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On the collection of points of a formal space

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Abstract

The collection of points of a locally compact regular formal space is shown to be isomorphic to a set in the context of Martin-Löf type theory. By introducing the notion of uniform formal space, this result is refined and generalized in the subcategory of open formal spaces.

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0. Introduction

The notion of locale is generally regarded as furnishing the proper concept of topology in topos-theoretic (intuitionistic) contexts [21,20,17]. Formal spaces [30] provide a presentation of locales which is furthermore adequate to be expressed within constructive (intuitionistic and predicative) settings such as Martin-Löf type theory [26,25], and Aczel's constructive set theory [1].

In contrast with what happens with locales in topoi, considered in such settings, the category of formal spaces is not locally small (the class $hom(\mathcal{S}_1, \mathcal{S}_2)$ of continuous functions between two given formal spaces $\mathcal{S}_1, \mathcal{S}_2$ need not be a set): the assumption that all homsets are small is easily seen to imply the powerset axiom [13]. More specifically

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(as observed by T. Coquand and P. Martin-Löf), nor even the collection of points $\mathsf{Pt}(\mathcal{S})$ of a given formal space $\mathcal{S}$ – i.e. the collection of continuous functions from the terminal object to $\mathcal{S}$ – may be assumed to form a set in general.

This paper is principally devoted to showing that, for $\mathcal{S}$ belonging to some important classes of formal spaces, the collection $\mathsf{Pt}(\mathcal{S})$ actually is isomorphic to a set. The first class for which this result is established (in Section 2) is that of compact regular formal spaces. More generally, this holds true for the locally compact regular formal spaces. Compact regular locales/formal spaces are the point-free constructive counterpart of compact Hausdorff spaces [21,9]. Examples of compact regular formal spaces that may illustrate the role played by this class even in constructive settings are the formal space $\mathcal{L}(A)$ of linear functionals of norm less than one over a given semi-normed space A, the Vietoris hyperspace of a given compact regular formal space [7,8], the Stone–Čech compactification of every formal space $\mathcal{S}$ for which the class $hom(\mathcal{S}, [0, 1])$ of continuous functions from $\mathcal{S}$ to the – compact regular – formal unit interval forms a set [13].[2]

In Section 3 the notion of (open) *uniform formal space* is introduced as a natural generalization of that of metric formal space [12], and the relation between complete regularity and uniformizability is analyzed. Uniform formal spaces are organized in a category in Section 4, where also the relation of this category with that of uniform spaces is sketched. These definitions and results are applied (in Section 5) to yield a generalization (in the subcategory of open formal spaces) of the representation of the collection of points of a locally compact regular formal space proved in Section 2. Intuitively, one would like to identify points with 'shrinking' sequences of 'regions'. In [12] it is shown that this may be done for locally compact metric formal spaces; there 'shrinking' means having a vanishing diameter, and 'regions' are basic opens. However, metrizable spaces necessarily satisfy forms of countability assumptions (e.g., first-countability), in general not enjoyed by arbitrary compact Hausdorff spaces. To capture quantitatively arbitrary compact regular formal spaces one is thus led to consider uniformizability, that is, a concept of proximity defined by the point-free equivalent of possibly many different (pseudo-)metrics.

The point-free analogue of a Cauchy complete uniform space, then, enjoys the property that its points may be identified with generalized sequences of neighborhoods (nets), shrinking according to the concept of proximity defined by the given uniformity, and forming a set. This is true, more generally, for the class of 'weakly complete' uniform formal spaces to be defined, comprising both complete uniform formal spaces and locally compact uniform formal spaces. The concept of weak completeness allows us to regard the representation of the collection of points of a compact regular formal space in Section 2, and the analogous fact proved for locally compact metric formal spaces in [12], uniformly in terms of a notion of completeness.

No familiarity with type theory is actually needed to read this paper: the arguments involved in the constructions of the considered classes of points as sets are essentially geometrical, and the proof that these constructions indeed yield sets may be formulated in terms of Bishop's naïve set and subset theory [4]. In this sense, the proof is claimed to be

[2] The (generalized) Stone–Čech compactification of a formal space $\mathcal{S}$ exists constructively exactly when $\mathcal{S}$ enjoys this property. The class of formal spaces for which $hom(\mathcal{S}, [0, 1])$ is a set comprises the locally compact formal spaces [13].

independent from the adopted foundation, provided that this is adequate for constructive mathematics in the sense of Feferman [16]. A large part of the material in this paper (particularly that concerning uniformizability) is intended to be valid also in the (choice-free) context of topoi and in the context of Aczel's constructive set theory, so care is taken to indicate (with an asterisk) those results that depend on a principle of choice.

1. Preliminaries

1.1. A *formal topology* (often simply a *topology*) is a pair $\mathcal{S} \equiv (S, \lhd)$ where S is a set, called the *base*, and the *cover relation* $\lhd$ is a relation between elements and subsets of S satisfying the following conditions (read $a \lhd U$ as 'a is covered by U', or 'U covers a'):

i. $a \in U$ implies $a \lhd U$ (*reflexivity*)
ii. if $a \lhd U$ and $U \lhd V$, then $a \lhd V$ (*transitivity*)
iii. $a \lhd U,\ a \lhd V$ imply $a \lhd U \downarrow V$ ($\downarrow$- *right*)

where $U \lhd V$ stands for $(\forall u \in U) u \lhd V$, and $U \downarrow V$, the *formal intersection* of U, V, is given by $\{b \in S : (\exists u \in U)\ (b \lhd \{u\})\ \&\ (\exists v \in V)\ (b \lhd \{v\})\}$ [10,31].

Two subsets U, V of S are the same *formal open*, $U =_{\mathcal{S}} V$, exactly when $U \lhd V$ & $V \lhd U$. Denote with $Open(\mathcal{S}) \equiv (\mathcal{P}(S), =_{\mathcal{S}})$ the collection of formal opens endowed with this equality. If V, W, Z are subsets of S, and U_i $(i \in I)$ is a family of subsets of S, one has $V \cup (W \downarrow Z) =_{\mathcal{S}} (V \cup W) \downarrow (V \cup Z)$, and $(\bigcup_i U_i) \downarrow V = \bigcup_i (U_i \downarrow V)$. In impredicative contexts, $Open(\mathcal{S})$ forms a set; thus, with $U \wedge V \equiv U \downarrow V$ and $\bigvee_{i \in I} U_i \equiv \bigcup_{i \in I} U_i$, $(Open(\mathcal{S}), \bigvee, \wedge)$ is a frame, and each frame may be obtained as $Open(\mathcal{S})$ for some $\mathcal{S}$ [30].

1.2. A *morphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ of formal topologies is a map $f : S_1 \to \mathcal{P}(S_2)$ such that

i. $S_2 \lhd_2 f(S_1)$,
ii. $f(a \downarrow^1 b) =_{\mathcal{S}_2} f(a) \downarrow^2 f(b)$,
iii. $a \lhd_1 U \to f(a) \lhd_2 f(U)$

(where , for $V \subseteq S$, $f(V) \equiv \bigcup_{b \in V} f(b)$). With this notion of morphism, formal topologies form the category **FT**. Within the topos-theoretic (impredicative) context, **FT** is equivalent to the category of frames [30], so its dual, to be called the category **FSp** of *formal spaces* and *continuous functions*, is equivalent to the category of locales.

1.3. A *formal point* of a formal topology $\mathcal{S}$ is a continuous function from the formal space $T \equiv (\{1\}, \in)$ to $\mathcal{S}$ (i.e. a morphism from $\mathcal{S}$ to T). This may alternatively be described as a subset $\alpha \subseteq S$ such that

i. $(\exists a \in S) a \in \alpha$,
ii. $a \in \alpha$ & $b \in \alpha$ imply $(\exists c)(c \in a \downarrow b$ & $c \in \alpha)$,
iii. $a \in \alpha$ and $a \lhd U$ imply $(\exists b \in U)(b \in \alpha)$.

The class of formal points is denoted by $\mathsf{Pt}(\mathcal{S})$. In contrast with what happens in topoi, $\mathsf{Pt}(\mathcal{S})$ in general forms a proper class in constructive contexts.

1.4. A formal topology $\mathcal{S}$ is said to be *compact* if, whenever $S \lhd U$, there exists a finite subset $\bar{u} = \{u_1, \ldots, u_n\} \subseteq U$ such that $S \lhd \bar{u}$.

For $U \subseteq S$, the *(pseudo-)complement* U^* of U is defined by $U^* \equiv \{b : (b \downarrow U) \lhd \emptyset\}$. For simplicity we will often improperly confuse elements a with singleton subsets $\{a\}$, so that, for instance, the notation a^* is used in place of $\{a\}^*$. Observe that, for U_i $(i \in I)$ a family of subsets of S, $(\bigcup_i U_i)^* = \bigcap_i (U_i^*)$; in particular, for $U, V \subseteq S$, $(U \cup V)^* = U^* \cap V^* = U^* \downarrow V^*$.

A topology $\mathcal{S}$ is said to be *regular* if, for all a in S, $a \lhd wc(a)$, with $wc(a) \equiv \{b : S \lhd a \cup b^*\}$, the subset of elements that are well covered by a.

Finally, for $U, V \subseteq S$, say V is *way-below* U if given any $W \subseteq S$ such that $U \lhd W$ there is a finite subset $\bar{w}$ of W such that $V \lhd \bar{w}$. $\mathcal{S}$ is *locally compact* if it may be endowed with an indexed family $wb(a)(a \in S)$ of subsets of S such that

wb_1: for all $a \in S$ $a =_{\mathcal{S}} wb(a)$,
wb_2: for all $b \in wb(a)$, $\{b\}$ is *way-below* $\{a\}$.

It is non-restrictive [12] to assume that the family of subsets $wb(x)$ for $x \in S$ satisfies the following property, trivially enjoyed also by the family $wc(x)$:

$$b' \lhd b,\ b \in \Box(a) \textit{ and } a \lhd a' \textit{ imply } b' \in \Box(a') \qquad (*)$$

$(\Box \in \{wc, wb\})$.

