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Fuzzy simple expansion

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Abstract
In this paper we generalize the concept of simple expansion due to Levine (1964) to the fuzzy setting. Also, we introduce new classes of fuzzy sets, namely fuzzy η-preopen sets, fuzzy weakly η-open sets and fuzzy weakly η-preopen sets. This families not only depend on the fuzzy topology τ but also on its simple expansion and we study their fundamental properties.

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1. Introduction

The concept of fuzzy operations and fuzzy set operations was first introduced by Zadeh (1965). A fuzzy set on a non-empty set X is a mapping from X into the unit interval I = [0, 1]. The null set 0X is the mapping from X into I which assumes the only the value 0 and the whole fuzzy set 1X is the mapping from X into I which takes value 1 only. A family τ of fuzzy sets of X is called a fuzzy topology (Chang, 1968) on X if 0X and 1X belong to τ and τ closed under arbitrary union and finite intersection. The members of τ are called fuzzy open sets and their complements are fuzzy closed sets.

In 1982, Mashhour et al. (1982) defined preopen sets in topological spaces. In 1991, Singal and Prakash (1991) generalized the concept of preopen sets to the fuzzy setting. In 1991, Bin Shahna (1991) introduced the concept of preopen function between two fuzzy topological spaces. In 1985, Rose (1984) defined weakly open functions in topological spaces. In 1997, Park et al. (1997) introduced the notion of weakly open functions between two fuzzy topological spaces. In 2004, Caldas et al. (2004) introduced the concept of fuzzy weakly preopen functions which is weaker than fuzzy preopen sets. Simple expansion was first introduced in 1964 by Levine (1964) as “Let (X, τ) is a topological space and B ⊆ X, B ≠ τ, then the topology τ(B) = {O ∪ (O ∩ B) : O, O' ∈ τ} is called simple expansion of τ by B‘. Also, a similar concept discussed by Hewit in 1943 (Hewit, 1943) as “If (X, τ) be a topological space and B ⊆ X, B ≠ τ, then the class τ U {B} is a subbase for a topology τ‘ finer than τ, τ‘ is called a simple expansion of τ by B‘. In this paper we generalized the concept of simple expansion due to Levine (1964) to the fuzzy setting. Also, we introduced new classes of fuzzy sets, namely fuzzy η-preopen sets, fuzzy weakly η-open sets and fuzzy weakly η-preopen sets. This families not only depend on the fuzzy topology τ but also on its simple expansion and we study their fundamental properties.
2. Expansion of fuzzy topology

In this paper we generalize the concept of simple expansion due to Levine (1964) to the fuzzy setting as the following:

**Definition 2.1.** Let \((X, \tau)\) be a fuzzy topological space, \(\eta \in P^X\) such that \(\eta \not\in \tau\). Then the fuzzy topology \(\tau_\eta = \{\rho \cap (\rho' \cap \eta): \rho, \rho' \in \tau\} \) is called a simple expansion of \(\tau\) by \(\eta\).

**Example 2.1.** Let

\[
a(x) = \begin{cases} 
1 - 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{2}, & \text{if } \frac{1}{2} < x \leq 1 
\end{cases}
\]

\[
b(x) = \begin{cases} 
1 - 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{2}, & \text{if } \frac{1}{2} < x \leq 1 
\end{cases}
\]

\[
c(x) = \begin{cases} 
\frac{3x}{2}, & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{2}, & \text{if } \frac{1}{2} < x \leq 1 
\end{cases}
\]

**Theorem 2.1.** If \((X, \tau)\) is a fuzzy topological space and \(\eta \in P^X\), \(\eta \not\in \tau\), then the topology \(\tau_\eta = \{\rho \cap (\rho' \cap \eta): \rho, \rho' \in \tau\} \) coincides with the fuzzy topology \(\tau'\) generated by the class \(\tau \cup \eta\).

