantage of this method is its simplicity and loop-free feature. On the other hand, it is not computationally feasible to obtain the Kth most reliable path once K becomes larger than three.

7. Discussion

In this paper we showed how we can handle the failure problem. Other problems involved in a communication network such as the capacity of a network [7], queueing problems [8], and any question concerning the maximum flow such as that considered by Ford and Fulkerson [9] have not been discussed here. Interested readers can find many papers in the literature about these subjects. Thus there are many problems concerning networks which have not been treated here, nor have we considered any of the applications of network theory except to communication theory.

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