Remark. Aczel [3] has observed that every locally compact topology $\mathcal{S}$ is *set-presented* in the constructive set theory CZF, i.e. that there are families of sets, $I(a)(a \in S)$, and $C(a,i)(a \in S, i \in I(a))$, $C(a,i) \subseteq S$, such that $a \lhd U \iff (\exists i \in I(a))C(a,i) \subseteq U$ (the same also holds in Martin-Löf type theory [14]).

Note that this allows us to simplify the definition of local compactness: we may say that a locally compact formal topology is a set-presented topology such that, for all $a \in S$, $a \lhd wb'(a)$, where $wb'(a)$ is the subset of elements that are way-below a with respect to just the subsets $C(a,i)$: $b \in wb'(a) \iff (\forall i)(\exists \bar{v}) b \lhd \bar{v}$, with $\bar{v}$ finite subset of $C(a,i)$.

2. Maximal regular subsets form a set

The theorem recalled below was proved in [11]. It asserts that the points of a compact regular formal space may be characterized as subsets of basic neighborhoods (the maximal regular ones) satisfying certain purely first-order conditions. This characterization is the first step in the proof that the class of points of a compact regular formal space is isomorphic to a set.

2.1. Given a formal space $\mathcal{S}$, we call a subset α of S (filtering and) *regular* if it satisfies

1. $(\exists a)(a \in \alpha)$,
2. $(a \in \alpha \ \&\ b \in \alpha) \leftrightarrow (\exists c \in S)(c \in a \downarrow b \ \&\ c \in \alpha)$,
3. $a \in \alpha \rightarrow a \not\lhd \emptyset$,
4. $a \in \alpha \rightarrow (\exists b \in S)(b \in wc(a) \ \&\ b \in \alpha)$.

We say that α is a *maximal regular subset*[3] of $\mathcal{S}$ if moreover, for all a, b in S,

5. $b \in wc(a) \to ((\exists c \in S)(c \in \alpha \;\&\; (b \downarrow c) \lhd \emptyset) \vee a \in \alpha)$,

that is, if a neighbourhood b is well covered by a neighbourhood a, either a is a neighbourhood of α, or one can find a neighbourhood c of α disjoint from b.

Observe that maximal regular subsets are maximal (between regular subsets) in the usual sense: let $\mathcal{S}$ be any formal topology and let $\alpha \subseteq S$ be a maximal regular subset. If $\beta \subseteq S$ is a regular subset and $\alpha \subseteq \beta$ we have $\alpha = \beta$: indeed, let $a \in \beta$; by regularity of β there is $b \in \beta$ such that $b \in wc(a)$. Then (by maximality) either there is $c \in \alpha$ such that $(b \downarrow c) \lhd \emptyset$, or $a \in \alpha$; but $\alpha \subseteq \beta$ implies that $(b \downarrow c) \not\lhd \emptyset$ for all $c \in \alpha$ (since, by 2, from $b, c \in \alpha$ one has $d \in \alpha$ with $d \in b \downarrow c$, and, by 3, $d \not\lhd \emptyset$), and hence $a \in \alpha$.

The following result was proved for topologies with a positivity predicate in [11]. The extension to general topologies indicated here is straightforward (a proof can be found in [14]).

Theorem. *In a compact regular formal topology $\mathcal{S}$, the formal points of $\mathcal{S}$ are precisely the maximal regular subsets of $\mathcal{S}$.*

Note that no second-order object (other than the 'parameter' α) appears in the expression of conditions 1–5.

2.2. In order to emphasize the topological aspects (as opposed to the type-theoretical or set-theoretical ones) of the argument we are to carry out, and at the same time not to bind it to a particular setting, we keep using the common informal mathematical language. This informal language is similar to the one adopted in Bishop's style mathematics, and should allow a reader acquainted with type theory to easily imagine the type-theoretic (formal) version of the proof (a hint is given in 2.3 below).

Since just the principles of Bishop's naive foundation are exploited in the following, we expect the construction to be presented to be valid in every setting in which these principles are formally represented (cf. Feferman's notion of adequate formalization [16]).

Let $\mathcal{S}$ be any topology. For any $x, y \in S$, consider the set $W = \{(x, y) \in S \times S : y \in wc(x)\}$. Observe that, formally, the elements of W will be triples (x, y, p), where p is a proof that $y \in wc(x)$ [16]. We define a predicate D (the *domain* for $\mathcal{S}$) on the *set* $W \to S$ by stipulating that a function $f : W \to S$ satisfies D if and only if

$$\begin{aligned}
&(\forall w_1, w_2 \in W)(\exists w \in W)(f(w) \in f(w_1) \downarrow f(w_2)) \;\& \\
&\& \;(\forall w \in W)[\\
&\qquad f(w) \not\lhd \emptyset \;\& \\
&\qquad \& \;(\exists w' \in W)\, f(w') \in wc(f(w)) \;\& \\
&\qquad \& \;((f(w) \downarrow b_w) \lhd \emptyset \vee (f(w) = a_w))]
\end{aligned}$$

with $(a_w, b_w) = w$. One may think of the elements f of $W \to S$ as of 'sequences' (or, better, nets; cf. 5.4) of elements of S indexed by the pairs (a, b) such that $b \in wc(a)$. Then

[3] The idea for defining maximality for regular subsets in this way was inspired by the notion of *maximal approximation*, as formulated in [23]. This form of maximality condition has been exploited recently also in connection with R-structures [32].

a ‘sequence’ f satisfies D essentially if the set of its elements has the properties required in the conditions defining a maximal regular subset of $\mathcal{S}$. It will thus not come as a surprise that, defining for $f \in W \to S$, f satisfying D,

$$F_f = \{a \in S : (\exists w \in W)\, f(w) \lhd a\},$$

F_f is a maximal regular subset of $\mathcal{S}$ (to check this, observe that in every topology $\mathcal{S}$, each neighbourhood is well covered by the whole space S, and that one may always assume that $\mathcal{S}$ has a ‘top’ basic element 1, $1 =_{\mathcal{S}} S$. Thus, W may be assumed to be non-empty).

Conversely, given a maximal regular subset α of $\mathcal{S}$ we can extract from α a ‘sequence’ f_α as follows. Informally, we define f_α by using the maximality condition and relying on the constructive reading of existential and disjunctive statements: given $w = (a, b) \in W$, by condition 5 we get a proof $\pi(w)$ of

$$((\exists c \in S)(c \in \alpha \;\&\; (b \downarrow c) \lhd \emptyset) \vee a \in \alpha).$$

This means that either we have a proof that a belongs to α, or we have an element c and a proof that it satisfies the first disjunct.

The value of $f_\alpha((a, b))$ is defined accordingly as being either a or c. Thus, in particular $f_\alpha((a, b)) \in \alpha$.

Observe that f_α is a ‘choice’ function,[4] and that its values actually depend not just on the pair (a, b), but also on the particular proof that $b \in wc(a)$ (the formal definition appears in the next paragraph). This dependence is not mentioned explicitly, since it has no effect on what follows.

2.3. Here is a sketch of how this definition may be given formally in type theory (the reader uninterested in the formalization in type theory may safely skip this paragraph). We adopt here the notation in [24]. We also assume that subsets are treated as propositional functions [24, pg. 64], so that, in particular, a point α is a propositional function $\alpha(x)$ $(x \in S)$ ($a \in \alpha$ will stand for $\alpha(a)$ true, and, for $a, b \in S$, $b \in wc(a)$ is the proposition $S \lhd b^* \cup a$). The set W is defined as $(\Sigma x \in S \times S)\mathbf{q}(x) \in wc(\mathbf{p}(x))$ ($\mathbf{p}$, $\mathbf{q}$ being left and right projection, respectively [24, pg. 45]). Observe that, for $w \in W$, $\mathbf{p}(w)$ is an element of $S \times S$, and $\mathbf{q}(w)$ is a proof that $\mathbf{q}(\mathbf{p}(w)) \in wc(\mathbf{p}(\mathbf{p}(w)))$.

For $w \in W$ we set

$$f_\alpha(w) \equiv \mathbf{D}(\pi(w), (u)\mathbf{p}(u), (v)\mathbf{p}(\mathbf{p}(w))),$$

where:

$\mathbf{D}$ is the constant pertaining to the rule of $+$-elimination,
$\pi(w)$ is an element (yielded by condition 5, through the propositions-as-sets interpretation) of the sum $(\sum c \in S)(c \in \alpha \;\&\; (\mathbf{q}(\mathbf{p}(w)) \downarrow c) \lhd \emptyset) + \mathbf{p}(\mathbf{p}(w)) \in \alpha$,
u is a generic element of the first addendum, $(\sum c \in S)(c \in \alpha \;\&\; (\mathbf{q}(\mathbf{p}(w) \downarrow c) \lhd \emptyset)$,
v is a generic element of the second addendum, $\mathbf{p}(\mathbf{p}(w)) \in \alpha$.

[4] In the development of constructive mathematics in Bishop’s style this function would have been part of the definition of maximality of a regular subset, although it may be obtained using the constructive principle of choice available in Bishop’s setting. A discussion of this topic may found in [16, 15.2, 15.3].

2.4. We continue in the previous more informal style. Let α be a maximal regular subset.

Proposition. *$D(f_\alpha)$ true and $F_{f_\alpha} = \alpha$.*

Proof. Note, first, that by definition of f_α one has that, for all a, b such that $b \in wc(a)$, $(f_\alpha((a, b)) \downarrow b) \lhd \emptyset \vee f_\alpha((a, b)) = a$; thus, the last condition in D is satisfied. Moreover, since $f_\alpha((a, b)) \in \alpha$ for all (a, b) in W, $f_\alpha((a, b)) \not\lhd \emptyset$ also holds.