**Proof.** Let \(z_0 \in \tau'\). Then \(z_0 \in \bigvee_{i \in I} \bigwedge_{j \in J} \Delta_i \cap \Delta_j \), \(\Delta_j \in \tau \cup \{\eta\}\) and \(\Delta_i \in \Delta_0\) is an arbitrary (finite) index set. If \(\Delta_j \in \tau\) for each \(i \in I, j \in J\), then \(z_0 \in \tau_\eta\). Assume that there exist \(i_0 \in I, j_0 \in J\) such that \(\Delta_{i_0} \cap \Delta_{j_0} = \eta \cap \rho\) for some \(\rho, \rho' \in \tau\). Hence \(z_0 \in \tau_\eta\).

Conversely, if \(z_0 \in \tau_\eta\), then there exist \(\rho, \rho' \in \tau\) such that \(z_0 \in (\rho \cap \eta) \cup (\rho' \cap \eta)\). Therefore \(z_0 \in \tau'\). \(\square\)

**Definition 2.2.** Singal and Prakash, 1991. Let \(\mu\) be any fuzzy set of a fts \((X, \tau)\)

(1) \(\mu\) is called fuzzy preopen set of \(X\) if \(\mu \subseteq \text{int}_\eta(\text{cl}(\mu))\).
(2) \(\mu\) is called fuzzy preclosed set of \(X\) if \(\mu \supseteq \text{cl}(\text{int}_\mu)\).

We denote the set of all fuzzy preopen (resp. fuzzy preclosed) sets by \(FPO\(X)\) (resp. \(FPC\(\X)\)).

**Definition 2.3.** Singal and Prakash, 1991. Fuzzy preclosure and fuzzy preinterior of a fuzzy set \(\mu\) of a fts \((X, \tau)\) are defined as follows:

(1) \(\text{pcl}(\mu) = \bigwedge\{\rho: \rho\text{ is a fuzzy preclosed and }\mu \subseteq \rho\}\).
(2) \(\text{pint}(\mu) = \bigvee\{v: v\text{ is a fuzzy preopen and }\mu \supseteq v\}\).

In this paper we introduce new classes of fuzzy sets called fuzzy \(\eta\)-preopen sets. This family of fuzzy sets not only depends on the fuzzy topology \(\tau\) but also on \(\eta\). We study some of its properties and its characterizations.

**Definition 2.4.** For any fts \((X, \tau)\). A fuzzy subset \(\mu\) is called a fuzzy \(\eta\)-preopen set if \(\mu \subseteq \text{int}_\eta(\text{cl}(\mu))\).

The complement of fuzzy \(\eta\)-preopen set is called fuzzy \(\eta\)-preclosed set. The family of all fuzzy \(\eta\)-preopen sets is denoted by \(FP_{\eta}\(\O\(\X)\)\) and the family of all fuzzy \(\eta\)-preclosed sets is denoted by \(FP_{\eta}\(\C\(\X)\)\).

The following example show that a fuzzy preopen set is a fuzzy \(\eta\)-preopen but not conversely.

**Example 2.2.** Consider the following fuzzy sets

\[
a(x) = \begin{cases} 
x - \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2} \\
0, & \text{otherwise} 
\end{cases}
\]

\[
b(x) = \begin{cases} 
\frac{1}{2} - x, & \text{if } \frac{1}{2} < x \leq 1 \\
\frac{1}{2}, & \text{if } \frac{1}{2} < x < 1 
\end{cases}
\]

\[
c(x) = \begin{cases} 
\frac{3x}{2}, & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{2}, & \text{if } \frac{1}{2} < x \leq 1 
\end{cases}
\]

\[
\eta(x) = \begin{cases} 
3x, & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{2}, & \text{if } \frac{1}{2} < x < 1 
\end{cases}
\]

Let \(\tau = \{0, 1\}, \eta \not\in \tau\). Then \(\tau_\eta = \{0, 1\}, \eta \not\in \tau\). We have the fuzzy set \(c\) is a fuzzy \(\eta\)-preopen but it is not fuzzy preopen, since \(\text{int}(\text{cl}(c)) = b \not\subseteq c\).