Now let a_1, b_1, a_2, b_2 be such that $b_1 \in wc(a_1)$ and $b_2 \in wc(a_2)$. Since $f_\alpha((a_1, b_1))$ and $f_\alpha((a_2, b_2))$ belong to α, by condition 2 there is $c \in \alpha$ such that $c \in f_\alpha((a_1, b_1)) \downarrow f_\alpha((a_2, b_2))$. Moreover, by condition 4, there is $c' \in \alpha$, $c' \in wc(c)$. As noted, we have $(f_\alpha((c, c')) \downarrow c') \lhd \emptyset \vee f_\alpha((c, c')) = c$. The first disjoint is false however, since c' belongs to α, and thus has non-empty formal intersection with all other neighborhoods in α (recall that $f_\alpha((a, b)) \in \alpha$, for all (a, b) in W). Then $f_\alpha((c, c')) = c \in f_\alpha((a_1, b_1)) \downarrow f_\alpha((a_2, b_2))$, as required.

Now, let a, b be such that $b \in wc(a)$. Since $f_\alpha((a, b)) \in \alpha$, by condition 4 there are $c, c' \in \alpha$, $c \in wc(f_\alpha((a, b)))$, and $c' \in wc(c)$. Then we have $(f_\alpha((c, c')) \downarrow c') \lhd \emptyset \vee f_\alpha((c, c')) = c$. Reasoning as above, we obtain $f_\alpha((c, c')) = c$, which is what we wanted, since $c \in wc(f_\alpha((a, b)))$.

Finally, to prove that $F_{f_\alpha} = \alpha$, we just need to show that $F_{f_\alpha} \subseteq \alpha$, since F_{f_α} and α are two maximal regular subsets. But $F_{f_\alpha} \subseteq \alpha$ is obvious (since $f_\alpha((a, b))$ belongs to α for all $(a, b) \in W$, and since $f_\alpha((a, b)) \lhd c$ implies $c \in \alpha$ by condition 2 on regular subsets). □

Summing up, we have shown that the maximal regular subsets of $\mathcal{S}$ may be identified with (equivalence classes of) effectively defined functions satisfying D. Set $f =_D g$ if and only if $\alpha_f = \alpha_g$.

Theorem*. *The collection of maximal regular subsets of a formal topology $\mathcal{S}$ is isomorphic to the set $\Sigma D \equiv \{f \in W \to S : D(f)\}$, endowed with the equality $=_D$.*

Corollary*. *The collection of points of a compact regular formal space $\mathcal{S}$ is isomorphic to a set.*

2.5. Remarks. i. In [11] a characterization analogous to that recalled at the beginning of this section for the points of a compact regular topology is obtained for *locally* compact regular topologies. On the basis of this characterization, and arguing essentially as above, one proves that more generally the points of locally compact regular formal spaces form a set. This result will also be obtained (for *open* locally compact regular formal topologies) as a corollary of a more general result in Section 5.

ii. In [13] the Cečh–Stone compactification of a formal space is defined. In particular, this associates with a topology $\mathcal{S}$ a compact completely regular topology $\mathcal{S}_\gamma$ in such a way that, if $\mathcal{S}$ is completely regular, the collection of points of $\mathcal{S}$ forms a 'dense' subclass of the points of its compactification. More formally, a pair $(\mathcal{S}', f)$ is a *compactification* of $\mathcal{S}$ if $\mathcal{S}'$ is compact and (completely) regular and $f : \mathcal{S}' \to \mathcal{S}$ is *onto* (for $U \subseteq S$ there is $U' \subseteq S'$ such that $f(U') = U$) and *dense* (for all a, $f(a) \lhd \emptyset \to a \lhd \emptyset$). This is just the point-free way of expressing the standard definition of compactification. In [13], we have in particular that $f(a) = \{a\}$, whence 'dense' means that if a neighbourhood a is covered by the empty set in $\mathcal{S}$ so it is in $\mathcal{S}_\gamma$. By the results in this section, thus, the collection of points of every completely regular formal space may be embedded into a set as a dense subclass in the above sense.

iii. In recent papers (e.g. [31]), formal spaces endowed with a positivity predicate with two arguments $\mathsf{Pos}(a, F)(a \in S, F \subseteq S)$ have been considered. Intuitively, $\mathsf{Pos}(a, F)$ means "there is a point in a whose neighborhoods belong to F". The above result allows us to show that in every (locally) compact regular formal topology $(S, \triangleleft)$ a binary positivity predicate is definable by simply formalizing its meaning, and of course substituting elements of ΣD for points: $\mathsf{Pos}(a, F) \equiv (\exists f \in \Sigma D)(\exists (c, d) \in W)(f((c, d)) \triangleleft a \ \& \ F_f \subseteq F)$.

3. Uniform formal spaces

The representation obtained in the previous section may appear as somewhat unclear. It would be nice to be able to identify (maximal) points with shrinking sequences of neighborhoods (possibly indexed by the natural numbers), where 'shrinking' means that the neighborhoods in the sequence have a vanishing 'diameter'. This is indeed what one obtains in the presence of stronger topological conditions – necessarily involving some kind of countability assumption on the topology – that make it possible to carry out a metrization.[5]

The standard way to get rid of countability assumptions is to make use of a more general criterion of uniform proximity than that determined by just one metric. This leads to the notion of uniformity. Recall [15] that no assumption other than complete regularity is needed for a topological space to be uniformizable (and, with dependent choice, compact regular spaces are completely regular).

A uniformity may always be thought of as defined by a family of pseudo-metrics (*gauge* structure). Thus, uniformizability of a space corresponds to having a notion of proximity defined by as many different criteria as there are pseudo-metrics defining the uniformity.

We then need a point-free analogue of these facts, beginning with a notion of *uniform formal topology*. This will be just a topology endowed with a family of a particular kind of diameter (playing here the role of pseudo-metrics), compatible with the given topology in a natural sense to be specified. The (constructive analogue of the) equivalence between complete regularity and uniformizability may then be re-obtained in this point-free setting. This will lead (in the next section) to achieving our original goal, that is, to identifying the points of any compact (completely) regular open topology with generalized 'sequences' of shrinking neighborhoods, and also to opening the way to a natural generalization of this result.

3.1. For simplicity, we restrict ourselves here to the full subcategory of open formal spaces (a 'conservative' generalization to non-open spaces is work in progress; note, however,

[5] In [12] we proved that the points of (locally) compact metric formal topologies may be indexed precisely by a set made up of such sequences. A metrization result for enumerably completely regular open formal topologies (a constructive point-free version of the Urysohn metrization theorem) thus implies that such an indexing obtains for locally compact enumerably completely regular open topologies (whence, in particular, for compact enumerably regular open topologies) [12].

that the property of being open has been shown to be often needed in connection with uniformizability [22]). The equivalent of *open* locales in the present setting is given by formal spaces with a *positivity predicate* [30], namely a predicate $\mathsf{Pos}(x)$, for x in S, satisfying

i. $\mathsf{Pos}(a)$ and $a \lhd U$ imply $(\exists b \in S) b \in U$ & $\mathsf{Pos}(b)$ (*monotonicity*),
ii. $a \lhd U$ implies $a \lhd U^+$ (*positivity*),

where $U^+ \equiv \{b \in U : \mathsf{Pos}(b)\}$. We write $\mathsf{Pos}(U)$ for $(\exists a \in U)\mathsf{Pos}(a)$. Classically, all formal spaces (frames) are open (with $\mathsf{Pos}(a) \equiv a \not\lhd \emptyset$). Note that one has $\neg\mathsf{Pos}(U) \iff U =_{\mathcal{S}} \emptyset$, and $\mathsf{Pos}(U) \Rightarrow U \neq_{\mathcal{S}} \emptyset$, while $U \neq_{\mathcal{S}} \emptyset \Rightarrow \mathsf{Pos}(U)$ in general cannot be obtained constructively. The positivity predicate, thus, may be thought of as a way of expressing positively the information that an open subset is non-empty.

3.2. The following definition of uniform formal space naturally generalizes that of metric formal space [12]. We recall first some definitions from [12]:

Let $\mathbf{Q}^+$ be the set of positive rational numbers, and let $d(x, r)$, be a relation satisfying

1. $d(a, r)$ & $r < r' \to d(a, r')$, 2. $d(a, r) \to \exists r'(r' < r$ & $d(a, r'))$

for all a in S, $r \in \mathbf{Q}^+$. We say that d is an *elementary diameter* if

(o) $a \lhd \emptyset \to (\forall r) d(a, r)$,
(i) $b \lhd a \to (d(a, r) \to d(b, r))$,
(ii) $S \lhd \{a : d(a, \epsilon)\}$, for all ϵ in $\mathbf{Q}^+$.

Intuitively, $d(a, r)$ holds true when the diameter of the basic neighbourhood a is less than r. Let $Ch(S) \equiv \{(a_1, \ldots, a_n) \in \mathcal{P}_w(S) : \mathsf{Pos}(a_i \downarrow a_{i+1}), i = 1, \ldots, n-1\}$ be the set of *chains* of $\mathcal{S}$, and define, for $(z_1, \ldots, z_n) \in Ch(S)$ and $r \in \mathbf{Q}^+$,

$$lg((z_1, \ldots, z_n), r) \equiv (\exists r_1, \ldots, r_n \in \mathbf{Q}^+)(\forall i) d(z_i, r_i) \ \& \ \Sigma_1^n r_i < r.$$

Denote the set of chains beginning with a and ending with b (i.e., the chains $(z_0, \ldots, z_n) \in Ch(S)$ such that $z_0 = a, z_n = b$) by $Ch(a, b)$. The following relation may be used to evaluate distances:

$$\nu_d(x, y, r) \equiv (\exists (z_0, \ldots, z_n) \in Ch(x, y))(lg((z_1, \ldots, z_{n-1}), r);$$

that is, one has $\nu_d(x, y, r)$ when a chain of length less than r exists connecting x and y. We say that the distance between x and y is less than r.

A *gauge structure* on a open formal topology $\mathcal{S}$ is just a family $\mathcal{D} \equiv \{d_i\}_{i \in \mathcal{I}}$ of elementary diameters on $\mathcal{S}$. Let lg_i and ν_i be the length and distance relations associated with each d_i, and let, for $\bar{i} = \{i_1, \ldots, i_k\} \subseteq I$, $\delta \in \mathbf{Q}^+$,

$$W_{\bar{i}}^{\delta} \equiv \{(c_1, c_2) : (\forall i \in \bar{i}) \exists (c_1, z_1 \ldots, z_n, c_2) \in Ch(c_1, c_2) lg_i((c_1, z_1, \ldots, z_n, c_2), \delta)\}$$

(observe that $(c_1, c_2) \in W_{\bar{i}}^{\delta}$ if and only if, for all $i \in \bar{i}$, $\nu_i(c_1, c_2, r_i)$, $d_i(c_1, s_i)$, $d_i(c_2, t_i)$, for some r_i, s_i, t_i such that $r_i + s_i + t_i < \delta$, i.e. if and only if the distance between c_1 and c_2 plus their diameters is less than δ for every $i \in \bar{i}$).

Define

$$b \lhd_{\delta}^{\bar{i}} a \;\equiv\; (\forall (c_1, c_2) \in W_{\bar{i}}^{\delta}) c_1 \lhd b \rightarrow c_2 \lhd a.$$

We say that b is *uniformly covered* by a, and write

$$b \in uc(a),$$

if

$$(\exists i_1, \ldots, i_k \in \mathcal{I})(\exists \delta \in \mathbf{Q}^+) b \lhd_{\delta}^{\bar{i}} a.$$

A gauge structure $\mathcal{D}$ on a topology $\mathcal{S}$ is *compatible* if, for all a in S,

$$a \lhd uc(a).$$

If $\mathcal{D}$ is a compatible gauge structure on $\mathcal{S}$, the pair $(\mathcal{S}, \mathcal{D})$ is called a (open) *uniform formal topology*, and $\mathcal{S}$ is said to be *uniformizable*. Note that the family of subsets $uc(x)$ for $x \in S$ satisfies property 1.4, $(*)$ (with $\Box = uc$).

It is easy to see that in the particular case of a compatible gauge structure consisting of a single elementary diameter, the uniformly covered relation collapses to the measurably covered relation, and thus the resulting uniform topology is just a metric formal topology [12]; thus, the concept of uniformizability naturally generalizes the notion of metrizability in formal spaces.

The following simple lemma will be applied repeatedly.

Lemma. *Assume $b \lhd_{\delta}^{\bar{i}} a$, for some finite $\bar{i} \subseteq I$ and $\delta > 0$, and let c be such that $d_i(c, \delta/2)$ for all $i \in \bar{i}$, and $\mathsf{Pos}(b \downarrow c)$. Then $c \lhd a$.*

Proof. From $\mathsf{Pos}(b \downarrow c)$, one obtains $c' \lhd b$, $c' \lhd c$, $\mathsf{Pos}(c')$. Since $c' \lhd c$, one has $d_i(c', \delta/2)$ for all $i \in \bar{i}$. Thus (c', c) is a chain shorter than δ for all $i \in \bar{i}$, whence, by $b \lhd_{\delta}^{\bar{i}} a$, $c \lhd a$. $\Box$

3.3. Remark. Uniform locales were first studied (classically) in Isbell [19], where these are point-free versions of uniform spaces in the presentation via systems of coverings (see also [28,29], where a non-constructive diametrization is carried out for Isbell's definition). Johnstone [22] presents an intuitionistic theory of uniformizability and uniform locales, described again in terms of coverings. This is probably the more natural way to define uniformities in a point-free setting. However, it is still not clear (at least not to the author) whether classically equivalent ways of introducing uniformities are such also in the predicative and intuitionistic sense, and the definition given above, deriving from that of gauge structure, seems more suited to a predicative context. This very naturally generalizes the notion of constructive metrizability introduced in [12], allows us to obtain the quantitative characterization of complete regularity to follow, and is the kind of approach needed to carry the standard meaning to the notion of uniform continuity in metric spaces (cf. 4.1).

The concept of point-free uniformity just introduced was conceived with the aim, recalled in the introduction, of describing points as Cauchy nets of neighborhoods, and thus of giving a unified treatment of the set-indexing of classes of points presented here and in [12]. The notions in this section constitute just a sketch, far from being exhaustive, of a constructive theory of uniformizability. A broader treatment of the topic, as well as the analysis of the relationship between these definitions and those in Johnstone [22], are the subject of a work in progress.

3.4. Now we show that gauge structures serve our purpose, namely they allow us to regard complete regularity in quantitative terms.

Let $\mathbf{I} \equiv \{p \in \mathbf{Q} : 0 \leq p \leq 1\}$. Given two subsets U, V of S, a *scale* from U to V is a family of subsets U_p of S, indexed by $\mathbf{I}$, with $U_0 = U$, $U_1 = V$, and such that, for all p, q in $\mathbf{I}$, $p < q$ implies $S \lhd U_p^* \cup U_q$ (i.e. U_p well covered by U_q). If a scale exists from U to V we say that U is *really covered* by V. $\mathcal{S}$ is *completely regular* if it comes equipped with an indexed family $rc(a)$ $(a \in S)$ of subsets of S such that

rc_1: for all $a \in S$, $a =_{\mathcal{S}} rc(a)$, and
rc_2: for all a, b in S, if $b \in rc(a)$, $\{b\}$ is really covered by $\{a\}$.

As for the family wb, one may always assume that the family rc satisfies property 1.4, $(*)$ with $\Box = rc$ [12]. Observe that a completely regular topology is also regular, since for all a, $rc(a) \subseteq wc(a)$.

Remark. A more satisfactory equivalent of the notion of complete regularity may be given for those formal spaces $\mathcal{S}$ for which the class of real-valued continuous functions forms a set [13]. Observe also that in type theory $\mathcal{S}$ completely regular *means* that one has a set-indexed family $(U_{(a,b,i)})_{b \in rc(a), i \in \mathbf{I}}$ that, for fixed a, b with $b \in rc(a)$, is a scale from b to a. In choice-free contexts, complete regularity does not come with such a choice of a scale for each a, b as above (P. Aczel drew the author's attention to this point). We may define this version of the notion 'strong complete regularity'.

Now let $\mathcal{S}$ be a (strongly) completely regular formal topology. With each pair (a, b) in S such that $b \in rc(a)$ one may associate an elementary diameter as follows (cf. [12]): from $b \in rc(a)$ one obtains a scale U_p $(p \in \mathbf{I})$ from b to a. Define V_p $(p \in \mathbf{I})$ by $V_p = U_p$ for all $p < 1$, and $V_1 = S$. The elementary diameter $d_{a,b}$ associated with a, b is given by $d_{a,b}(x, r)$ if and only if

$$(\exists p \in \mathbf{I})(x \lhd V_p \ \&\ p < r) \\ \vee (\exists p_1, p_2 \in \mathbf{I})(x \lhd V_{p_1} \ \&\ (x \downarrow V_{p_2}) \lhd \emptyset \ \&\ p_1 - p_2 < r).$$

This association leads to the *uniformization* of completely regular topologies. Similarly to the metrization result in [12], this consists in 'transforming' the really covered relation into the uniformly covered relation. First, we need to 'extend' the diameters $d_{a,b}$ to finite subsets: for $u = (u_1, \ldots, u_n) \in \mathcal{P}_\omega(S)$ and $r \in \mathbf{Q}^+$, define $\bar{d}_{a,b}(u, r)$ as

$$(\exists p \in \mathbf{I})(u \lhd V_p \ \&\ p < r) \\ \vee (\exists p_1, p_2 \in \mathbf{I})(u \lhd V_{p_1} \ \&\ (u \downarrow V_{p_2}) \lhd \emptyset \ \&\ p_1 - p_2 < r).$$

In [12, 4.3 and 4.4] we proved that:

i. *For all u in $\mathcal{P}_\omega(S)$, $(b \downarrow u) \not\lhd \emptyset$ and $\bar{d}_{a,b}(u, 1)$ implies $u \lhd a$.*

ii. *For all* u, v *in* $\mathcal{P}_\omega(S)$ *such that* $(u \downarrow v) \not\lhd \emptyset$, $\bar{d}_{a,b}(u, r_1)$ *and* $\bar{d}_{a,b}(v, r_2)$ *imply* $\bar{d}_{a,b}(u \cup v, r_1 + r_2)$.

We then have the announced:

Theorem. *Given any (strongly) completely regular open topology* $(\mathcal{S}, rc)$*, the pair* $(\mathcal{S}, \{d_{a,b}\}_{b \in rc(a)})$ *is a uniform formal topology.*

Proof. Since $\mathcal{S}$ is completely regular, it is enough to prove that

$$b \in rc(a) \to b \in uc(a).$$

Given $a, b \in S$ such that $b \in rc(a)$, we show that $b \lhd_1^{(a,b)} a$, i.e. that, if $(c_1 = z_0, \ldots, z_n = c_2)$ is a chain in $Ch(c_1, c_2)$, with $c_1 \lhd b$ and $lg_{(a,b)}((z_0, \ldots, z_n), 1)$, then $c_2 \lhd a$. By $lg_{(a,b)}((z_0, \ldots, z_n), 1)$, one has

$$(\exists r_0, \ldots, r_n \in \mathbf{Q}^+)(\forall k \leq n) d_{a,b}(u_k, r_k) \ \& \ \Sigma_0^n r_k < 1.$$

By ii above, this implies $\bar{d}_{a,b}((u_1, \ldots, u_n), 1)$, whence, by i, we conclude that $\{u_1, \ldots, u_n\} \lhd a$. □

3.5. Classically, uniformizability *coincides* with complete regularity. Intuitionistically, a counterexample to 'uniformizable implies regular' is given in [22], with respect to the notion of uniformity there adopted: every discrete locale is there shown to have a uniformity. We can show that actually this is even metrizable, whence in particular uniformizable in the present sense: define on the discrete formal space $\mathcal{D}(X) \equiv (X, \in, \mathsf{Pos} \equiv \mathit{True})$ an elementary diameter by letting $d(x, r) \equiv \mathit{True}$. We leave it as an easy exercise to verify that this is indeed a compatible elementary diameter.