**Theorem 2.2.** Let \((X, \tau)\) be a fts, \(\eta\) be a fuzzy set such that \(\eta \not\in \tau\). We have:

(1) An arbitrary union of fuzzy \(\eta\)-preopen sets is a fuzzy \(\eta\)-preopen set.
(2) An arbitrary intersection of fuzzy \(\eta\)-preclosed sets is a fuzzy \(\eta\)-preclosed set.

**Proof.** (1) Let \(\{V_s\}\) be a collection of fuzzy \(\eta\)-preopen sets. Then, for each \(x, V_s \subseteq \text{int}_\eta(\text{cl}(V_s))\). Now,

\[
\bigvee V_s \subseteq \bigvee \text{int}_\eta(\text{cl}(V_s)) \subseteq \text{int}_\eta \bigvee (\text{cl}(V_s)) = \text{int}_\eta \text{cl} (\bigvee V_s)
\]

Hence \(\bigvee V_s\) is a fuzzy \(\eta\)-preopen set.

(2) Follows easily by taking complements. \(\square\)

**Definition 2.5.** Let \((X, \tau)\) be a fts and \(\eta\) be a fuzzy set on \(X\), \(\eta \not\in \tau\). Fuzzy \(\eta\)-preclosure \((p_{\eta}\text{cl})\) and fuzzy \(\eta\)-preinterior \((p_{\eta}\text{int})\) of a fuzzy set \(\varrho\) are defined as follows:

(1) \(p_{\eta}\text{cl}(\varrho) = \bigwedge\{\mu: \mu \in FP_{\eta}\text{O}(X, \tau) \text{ and } \varrho \subseteq \mu\}\).
(2) \(p_{\eta}\text{int}(\varrho) = \bigvee\{v: v \in FP_{\eta}\text{O}(X, \tau) \text{ and } v \supseteq \varrho\}\).

**Theorem 2.3.** Let \(\varrho\) be any fuzzy set in a fts \((X, \tau)\), \(\eta\) be a fuzzy set such that \(\eta \not\in \tau\). Then \(p_{\eta}\text{cl}(1 - \varrho) = 1 - p_{\eta}\text{int}(\varrho)\) and \(p_{\eta}\text{int}(1 - \varrho) = 1 - p_{\eta}\text{cl}(\varrho)\).

**Proof.** We see that a fuzzy \(\eta\)-preopen set \(\mu \subseteq \varrho\) is precisely the complement of fuzzy \(\eta\)-preopen set \(\varrho \supseteq 1 - \varrho\). Thus
implies that

\[ p_q(int(q)) = \sqrt{\{1 - v : v \in FP_qC(X, \tau) \text{ and } v \geq 1 - q}\} \]
\[ = 1 - \bigwedge\{v : v \in FP_qC(X, \tau) \text{ and } v \geq 1 - q}\]  
\[ = 1 - p_q(cl(1 - q)) \]

Similarly, \( p_q(cl(1)) = 1 - p_q(int(1 - q)). \)

**Theorem 2.4.** For a fts \((X, \tau)\), \(\eta\) be a fuzzy set such that \(\eta \notin \tau\), then \(\mu\) is called a fuzzy \(\eta\)-preclosed (resp. fuzzy \(\eta\)-preopen) if and only if \(\mu = p_q(cl(\mu))\) (resp. \(\mu = p_q(int(\mu))\)).

**Proof.** Let \(\mu \in FP_qC(X, \tau)\). Since \(\mu \leq \mu\), \(\mu \in \{f : f \in FP_qC(X, \tau) \text{ and } \mu \leq f\}\), and \(\mu \leq f\) implies that \(\mu = \bigwedge\{f : f \in FP_qC(X, \tau) \text{ and } \mu \leq f\}\), i.e., \(\mu = p_q(cl(\mu))\). Conversely, suppose that \(\mu = p_q(cl(\mu))\), i.e., \(\mu = \bigwedge\{f : f \in FP_qC(X, \tau) \text{ and } \mu \leq f\}\). This implies that \(\mu \in \{f : f \in FP_qC(X, \tau) \text{ and } \mu \leq f\}\). Hence \(\mu \in FP_qC(X, \tau)\). \(\Box\)