Now let $\mathcal{S}$ be a formal topology, and $\{d_i\}_{i \in \mathcal{I}}$ be any gauge structure on $\mathcal{S}$. For $\bar{i} = i_1, \ldots, i_t \in \mathcal{I}$, $b \in S$ and $p > 0$, let $U^p_{b,\bar{i}} \equiv \{c_2 \in S : (\exists c_1 \lhd b)(\forall i \in \bar{i})(\exists (b_0, \ldots, b_n) \in Ch(c_1, c_2)) lg_i((b_0, \ldots, b_n), p)\}$, $U^0_{b,\bar{i}} \equiv \{b\}$. Then, if for all such $\bar{i}$, p and for all $b, c \in S$, $\mathsf{Pos}(c \downarrow U^p_{b,\bar{i}})$ is decidable, we have that

$$b \in uc(a) \text{ implies } b \text{ really covered by } a$$

(the scale U_p $(p \in \mathbf{I})$ is constructed as follows: let $b \in uc(a)$, i.e. $(\exists i_1, \ldots, i_k \in \mathcal{I})(\exists \delta \in Q^+) b \lhd^{\bar{i}}_{\delta} a$. Then let $\delta^* = min\{\delta, 1\}$ and let $U_0 = \{b\}$, $U_1 = \{a\}$, $U_p \equiv U^{p\delta^*}_{b,\bar{i}}$, for $0 < p < 1$. Recalling that, for each i, d_i is an elementary diameter, and hence that $S \lhd \{a : d_i(a, \epsilon)\}$, for all ϵ in $\mathbf{Q}^+$, it is an easy exercise to prove that this family indeed yields the required scale).

Thus, *a uniform open topology in which* $\mathsf{Pos}(c \downarrow U^p_{b,\bar{i}})$ *is decidable for all* $\bar{i} \in \mathcal{I}$, $p > 0$ *and for all* $b, c \in S$*, is completely regular*.[6] These considerations show in particular that (classically) uniformizability is invariant under isomorphisms since complete regularity is (constructively) such; this also holds true constructively (but the proof is omitted).

[6] Note that this generalizes [12, 3.7]. Note also that the decidability of $\mathsf{Pos}(x \downarrow y)$ suffices for having that a uniformizable topology is regular.

3.6. The following proposition establishes a connection between the notions of way-below and uniformly covered, refining that between way-below and measurably covered proved in [12].

Proposition. *Let $\mathcal{S}$ be any uniform open topology. If b is way-below a, then $b \in uc(a)$. Thus, if $\mathcal{S}$ is locally compact, we have*

$$wb(a) \subseteq uc(a),$$

for all a.

Proof. Let b be way-below a. Since $a \lhd uc(a)$, there are $b_1, \ldots, b_k$ in $uc(a)$ such that $b \lhd \{b_1, \ldots, b_k\}$. Thus, for $s = 1, \ldots, k$, $b_s \lhd_{\delta^s}^{\bar{i}^s} a$ for some $\bar{i}^s, \delta^s$. Let $\delta = min_s\{\delta^s\}/2$, $\bar{i} = \bigcup_s \bar{i}^s$. Then $b \lhd_\delta^{\bar{i}} a$: assume that, for $c_1 \lhd b$, $c_2 \in S$, and for all $i \in \bar{i}$ there is $(c_1 = z_0, \ldots, z_m = c_2)$ in $Ch(c_1, c_2)$ with $lg_i((z_0, \ldots, z_m), \delta)$. Since $c_1 \lhd b$, $b \lhd \{b_1, \ldots, b_k\}$, and $\mathsf{Pos}(c_1)$, by monotonicity of the positivity predicate one obtains $\mathsf{Pos}(b_s \downarrow c_1)$ for some s. Thus there is $c_1' \lhd c_1$, $c_1' \lhd b_s$ such that $\mathsf{Pos}(c_1')$. Since $c_1' \lhd c_1$, we have $d_i(c_1', r)$ for all r for which $d_i(c_1, r)$. In particular, $d_i(c_1', \delta)$ for all $i \in \bar{i}$. But then we have, for all $i \in \bar{i}_s$, a chain $(c_1', c_1 = z_0, \ldots, z_m = c_2)$, with $lg_i((c_1', z_0, \ldots, z_m), 2\delta)$. Since $2\delta \leq \delta_s$, by $b_s \lhd_{\delta^s}^{\bar{i}^s} a$ we may conclude $c_2 \lhd a$. Thus, $b \lhd_\delta^{\bar{i}} a$. □

3.7. A compact regular topology is (strongly) completely regular (using dependent choice; cf. [12]), and hence uniformizable. We conclude this section by showing that – again invoking countable dependent choice – in a locally compact regular topology, 'way-below' implies 'really covered', and hence also that *locally* compact regular topologies are completely regular.

It is a well known fact that in the lattice-theoretic context the way-below relation interpolates. The following lemma establishes (little more than) the corresponding property in our setting.

Lemma. *Let $\mathcal{S}$ be locally compact, and let U, V be subsets of S with V way-below U. Then there is a finite subset $\bar{z}$ of S such that V is way-below $\bar{z}$ and $\bar{z}$ is way-below U.*

Proof. By local compactness, $U \lhd \bigcup_{a \in U} \bigcup_{b \in wb(a)} wb(b)$. Since V is way-below U, there is a finite subset $\bar{w} \subseteq \bigcup_{a \in U} \bigcup_{b \in wb(a)} wb(b)$ such that $V \lhd \bar{w}$. If $w = \emptyset$, the thesis holds true trivially. Let $\bar{w} = \{c_1, \ldots, c_k\}$, with $c_i \in wb(b_i)$, $b_i \in wb(a_i)$ for some $a_i \in U$. Then the required $\bar{z}$ is given by $\bar{z} \equiv \{b_1, \ldots, b_k\}$. Indeed, let $M \subseteq S$ be such that $\bar{z} \lhd M$. For all $c_i \in \bar{w}$ there is a finite subset $\bar{m}_i \subseteq M$ such that $c_i \lhd \bar{m}_i$ (since each $c_i \in wb(b_i)$). Thus $\bar{w} \lhd \bigcup_i m_i \equiv \bar{m}$, whence $V \lhd \bar{w} \lhd \bar{m}$, that proves that V is way-below $\bar{z}$. To prove that $\bar{z}$ is way-below U, let $N \subseteq S$ be such that $U \lhd N$. Then for each $b_i \in \bar{z}$, there is a finite $\bar{n}_i \subseteq N$ such that $b_i \lhd \bar{n}_i$, whence $\bar{z} \lhd \bigcup_i \bar{n}_i \equiv \bar{n}$. □

3.8. Now we just need to prove the following:

Proposition. *Let $\mathcal{S}$ be regular. Then, given U, V subsets of S, V way-below U implies V well covered by U.*

Proof. By regularity, $U \lhd \bigcup_{a \in U} wc(a)$. Then (since V is way-below U), $V \lhd \bar{m}$, with $\bar{m} \equiv \{b_1, \ldots, b_k\}$, a finite subset of $\bigcup_{a \in U} wc(a)$. Then it suffices to prove that $S \lhd \bar{m}^* \cup U$. For each $b_i \in \bar{m}$ there is $a_i \in U$ such that $b_i \in wc(a_i)$, i.e. $S \lhd b_i^* \cup a_i$. Thus, $S \lhd (b_1^* \cup U) \downarrow \ldots \downarrow (b_k^* \cup U)$. Therefore, $S \lhd (b_1^* \downarrow \ldots \downarrow b_k^*) \cup U$, whence $S \lhd \{b_1, \ldots, b_k\}^* \cup U$, i.e. $S \lhd \bar{m}^* \cup U$. □

Corollary*. *A locally compact regular topology $\mathcal{S}$ is (strongly) completely regular with $rc(a) = wb(a)$ for all a.*

Proof. Given $b \in wb(a)$, using the interpolation property 3.7, Proposition 3.8, and applying dependent choice, construct the required scale U_p $(p \in \mathbf{I})$. □

Note that in considering locally compact (strongly) completely regular topologies in the absence of choice principles, one may as well take the family rc to coincide with the family wb (by [12, 2.8]).

4. The category of uniform formal topologies

In this section the category **UFT** of uniform formal topologies and uniform morphisms is introduced, and the relation of this with the standard category of uniform spaces is sketched.

4.1. Let $(\mathcal{S}, \{d_i\}_{i \in \mathcal{I}})$ be any uniform formal space; recall that we defined, for $\bar{i} = i_1, \ldots, i_k$ in I, δ in $\mathbf{Q}^+$,

$$W_{\bar{i}}^{\delta} \equiv \{(c, d) : (\forall i \in \bar{i}) \exists (c, z_1 \ldots, z_n, d) \in Ch(c, d) lg_i((c, z_1 \ldots, z_n, d), \delta)\}.$$

A morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ between two uniform topologies $(\mathcal{S}_1, \{d_i\}_{i \in \mathcal{I}})$ and $(\mathcal{S}_2, \{d_j\}_{j \in \mathcal{J}})$ is said to be *uniform* if, for all $i \in \mathcal{I}, \epsilon > 0$, there are $\bar{j} = \{j_1, \ldots, j_n\} \subseteq J, \delta > 0$ such that, for all $a, b \in S_1$, $c, d \in S_2$,

$$(c, d) \in W_{\bar{j}}^{\delta} \;\&\; c \lhd f(a) \;\&\; d \lhd f(b) \to \nu_i(a, b, \epsilon).$$

(This definition expresses in the present setting the condition requiring the counter-image of an element of the pre-base of the uniformity on the co-domain to contain a basic element of the uniformity on the domain.) Observe that, when the gauge structures consist of a single elementary diameter, the above condition reduces to the following one: for all $\epsilon > 0$ there is $\delta > 0$ such that, for all $a, b \in S_1$, $c, d \in S_2$,

$$\nu_2(c, d, \delta) \;\&\; c \lhd f(a) \;\&\; d \lhd f(b) \to \nu_1(a, b, \epsilon),$$

which gives the point-free version of the familiar ϵ–δ condition for uniform continuity in metric spaces. With this notion of morphism, uniform formal topologies form the category **UFT** (composition is defined as composition of the underlying morphisms; although posing no particular difficulty, the proof that the composite of two uniform morphisms is a uniform morphism requires some work).

The rest of this section is devoted to sketching some basic facts concerning the relation between **UFT** and the usual category of uniform spaces. Familiarity with the notions of sober space and spatial locale/formal space is presupposed here (see e.g. [18,12]).

4.2. Assume $(X, \mathcal{U})$ is any uniform space, where X is a set and the uniformity $\mathcal{U} = \mathcal{U}(\mathcal{D})$ is presented by the family of pseudo-metrics $\mathcal{D} = \{\rho_i\}_{i \in \mathcal{I}}$ (cf. [4]). One associates with $(X, \mathcal{U})$ a uniform formal space $(\mathcal{S}_X, \{d_i\}_{i \in \mathcal{I}})$ as follows: assume B_X is any set indexing a base of the uniform topology on X (observe that when X is a set, the set of finite inhabited subsets of $X \times \mathbf{Q}^+ \times I$ is a set of indices for the standard base). Then the base whose elements are the unions of any pair of elements of the base indexed by B_X is set-indexed by $B_X \times B_X$. Define $\mathcal{S}_X$ as $B_X \times B_X$, and let, for $x \in S_X$, $ext(x)$ be the basic open indexed by x. Define the cover $\lhd_X$ as point-set inclusion (that is, $a \lhd_X U$ if and only if $ext(a) \subseteq \bigcup_{b \in U} ext(b)$), and $\mathsf{Pos}(a)$ true if and only if $ext(a)$ contains a point. Finally, let $\{d_i\}_{i \in \mathcal{I}}$ be the family of elementary diameters defined by: for $i \in \mathcal{I}, a \in S_X$, $d_i(a, r)$ if and only if for all x, y in $ext(a)$, one has $\rho_i(x, y) < r' < r$ (i.e., iff the standard point-set diameter associated with the pseudo-metric ρ_i is strictly less than r on $ext(a)$). It is not difficult to check that the pair $(\mathcal{S}_X, \{d_i\}_{i \in \mathcal{I}})$ thus defined is a uniform formal space.

4.3. For $i \in I, \alpha, \beta \in \mathcal{P}t(\mathcal{S}_X), r \in \mathbf{Q}^+$, let

$$\bar{\rho}_i(\alpha, \beta, r) \equiv (\exists r' < r)(\forall a \in \alpha, b \in \beta)\nu_i(a, b, r').$$ [7]

If X is sober, one has $\mathcal{P}t(\mathcal{S}_X) \cong X$. If α_x is the formal point associated with x under the bijection $X \cong \mathcal{P}t(\mathcal{S})$, one also has $\rho_i(x, y) < r \iff \bar{\rho}_i(\alpha_x, \alpha_y, r)$, so that $\bar{\rho}_i(\alpha_x, \beta_y) \equiv inf\{r : \bar{\rho}_i(\alpha_x, \beta_y, r)\}$ (exists and) coincides with $\rho_i(x, y)$ (recall that the reals are not order-complete intuitionistically). Thus, for every sober uniform space $(X, \{\rho_i\}_{i \in I})$, $(X, \{\rho_i\}_{i \in I}) \cong (\mathcal{P}t(\mathcal{S}_X), \{\bar{\rho}_i\}_{i \in I})$.

If $(\mathcal{S}, \{d_i\}_{i \in \mathcal{I}})$ is any uniform formal topology, the topology induced on $\mathsf{Pt}(\mathcal{S})$ by the family $\{\bar{\rho}_i(\alpha, \beta, r)\}_{i \in I}$, namely that having as base the finite intersections of balls of the form $B_i(\alpha, r) \equiv \{\beta : \rho_i(\alpha, \beta, r)\}$, coincides with the spatial topology on $\mathcal{P}t(\mathcal{S})$. Thus, *a sober space X is uniformizable by a family of pseudo-metrics $\mathcal{D} = \{\rho_i\}_{i \in \mathcal{I}}$ if and only if $\mathcal{S}_X$ is uniformizable by a family $\{d_i\}_{i \in \mathcal{I}}$ of elementary diameters such that $\bar{\rho}_i(\alpha, \beta) \equiv inf\{r : \bar{\rho}_i(\alpha, \beta, r)\}$ exists in $\mathbf{R}$ for all $i \in I, \alpha, \beta \in \mathcal{P}t(\mathcal{S})$.*

4.4. If $f : X \to Y$ is any uniformly continuous function between two uniform spaces X, Y, then, in particular, it is continuous. One easily checks that the morphism f^* between $\mathcal{S}_Y$ and $\mathcal{S}_X$ associated with f ($f^*(a) = f^{-1}[ext(a)]$) is uniform. Conversely, if $g : (\mathcal{S}_2, \{d_j\}_{j \in J}) \to (\mathcal{S}_1, \{d_i\}_{i \in I})$ is a uniform morphism, the continuous map (for the spatial topologies) induced on the collection of points $g^{pt} : \mathcal{P}t(\mathcal{S}_1) \to \mathcal{P}t(\mathcal{S}_2)$ ($g^{pt}(\alpha) = \bigcup_{a \in \alpha}\{b \in S_2 : a \in g(b)\}$; cf. [30]) is uniformly continuous (in the sense that for all $j \in J, \epsilon > 0$ there is a finite $\bar{i} \subseteq I$ and $\delta > 0$ such that, if $\bar{\rho}_i(\alpha, \beta, \delta)$ for all $i \in \bar{i}$, then $\bar{\rho}_j(g^{pt}(\alpha), g^{pt}(\beta), \delta)$). Thus, a map $f : X \to Y$ between two sober uniform spaces X, Y is uniformly continuous if and only if $f^* : \mathcal{S}_Y \to \mathcal{S}_X$ is a uniform morphism.

Denote with $\mathbf{UFT}^*_{\mathbf{Sp}}$ the full subcategory of **UFT** whose objects are those spatial uniform topologies $(\mathcal{S}, \{d_i\}_{i \in \mathcal{I}})$ for which $\mathcal{P}t(\mathcal{S})$ is (isomorphic to) a set and $\bar{\rho}_i(\alpha, \beta)$ exists in $\mathbf{R}$ for all $i \in I, \alpha, \beta \in \mathcal{P}t(\mathcal{S})$. A few more steps show that there is a duality between the category of sober uniform spaces and $\mathbf{UFT}^*_{\mathbf{Sp}}$. A more detailed and exhaustive

[7] Note that these relations may be regarded as pseudo-metrics [12].

discussion on the relationships between these two categories, and between **UFT** and existing categories of uniform locales, will be presented elsewhere.

5. Weakly complete formal spaces

Now we are in a position to show that the points of a compact regular open formal topology may be identified with generalized sequences that contain, for each diameter in the associated gauge structure, an arbitrarily small neighbourhood (Cauchy nets of basic neighborhoods). This corresponds to Cauchy completeness of the uniformity associated with a compact Hausdorff space. One may thus expect this result to hold for (the point-free counterpart of) all Cauchy complete uniform spaces; in this section the result is proved indeed for an even wider class of uniform topologies, that of weakly Cauchy complete topologies, which contains, beside Cauchy complete topologies, locally compact uniform formal topologies. The notion of weak completeness plays here just an auxiliary role. It allows us to regard uniformly, in terms of a concept of completeness, the solution to the problem of representing the collection of points as a set for formal spaces belonging to different classes.

5.1. Let $\mathcal{S}_{\mathcal{D}} \equiv (\mathcal{S}, \{d_i\}_{i\in\mathcal{I}})$ be any uniform topology, and let $uc'(a)(a \in S)$ be a family of subsets of S such that $uc'(a) \subseteq uc(a)$, $a \lhd uc'(a)$ for all a, and satisfying condition 1.4 $(*)$ (with $\Box = uc'$). We call a subset α of S (filtering and) *uniform with respect to* uc' (often simply uniform) if it satisfies conditions 1, 2 of Section 2, and

$3'$. $a \in \alpha \to \mathsf{Pos}(a)$,

$4'$. $a \in \alpha \to (\exists b)(b \in uc'(a) \ \& \ b \in \alpha)$.

We say that a uniform subset is a *Cauchy uniform subset of* $\mathcal{S}_{\mathcal{D}}$ *with respect to* uc' (often simply a Cauchy uniform subset) if it also satisfies

$5'$. $(\forall \epsilon \in \mathbf{Q}^+)(\forall i \in \mathcal{I})(\exists a)(a \in \alpha \ \& \ d_i(a, \epsilon))$.

Cauchy uniform subsets may be regarded as an elementary alternative to the classical notion of round Cauchy filters.

Clearly, given any uniform topology $\mathcal{S}_{\mathcal{D}}$, the points of $\mathcal{S}_{\mathcal{D}}$ are Cauchy uniform subsets with respect to uc', for every uc' as above (by $a \lhd uc'(a)$, 3.2, ii, and by 1.3, iii). The converse may be false for *every* such uc'. When it is the case that the points of $\mathcal{S}_{\mathcal{D}}$ coincide with its Cauchy uniform subsets for a given family uc', $\mathcal{S}_{\mathcal{D}}$ is said to be *weakly (Cauchy) complete* (with respect to uc'). A formal topology is *weakly complete* if it is isomorphic to a uniform topology $(\mathcal{S}, \mathcal{D})$ weakly complete with respect to uc' for some family uc'.

Note, finally, that the Cauchy uniform subsets are maximal among uniform subsets, in the sense that, if β is a Cauchy uniform subset and α is any uniform subset, $\beta \subseteq \alpha$ implies $\alpha = \beta$: indeed, let $b \in \alpha$. By $4'$ there is $b' \in \alpha$ such that $b' \lhd_{\delta}^{\bar{i}} b$ for some δ and $\bar{i}$. Since $\beta \subseteq \alpha$, for all $c \in \beta$ we have $\mathsf{Pos}(c \downarrow b')$ (by $3'$, 2). Moreover, since β satisfies $5'$, we can choose $c \in \beta$ such that $d_i(c, \delta/2)$ for all $i \in \bar{i}$ (by $5'$, 2 and 3.2, i). Since $b' \lhd_{\delta}^{\bar{i}} b$, by Lemma 3.2 one has $c \lhd b$, and by 2 (with $a = b$), we conclude that $b \in \beta$.