**Theorem 2.5.** For a fts \((X, \tau)\), \(\eta\) is a fuzzy set such that \(\eta \notin \tau\), the following hold for \(\eta\)-preclusion

1. \(p_q(cl(O_X)) = O_X\).
2. \(p_q(cl(\mu)) \subseteq FP_qC(X, \tau)\).
3. \(p_q(cl(v) \leq p_q(cl(v))\) if \(\mu \leq v\).
4. \(p_q(cl(p_q(cl(\mu)))) = p_q(cl(\mu)).\)

**Proof.** Obvious. \(\Box\)

**Theorem 2.6.** For a fts \((X, \tau)\), \(\eta\) be a fuzzy set such that \(\eta \notin \tau\), the following hold

1. \(p_q(cl(\mu) \vee v) \geq p_q(cl(\mu)) \vee p_q(cl(v))\).
2. \(p_q(cl(\mu \vee v) \leq p_q(cl(\mu)) \vee p_q(cl(v))\).

**Proof.** Obvious. \(\Box\)

**Definition 2.6.** Let \((X, \tau)\) and \((Y, \sigma)\) be two fts’s. A function \(f : (X, \tau) \to (Y, \sigma)\) is called:

1. Fuzzy preopen [Bin Shahna, 1991] if \(f(\lambda)\) is fuzzy preopen subset of \(Y\), for each fuzzy open set \(\lambda\) in \(X\).
2. Fuzzy open [Chang, 1968] if \(f(\lambda)\) is fuzzy open subset of \(Y\), for each fuzzy open set \(\lambda\) in \(X\).
3. Fuzzy weakly open [Park et al., 1997] if \(f(\lambda) \subseteq (int(f(cl(\lambda))))\), for each fuzzy open set \(\lambda\) in \(X\).
4. Fuzzy contra open [Caldas et al., 2004] (resp. fuzzy contra closed) if \(f(\lambda)\) is fuzzy closed (resp. fuzzy open) set of \(Y\), for each fuzzy open (resp. fuzzy closed) set \(\lambda\) in \(X\).
5. Fuzzy weakly preopen [Caldas et al., 2004] if \(f(\lambda) \subseteq (int(f(cl(\lambda))))\), for each fuzzy open set \(\lambda\) in \(X\).

3. New forms of fuzzy functions

Now, we define the generalized forms of preopen function in fuzzy setting.

**Definition 3.1.** Let \((X, \tau)\) and \((Y, \sigma)\) be two fts’s, \(\eta\) be a fuzzy set such that \(\eta \notin \sigma\). A function \(f : (X, \tau) \to (Y, \sigma)\) is called:

1. Fuzzy \(\eta\)-preopen if \(f(\lambda)\) is a fuzzy \(\eta\)-preopen subset of \(Y\), for each fuzzy open set \(\lambda\) in \(X\).
2. Fuzzy \(\eta\)-open if \(f(\lambda)\) is fuzzy \(\eta\)-open subset of \(Y\), for each fuzzy open set \(\lambda\) in \(X\).
3. Fuzzy weakly \(\eta\)-open if \(f(\lambda) \subseteq (\sigma, int(f(cl(\lambda))))\), for each fuzzy open set \(\lambda\) in \(X\).
4. Fuzzy weakly \(\eta\)-preopen if \(f(\lambda) \subseteq (\sigma, int(f(cl(\lambda))))\), for each fuzzy open set \(\lambda\) in \(X\).

**Theorem 3.1.** Let \((X, \tau)\) and \((Y, \sigma)\) be two fts’s, \(\eta\) be a fuzzy set such that \(\eta \notin \sigma\). For a function \(f : (X, \tau) \to (Y, \sigma)\) the following are equivalent.

1. \(f\) is a fuzzy weakly \(\eta\)-preopen.
2. \(f\) is a fuzzy \(\eta\)-preopen.
3. \((int(f(cl(\lambda)))) \subseteq (\sigma, int(f(cl(\lambda))))\) for each \(\lambda\) in \(FO(X)\).
4. \((f(\lambda)) \subseteq (\sigma, int(f(cl(\lambda))))\) for each \(\lambda\) in \(FO(\lambda)\).