5.2. For U in the frame *Open*$(\mathcal{S})$ (1.1), define $d_i(U, r) \iff (\exists r' < r)(\forall a, b \in Pos)(a \lhd U \,\&\, b \lhd U \to \nu_{d_i}(a, b, r'))$. A *Cauchy filter* on *Open*$(\mathcal{S})$ is a filter F on *Open*$(\mathcal{S})$ such that for all $i \in \mathcal{I}$, $\epsilon \in \mathbf{Q}^+$ there is U in F with $d_i(U, \epsilon)$. Rephrasing in this context the notion of Cauchy completeness for spaces and locales [5] one may define $(\mathcal{S}, \{d_i\}_{i\in\mathcal{I}})$ as being *Cauchy complete* if every Cauchy filter F on *Open*$(\mathcal{S})$ contains a point of $\mathcal{S}$. Clearly, every sober uniform space $(X, \mathcal{U})$ is complete if and only if its point-free representation as a uniform topology $(\mathcal{S}_X, \{d_i\}_{i\in\mathcal{I}})$ (4.2) is Cauchy complete.

However, the notion of Cauchy completeness seems of rare use in predicative contexts.[8] What matters here is that:

Proposition. *Every Cauchy complete uniform topology is weakly Cauchy complete for every family uc' as above.*

Proof. The Cauchy uniform subsets correspond to a subcollection of Cauchy filters on *Open*$(\mathcal{S})$, and since all the latter converge, so do the former: assume α is a Cauchy uniform subset on $\mathcal{S}$; define a filter $\mathcal{F}_\alpha \subseteq$ *Open*$(\mathcal{S})$ by $U \in \mathcal{F}_\alpha \iff (\exists a \in U) a \in \alpha$. Using the fact that α is Cauchy, one easily proves that F_α is a Cauchy filter. By Cauchy completeness, F_α contains a point β of $\mathcal{S}$. This implies that $\beta \subseteq \alpha$, but since β is also a Cauchy uniform subset, by maximality one concludes that $\beta = \alpha$. □

5.3. Now we show that every locally compact uniform topology — and hence, every locally compact (strongly completely) regular open topology is weakly Cauchy complete.

Let $uc' = wb$. By Proposition 3.6 (and by 1.4 $(*)$) uc' satisfies the required properties. Now, a formal point α of a locally compact formal topology trivially satisfies the condition $a \in \alpha \to (\exists b)(b \in wb(a) \,\&\, b \in \alpha)$, and hence is a uniform subset with respect to wb (the left implication in condition 2 follows from 1.3, iii, and 3′ is obtained by positivity and again 1.3, iii). Furthermore, in a uniform formal topology, formal points contain 'arbitrarily small' neighborhoods, according to each one of the diameters in the gauge structure (by 3.2, ii and 1.3, iii), i.e., condition 5′ is satisfied. Thus, in a locally compact uniform topology, the points are Cauchy uniform subsets with respect to wb. But also the converse holds true:

Theorem. *Let $(\mathcal{S}, \{d_i\}_{i\in\mathcal{I}})$ be a locally compact uniform formal topology. A subset α of S is a formal point of $\mathcal{S}$ if and only if it is a Cauchy uniform subset of $\mathcal{S}$ with respect to wb.*

Proof. Let α be a Cauchy uniform subset with respect to wb. Conditions 1.3, i, ii and iv are trivially satisfied. Then let $a \in \alpha$, $U \subseteq S$ and $a \lhd U$. Since α is uniform with respect to wb, the subset U may be assumed to be finite (cf. [12, Lemma 5.2]); then let $U = \{a_1, \ldots, a_t\}$. By 4′, there is b in α, $b \in wb(a)$. Since $\mathcal{S}$ is a uniform topology, for $s = 1, \ldots, t$, we have $a_s \lhd uc(a_s)$. Then, by local compactness, $b \lhd U_b$, with U_b finite subset of $\bigcup_{s=1}^{t} uc(a_s)$.

[8] This notion has anyway a rather peculiar behavior in point-free topology, even in topoi; see [5, pg. 75]. In [5] the relation of Cauchy completeness of a uniform locale with the following different, properly stronger, concept of completeness for localic uniformities is also analyzed: a uniform frame L is said to be *complete* (in the sense of Kříž) whenever every dense and surjective uniform homomorphism from a uniform frame L' to L is an isomorphism.

Let $U_b \equiv \{c_1, \ldots, c_k\}$ where, for all $j = 1, \ldots, k$, $c_j \in uc(a_s)$ for some s, i.e., $c_j \lhd^{\bar{i}^j}_{\delta^j} a_s$ for some δ^j, and $\bar{i}^j$. Let $\delta^* = min_{1\leq j\leq k}\{\delta^j\}/2$, $\bar{i} = \bigcup_{j=1}^{k} \bar{i}^j$.

Consider b' in α such that $d_i(b', \delta^*)$ for all $i \in \bar{i}$ and $b' \lhd b$ (this exists since for all $i \in \bar{i}$ there is $b_i \in \alpha$ such that $d_i(b_i, \delta^*)$; the set of these b_i is finite, thus, from b in α and condition 2 we obtain the required b'). Since $b' \lhd b \lhd U_b$, by $\downarrow$*-right* and monotonicity, one has $\mathsf{Pos}(b' \downarrow U_b)$, i.e., there is k such that $\mathsf{Pos}(b' \downarrow c_k)$. By $c_j \lhd^{\bar{i}^j}_{\delta^j} a_s$ for all j and some s, and by $d_i(b', \delta^*)$, with $\delta^* \leq min\{\delta^j\}/2$ for all $i \in \bar{i}$, applying Lemma 3.2 we get $b' \lhd a_s$ for some s, whence $a_s \in \alpha$ (by 2, with a, b both equal to a_s), as desired. □

5.4. Now we can prove that the Cauchy uniform subsets may be identified with a set of ‘shrinking’ generalized sequences (Cauchy nets of basic opens). Let $V \equiv \mathcal{P}^*_\omega(\mathcal{I}) \times \mathbf{Q}^+$, with $\mathcal{P}^*_\omega(\mathcal{I})$ the set of inhabited finite subsets of $\mathcal{I}$. Consider the following predicate D_c, defined on the set of functions from the set V to S: for $f : V \to S$, $D_c(f)$ is to be true if and only if

$$(\forall v_1, v_2 \in V)(\exists v \in V)(f(v) \in f(v_1) \downarrow f(v_2))\ \&$$
$$\&\ (\forall v \in V)[$$
$$\mathsf{Pos}(f(v))\ \&$$
$$\&\ (\exists v' \in V)\, f(v') \in uc'(f(v))\ \&$$
$$\&\ (\forall i \in \bar{i}_v) d_i(f(v), \epsilon_v)]$$

where $(\bar{i}_v, \epsilon_v) = v$ (compare 2.2). One easily checks that, given f such that $D_c(f)$ *true*, $F_f \equiv \{a \in S : (\exists v \in V)\, f(v) \lhd a\}$ satisfies conditions 1–5′, and thus is a Cauchy uniform subset of $\mathcal{S}$.

Conversely, given a Cauchy uniform subset α, define f_α as follows: for $\bar{i} \in \mathcal{P}^*_\omega(\mathcal{I})$, $\epsilon \in \mathbf{Q}^+$, by 5′, 2 we have $c \in \alpha$ such that $d_i(c, \epsilon/2)$ for all $i \in \bar{i}$. This yields a (choice) function defined by

$$f_\alpha(v) = c,$$

for $v = (\bar{i}, \epsilon)$ (the type-theoretic definition is $f_\alpha(v) = \mathbf{p}(k(v))$, where $k(v)$ is a proof of $(\exists c \in \alpha)(\forall i \in \mathbf{p}(v)) d_i(c, \mathbf{q}(v)/2)$, and $\mathbf{p}, \mathbf{q}$ are the projections related to the $\sum$ type).

Proposition. *$D_c(f_\alpha)$ true and $F_{f_\alpha} = \alpha$.*

Proof. The first property to check is $(\forall(\bar{i}_1, \epsilon_1), (\bar{i}_2, \epsilon_2))(\exists(\bar{i}_3, \epsilon_3))(f_\alpha((\bar{i}_3, \epsilon_3)) \in f_\alpha((\bar{i}_1, \epsilon_1)) \downarrow f_\alpha((\bar{i}_2, \epsilon_2)))$. Since $f_\alpha((\bar{i}_1, \epsilon_1))$ and $f_\alpha((\bar{i}_2, \epsilon_2))$ belongs to α, there is $c \in \alpha$, $c \in f_\alpha((\bar{i}_1, \epsilon_1)) \downarrow f_\alpha((\bar{i}_2, \epsilon_2))$; there is, moreover, c' in α, $c' \in uc'(c)$, whence $c' \lhd^{\bar{i}}_{\delta} c$ for some $\bar{i} \in \mathcal{P}^*_\omega(\mathcal{I})$, $\delta \in \mathbf{Q}^+$. We have $\mathsf{Pos}(f_\alpha((\bar{i}, \delta)) \downarrow c')$ (since $f_\alpha((\bar{i}, \delta)), c'$ both belong to α). Then, by $c' \lhd^{\bar{i}}_{\delta} c$ and Lemma 3.2, one obtains $f_\alpha((\bar{i}, \delta)) \lhd c$ (recall that, by definition of f_α, $d_i(f_\alpha((\bar{i}, \delta)), \delta/2)$ for all $i \in \bar{i}$). We may thus put $(\bar{i}_3, \epsilon_3) = (\bar{i}, \delta)$.