**Proof.** Obvious. \(\Box\)

**Definition 3.2** [Caldas et al., 2004]. A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be fuzzy strongly continuous, if for every fuzzy subset \(\lambda\) of \(X\), \(f(cl(\lambda)) \subseteq (\lambda)\).

**Theorem 3.2.** Let \((X, \tau)\) and \((Y, \sigma)\) be two fts’s, \(\eta\) be a fuzzy set such that \(\eta \notin \sigma\). If a function \(f : (X, \tau) \to (Y, \sigma)\) is a fuzzy weakly \(\eta\)-preopen and fuzzy strongly continuous. Then it is a fuzzy \(\eta\)-preopen.

**Proof.** Let \(\lambda\) be a fuzzy subset of \(X\). Since \(f\) is fuzzy weakly \(\eta\)-preopen, then we have \(f(\lambda) \subseteq (\sigma, int(f(cl(\lambda))))\). However, because \(f\) is strongly continuous, \(f(\lambda) \subseteq (\sigma, int(f(cl(\lambda))))\), i.e., \(f(\lambda) = (\sigma, int(f(cl(\lambda))))\). Consequently, \(f(\lambda) \subseteq FP_{\sigma}O(Y)\). Therefore \(f\) is fuzzy \(\eta\)-preopen function. \(\Box\)

**Theorem 3.3.** Let \((X, \tau)\) be a fuzzy regular space and \((Y, \sigma)\) be a fts, \(\eta\) be a fuzzy set such that \(\eta \notin \sigma\). Then the function \(f : (X, \tau) \to (Y, \sigma)\) is a fuzzy weakly \(\eta\)-preopen if and only if \(f\) is a fuzzy \(\eta\)-preopen.

**Proof.** The sufficiency is clear. For the necessity, let \(\lambda\) be a non-null fuzzy open subset of \(X\). For each \(x_{\lambda}\) fuzzy point in \(\lambda\), let \(\mu_{x_{\lambda}}\) be a fuzzy open set such that \(x_{\lambda} \in \mu_{x_{\lambda}} \subseteq cl(\mu_{x_{\lambda}}) \subseteq \lambda\). Hence we obtain that \(\lambda = \bigcup\{\mu_{x_{\lambda}} : x_{\lambda} \in \lambda\} = \bigcup\{cl(\mu_{x_{\lambda}}) : x_{\lambda} \in \lambda\}\) and
\(f(\lambda) = \bigcup \{f(\mu_x) : x_\in \lambda \} \)
\[\leq \bigcup \{p_q \text{int}(f(\text{cl}(\mu_x))) : x_\in \lambda \} \]
\[\leq p_q \text{int}(f(\bigcup \{\text{cl}(\mu_x)) : x_\in \lambda \})) = p_q \text{int}(f(\lambda)). \]
Thus \(f \) is a fuzzy \(q \)-preopen function. \( \Box \)

**Definition 3.3** Pu and Liu, 1980. An fuzzy set \(\mu\) is said be quasi-coincident \((q\)-coincident\) with an fuzzy set \(\lambda\), if there exists at least one point \(x \in X\) such that \(\mu(x) + \lambda(x) > 1\). It is denoted by \(\mu \lambda.\) \(\mu \lambda\) means that \(\mu\) and \(\lambda\) are not \(q\)-coincident. For two fuzzy \(\mu\) and \(\lambda\), \(\mu \leq \lambda\) if \(\mu(x) \leq \lambda(x)\) for each \(x \in X\). Note that \(\lambda \leq \mu\) iff \(\mu(q) \leq \lambda(q)\).

**Definition 3.4** Caldas et al., 2004. Two non-empty fuzzy sets \(\lambda\) and \(\beta\) in a fuzzy topological space \((X, \tau)\) \((i.e., \text{neither } \lambda \text{ nor } \beta \text{ is } \emptyset)\) are said be fuzzy preseparated if \(\lambda \not\in \text{cl}(\beta)\) and \(\not\in \text{cl}(\lambda)\) or equivalents if there exist two fuzzy preopen sets \(\mu\) and \(v\) such that \(\lambda \subseteq \mu, \beta \subseteq v, \lambda \not\in \text{cl}(v)\) and \(\beta \not\in \text{cl}(\mu)\).

A fuzzy topological space which cannot be expressed as the union of two fuzzy preseparated (fuzzy \(q\)-preseparated) sets is said to be a fuzzy preconnected (fuzzy \(q\)-preconnected).

**Theorem 3.4.** Let \((X, \tau)\) and \((Y, \sigma)\) are the fuzzy topological spaces, \(\eta\) be a fuzzy set such that \(\eta \notin \sigma\). If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is an injective fuzzy weakly \(q\)-preopen function of a space \(X\) onto a fuzzy \(q\)-preconnected space \(Y\), then \(X\) is fuzzy connected.

**Proof.** Let \(X\) be not fuzzy connected. Then there exist fuzzy separated sets \(\beta\) and \(\gamma\) in \(X\), such that \(X = \beta \cup \gamma\). Since \(\beta\) and \(\gamma\) are fuzzy separated, there exist two fuzzy open sets \(\mu\) and \(v\) such that \(\beta \subseteq \mu, \gamma \subseteq v, \beta \not\in \text{cl}(v)\) and \(\gamma \not\in \text{cl}(\mu)\). Hence we have \(f(\beta) \not\subseteq f(\mu), f(\beta) \not\subseteq f(v), f(\gamma) \not\subseteq f(v)\) and \(f(\gamma) \not\subseteq f(\mu)\). Since \(f\) is fuzzy weakly \(q\)-preopen, we have \(f(\mu) \subseteq p_q \text{int}(f(\text{cl}(\mu)))\) and \(f(v) \subseteq p_q \text{int}(f(\text{cl}(v)))\) and since \(\mu\) and \(v\) are fuzzy open and also fuzzy closed, we have \(f(\text{cl}(\mu)) = f(\mu), f(\text{cl}(v)) = f(v)\). Hence \(f(\mu)\) and \(f(v)\) are fuzzy \(q\)-preopen in \(Y\). Therefore, \(f(\beta)\) and \(f(\gamma)\) are fuzzy \(q\)-preseparated sets in \(Y\) and \(\beta \not\subseteq \text{cl}(\gamma)\) and \(\gamma \not\subseteq \text{cl}(\beta)\). Hence this distance contrary to the fact that \(Y\) is fuzzy \(q\)-preconnected. Thus \(X\) is fuzzy connected. \( \Box \)

**Definition 3.6** Caldas et al., 2004. Space \(X\) is said to be fuzzy hyper connected if every non-null fuzzy open subset of \(X\) is fuzzy dense in \(X\).

**Theorem 3.5.** Let \((X, \tau)\) and \((Y, \sigma)\) be two ftfs, \(\eta\) be a fuzzy set such that \(\eta \notin \sigma\). If \(X\) is a fuzzy hyper connected space. Then \(f : (X, \tau) \rightarrow (Y, \sigma)\) is fuzzy weakly \(q\)-preopen and only if \(f(X)\) is fuzzy \(q\)-preopen set in \(Y\).

**Proof.** The sufficiency is trivial. For the necessity observe that for any fuzzy open subset \(\lambda\) of \(X\), \(f(\lambda) \subseteq f(X) = p_q \text{int}(f(X)) = p_q \text{int}(f(\text{cl}(\lambda)))\). \( \Box \)

**Definition 3.7.** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be fuzzy weakly \(\eta\)-preclosed if \(p_q \text{cl}(f(\text{int}(\beta))) \subseteq f(\beta)\) for each fuzzy closed subset \(\beta\) of \(X\).