To show that $\forall(\bar{i}, \epsilon)\exists(\bar{j}, \epsilon')(f_\alpha((\bar{j}, \epsilon')) \in uc'(f_\alpha((\bar{i}, \epsilon)))$, one uses again the same trick: since $f_\alpha((\bar{i}, \epsilon))$ belongs to α, there are c, c' in α such that $c \in uc'(f_\alpha((\bar{i}, \epsilon)))$, $c' \in uc'(c)$. Reasoning as above, by $c' \lhd^{\bar{j}}_{\delta} c$ for some $\bar{j} \in \mathcal{P}^*_\omega(\mathcal{I})$ and $\delta \in \mathbf{Q}^+$, one obtains $f_\alpha((\bar{j}, \delta)) \lhd c$. Thus, since $c \in uc'(f_\alpha((\bar{i}, \epsilon)))$, one concludes that $f_\alpha((\bar{j}, \delta)) \in$

$uc'(f_\alpha((\bar{i}, \epsilon)))$. The verification of the other properties is straightforward. Finally, to prove $F_{f_\alpha} = \alpha$, argue as in the proof of Proposition 2.4. □

We have thus proved the following

Theorem*. *The collection of points of a weakly complete formal space $\mathcal{S}$ is isomorphic to a set.*

Remark. Recall that a compact (strongly completely) regular open topology is locally compact (strongly completely) regular (with $wc = rc = wb$ [12]). By 3.4, every such topology is uniform, and by 5.3 it is weakly complete with respect to wb; thus, the above result generalizes Corollary 2.4 (considered in the subcategory of open formal spaces). At the same time, this generalizes [12, 5.3].

The Cauchy uniform subsets may be regarded as Cauchy nets in the following sense: define a directed partial order on $\mathcal{P}^*_\omega(\mathcal{I}) \times \mathbf{Q}^+$ by letting $(\bar{i}_1, \epsilon_1) \leq (\bar{i}_2, \epsilon_2) \iff \bar{i}_1 \subseteq \bar{i}_2 \;\&\; \epsilon_2 \leq \epsilon_1$. We may thus regard a mapping $f : \mathcal{P}^*_\omega(\mathcal{I}) \times \mathbf{Q}^+ \to S$ as a net of basic neighborhoods. One may describe as *uniform* those nets on $\mathcal{P}^*_\omega(\mathcal{I}) \times \mathbf{Q}^+$ satisfying the clauses in D_c except the last one, and say that a uniform net is *Cauchy* if it moreover satisfies the following condition: for all i, ϵ there is $(\bar{i}', \epsilon')$ such that $d_i(f(\bar{i}'', \epsilon''), \epsilon)$ for all $(\bar{i}'', \epsilon'') \geq (\bar{i}', \epsilon')$. In this case we say that $D'_c(f)$. Clearly, given f such that $D'_c(f)$, F_f is a Cauchy uniform subset, and since $D_c(f) \Rightarrow D'_c(f)$, given any Cauchy uniform subset α, there is f_α with $D'_c(f_\alpha)$ and $F_{f_\alpha} = \alpha$.

6. Conclusion and related work

The identification of the points of a formal space with (uniform) Cauchy nets is considerably more informative than the representation in 2.4. In a weakly Cauchy complete metric formal space $\mathcal{S}$, for instance, it allows us not just to say that the points of $\mathcal{S}$ form a set, but also that (some) points *exist*: starting from any positive neighbourhood a, using the principle of dependent choice, one can easily build a Cauchy uniform net to which a belongs (this is more easily recognized by observing that, in these hypotheses, nets can be replaced by sequences; cf. [12]).

The construction of the collection of points of a regular (locally) compact formal space as a set in 2.4 has circulated in the form of a draft since June 2001. The result was first presented at a conference in April 2002. On that occasion, P. Aczel suggested that a choice-free version of the construction might probably be obtained in the constructive set theory CZF. This appears in [3], where also generalizations of this result (in particular to set-presentable and regular formal topologies) are described.

Another (type-theoretical) generalization of Corollary 2.4 is due to E. Palmgren: making use of a particular kind of type universe and exploiting (a form of) dependent choice, it is shown in [27] that the collection of points of a set-presented formal topology whose points are maximal forms a set in Martin-Löf type theory (a corresponding result is shown to hold in the constructive set theory CZF + uREA + DC in [3]).

Applications of the representations obtained in this paper may be in the following direction: in the formalization/development of mathematics in constructive type/set theories, the carriers of the mathematical structures under consideration (rings, metric or vector spaces, spaces of functionals,...) are generally required to be sets (see e.g. [2,6]. Note that this is always required also in the context of Bishop's constructive mathematics). It may be nontrivial to show that this requirement is actually met by a given class, or by the class yielded by a particular construction (as, for instance, for the completion of a uniform space). The results in this, and the related papers mentioned, may be instrumental in this sense.

References

[1] P. Aczel, The type-theoretic interpretation of constructive set theory, in: A. MacIntyre, L. Pacholski, J. Paris (Eds.), Logic Colloquium'77, North-Holland, 1979, pp. 55–66.

[2] P. Aczel, Galois: A Theory Development Project, Report for the 1993 Turin meeting on the Representation of Mathematics in Logical Frameworks.

[3] P. Aczel, Aspects of general topology in constructive set theory, Ann. Pure Appl. Logic (in press) doi:10.1016/j.apal.2005.05.016.

[4] E. Bishop, D. Bridges, Constructive Analysis, Springer-Verlag, Berlin, 1985.

[5] B. Banaschewski, A. Pultr, Samuel compactification and completion of uniform frames, Math. Proc. Camb. Phil. Soc. 108 (1990) 63–78.

[6] V. Capretta, Universal algebra in type theory, in: Y. Bertot, G. Dowek, A. Hirschowits, C. Paulin, Laurent Théry (Eds.), Theorem Proving in Higher Order Logics, 12th International Conference, TPHOLs'99, LNCS, vol. 1690, Springer, 1999, pp. 131–148.

[7] J. Cederquist, T. Coquand, Entailment relations and distributive lattices, in: S.R. Buss et al. (Eds.), Logic Colloquium'98, Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, Prague, Czech Republic, August 9–15, 1998, Lect. Notes Log., vol. 13, A K Peters, Ltd, Natick, MA, 2000, pp. 127–139.

[8] T. Coquand, A direct proof of the localic Hahn–Banach Theorem (unpublished).

[9] T. Coquand, Formal compact Hausdorff spaces, unpublished note.

[10] T. Coquand, G. Sambin, J. Smith, S. Valentini, Inductively generated formal topologies, Ann. Pure Appl. Logic 124 (1–3) (2003) 71–106.

[11] G. Curi, The points of (locally) compact regular formal topologies, in: P.M. Schuster et al. (Eds.), Reuniting the Antipodes—Constructive and Nonstandard Views of the Continuum, Synthèse Library, vol. 306, Kluwer Academic Publishers, Dordrecht, 2001, pp. 39–54.

[12] G. Curi, Constructive metrizability in point-free topology, Theoret. Comput. Sci. 305 (2003) 85–109.

[13] G. Curi, Exact approximations to Stone–Čech compactification (submitted for publication).

[14] G. Curi, Geometry of Observations. Some contributions to (constructive) point-free topology, Ph.D. Thesis, University of Siena, 2003.

[15] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[16] S. Feferman, Constructive theories of functions and classes, in: M. Boffa, D. van Dalen, K. McAloon (Eds.), Logic Colloquium'78, North-Holland, 1979, pp. 159–224.

[17] M. Fourman, R. Grayson, Formal Spaces, in: A.S. Troelstra, D. van Dalen (Eds.), The L.E.J. Brouwer Centenary Symposium, North-Holland, 1982, pp. 107–122.

[18] M. Fourman, D.S. Scott, Sheaves and logic, in: M. Fourman et al. (Eds.), Applications of Sheaves, Springer LNM, vol. 753, Springer-Verlag, 1979, pp. 302–401.

[19] J.R. Isbell, Atomless part of space, Math. Scand. 31 (1972) 5–32.

[20] P.T. Johnstone, Stone Spaces, Cambridge University Press, 1982.

[21] P.T. Johnstone, The points of pointless topology, Bull. Amer. Math. Soc. 8 (1) (1983).

[22] P.T. Johnstone, A constructive theory of uniform locales, I: uniform covers. General topology and applications, in: Proc. 5th Northeast Conf., New York, NY, USA, 1989, LNPAM, vol. 134, 1991, pp. 179–193.

[23] P. Martin-Löf, Notes on Constructive Mathematics, Almqvist and Wiksell, Stockholm, 1970.

[24] P. Martin-Löf, Intuitionistic Type Theory, Studies in Proof Theory. Lecture Notes, vol. 1, Bibliopolis, Napoli, 1984.
[25] B. Nördstrom, K. Peterson, J. Smith, Programming in Martin-Löf's Type Theory, Clarendon Press, Oxford, 1990.
[26] B. Nördstrom, K. Peterson, J. Smith, Martin-Löf's type theory, in: S. Abramsky, D.M. Gabbay, T.S.E. Taibaum (Eds.), Handbook of Logic in Computer Science, vol. 5, Clarendon Press, 2000, pp. 1–37.
[27] E. Palmgren, Maximal and partial points in formal spaces, Ann. Pure Appl. Logic (in press) doi:10.1016/j.apal.2005.05.024.
[28] A. Pultr, Pointless uniformities I. Complete regularity, Comm. Math. Univ. Carolinae 25 (1984) 91–104.
[29] A. Pultr, Pointless uniformities II. (Dia)metrisation, Comm. Math. Univ. Carolinae 25 (1984) 105–120.
[30] G. Sambin, Intuitionistic formal spaces — a first communication, in: D. Skordev (Ed.), Mathematical Logic and its Applications, Plenum, New York, London, 1987, pp. 187–204.
[31] G. Sambin, Some points in formal topology, Theoret. Comput. Sci. 305 (2003) 85–109.
[32] M.B. Smyth, The constructive maximal point space and partial metrizability, Ann. Pure Appl. Logic (in press) doi:10.1016/j.apal.2005.05.032.