**Definition 3.8.** Let \((X, \tau)\) and \((Y, \sigma)\) be two ftfs, \(\eta\) be a fuzzy set such that \(\eta \notin \sigma\). A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is called:

1. Fuzzy \(\eta\)-preclosed if \(f(\lambda)\) is a fuzzy \(\eta\)-preclosed subset of \(Y\), for each fuzzy closed set \(\lambda\) in \(X\).
2. Fuzzy \(\eta\)-closed if \(f(\lambda)\) is fuzzy \(\eta\)-closed subset of \(Y\), for each fuzzy closed set \(\lambda\) in \(X\).
3. Fuzzy weakly \(\eta\)-preclosed if \(\sigma_q \text{cl}(f(\text{int}(\lambda))) \subseteq f(\lambda)\), for each fuzzy open set \(\lambda\) in \(X\).
4. Fuzzy weakly \(\eta\)-preclosed if \(p_q \text{cl}(f(\text{int}(\lambda))) \subseteq f(\lambda)\), for each fuzzy open set \(\lambda\) in \(X\).

We have the following diagram and the converse of these implication do not hold.

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<table>
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<th>fuzzy closed function</th>
<th>fuzzy (\eta)-closed</th>
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</tr>
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<td>fuzzy preclosed function</td>
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<tr>
<td>fuzzy weakly preclosed function</td>
<td>fuzzy weakly (\eta)-preclosed</td>
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</tbody>
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**Theorem 3.6.** For a function \(f : (X, \tau) \rightarrow (Y, \sigma)\), \(\eta \notin \sigma\), the following conditions are equivalent:

1. \(f\) is fuzzy weakly \(\eta\)-preclosed.
2. \(p_q \text{cl}(f(\lambda)) \subseteq f(\text{cl}(\lambda))\) for every open set \(\lambda\) in \(X\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(\lambda\) be any fuzzy open subset of \(X\). Then \(p_q \text{cl}(f(\lambda)) = p_q \text{cl}(f(\text{int}(\lambda))) \subseteq p_q \text{cl}(f(\text{cl}(\text{int}(\lambda)))) \subseteq f(\text{cl}(\lambda)).\)

(2) \(\Rightarrow\) (1) Let \(\beta\) be any fuzzy closed subset of \(X\). Then \(p_q \text{cl}(f(\text{int}(\beta))) \subseteq f(\text{cl}(\text{int}(\beta))) \subseteq f(\text{cl}(\beta)) = f(\beta). \Box \)

**Theorem 3.7.** For a function \(f : (X, \tau) \rightarrow (Y, \sigma)\), \(\eta \notin \sigma\), the following conditions are equivalent:

1. \(f\) is fuzzy weakly \(\eta\)-preclosed.
2. \(p_q \text{cl}(f(\beta)) \subseteq f(\text{cl}(\beta))\) for each fuzzy open subset \(\beta\) of \(X\).
3. \(p_q \text{cl}(f(\text{int}(\beta))) \subseteq f(\beta)\) for each fuzzy closed subset \(\beta\) of \(X\).

**Proof.** Obvious. \( \Box \)

**Theorem 3.8.** Let \((X, \tau)\) and \((Y, \sigma)\) are two ftfs, \(\eta \notin \sigma\). Then:
(1) If \( f : (X, \tau) \to (Y, \sigma) \) is fuzzy preclosed and fuzzy contra closed, then \( f \) is fuzzy weakly \( \eta \)-preclosed.

(2) If \( f : (X, \tau) \to (Y, \sigma) \) is fuzzy contra open, then \( f \) is fuzzy weakly \( \eta \)-preclosed.

Proof. (1) Let \( \beta \) be a fuzzy closed subset of \( X \). Since \( f \) is a fuzzy preclosed, then \( \text{cl}(\text{int}(f(\beta))) \leq f(\beta) \) and since \( f \) is fuzzy contra closed, \( f(\beta) \) is fuzzy open. Therefore \( p_\eta \text{cl}(\text{int}(f(\beta))) \leq p_\eta \text{cl}(f(\beta)) \leq \text{cl}(\text{int}(f(\beta))) \leq f(\beta) \). Consequently \( f \) is a fuzzy weakly \( \eta \)-preclosed.

(2) Let \( \beta \) be a fuzzy closed subset of \( X \). Then \( p_\eta \text{cl}(\text{int}(f(\beta))) \leq (f(\text{int}(\beta))) \leq f(\beta) \). □

References