A note on some factorized groups

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1. Introduction

A group $G$ is (properly) factorizable if it contains two (proper) subgroups $A$ and $B$ such that $G = AB$, that is $G = \{ab \mid a \in A, b \in B\}$. A classical problem in group theory is to study how the structure of the two factors $A$ and $B$ determines the structure of the group $G$. For example, if $A$ and $B$ are abelian, $G$ is metabelian by a well-known result of Itô [1, Theorem 2.1.1].

In this paper we study the structure of a group $G = SH$ factorized by an elementary abelian group $S$ of exponent 2 and a periodic group $H$ without involutions. Our main result is

**Theorem.** Let $G = SH$ be a group factorized by $S$, a subgroup of exponent 2, and $H$, a periodic group without elements of even order. If $H$ is hypercentral then $G$ is hyper-abelian; moreover, if $H$ is soluble with derived length $d$, then $G$ has derived length at most $2d$.

The preceding theorem is one of the few results on factorized groups without further assumptions on $G$ (as, for example, the solubility or the residual finiteness), as can be easily verified, for example, in [1] and in its references.

2. Notation and preliminary results

If $G$ is a group, we denote by $\pi (G)$ the set of primes $p$ for which there exists an element of $G$ whose order is divisible by $p$. The other notation is standard (as in [1] or [3]).

We shall denote by $G$ a group satisfying the following condition:

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0021-8693/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
\textbf{Lemma 2.} \textit{Any normal subgroup of }\Delta\textit{ and therefore }\mathcal{X} = (\ast)\textit{ Lemma 1. Every }g \in G\textit{ can be uniquely written as }g = ax\textit{ with }a \in S\textit{ and }x \in H.\

\textbf{Proof.} Otherwise, there exist }b \in S,\ y \in H\textit{ with } (a, x) \neq (b, y)\textit{ and }g = by.\textit{ Then }ax = by,\ ba = yx^{-1}\textit{ and since }S\textit{ is an elementary abelian group, }yx^{-1}\textit{ should be an involution, against the hypothesis.} \quad \square

We recall that a subgroup }X\textit{ of }\Delta\textit{ is factorized if }X = (A \cap X)(B \cap X)\textit{ and }A \cap B \leq X\textit{ (see [1, Lemma 1.1.1]).}

\textbf{Lemma 2.} \textit{Any normal subgroup of }G\textit{ is factorized.}\

\textbf{Proof.} If }N\textit{ is a normal subgroup of }G,\textit{ in }\overline{G} = G/N\textit{ we have }\overline{S} \cap \overline{H} = \{1\}\textit{ because }\pi(\overline{S}) \cap \pi(\overline{H}) = \emptyset.\textit{ The statement is then a consequence of Lemma 1.1.2(iii) in [1].} \quad \square

We denote by }O_2(G)\textit{ the maximal normal periodic 2-subgroup of }G\textit{ (by the preceding lemma, }O_2(G) \subseteq S).\textit{ We also denote by }O_2'(G)\textit{ the maximal normal periodic 2' subgroup of }G\textit{ (by the preceding lemma, }O_2'(G) \subseteq H).\textit{ If }g \in G\textit{ is a periodic element of the group }G,\delta(g)\textit{ denotes the order of }g\textit{ and }\delta(g) = 2^a\delta'(g)\textit{ with } \langle 2, \delta'(g) \rangle = 1.\textit{ The following lemma is essential to obtain our results.}

\textbf{Lemma 3.} \textit{The group }\langle S, Z(H) \rangle\textit{ is factorized and }\langle S, Z(H) \rangle = SH_1\textit{ with }H_1\textit{ an abelian subgroup of }H.\textit{ In particular, }\langle S, Z(H) \rangle\textit{ is metabelian.}\

\textbf{Proof.} The group }\langle S, Z(H) \rangle\textit{ is certainly factorized since it contains }S,\textit{ one of the factors of }G.\textit{ Then we can write }\langle S, Z(H) \rangle = SH_1\textit{ with }H_1\textit{ a subgroup of }H.\textit{ As already observed, }S \cap H = \{1\}\textit{ and therefore, to prove that }H_1\textit{ is abelian, we can suppose }O_2(G) = \{1\}.\textit{ Let }a \in S,\ z \in Z(H)\textit{; then }a^z = bx\textit{ for some }b \in S\textit{ and }x \in H.\textit{ We put }\delta = \delta(x) = \delta'(x).\textit{ Since }a^z\textit{ has order 2, we have } (bx)^2 = 1\textit{ and }x^b = x^{-1};\textit{ then }b = xa^z\textit{ and therefore }x^a^z = x^{-1};\textit{ finally, since }z \in Z(H)\textit{ we also have }x^a = x^{-1}.\textit{ Therefore the element }ab\textit{ of }S\textit{ centralizes }x\textit{ and: }\[ [a, z] = aa^z = abx \quad \text{with } \delta = \delta'(\{a, z\}).\]

We consider the set }\Delta = \{[a, z]^y | a \in S,\ z \in Z(H),\ \delta = \delta'(\{a, z\})\}.\textit{ We have }\Delta \subseteq S\textit{ and therefore }\Delta\textit{ is normalized by }S\textit{ (which is abelian). Let }a \in S,\ z \in Z(H),\ y \in H\textit{ and }a^y = y'\textit{, for some }a' \in S\textit{ and }y' \in H,\textit{ then }[a, z]^y = [a^y, z] = [a', z].\textit{ Moreover, since }[a, z]\textit{ and }[a', z]\textit{ are conjugate, they have the same order and therefore }\delta'(\{a, z\}) = \delta'(\{a', z\}) = \delta.\textit{ Then }([a, z]^y)^y \in \Delta\textit{ for any }y \in H\textit{ and }\Delta\textit{ is normalized by }H.\textit{ The subgroup }
\( \langle \Delta \rangle \) is a normal subgroup of \( G \). Since by hypothesis we have \( O_2(G) = \{1\} \), and also that \( \Delta = \{1\} \) and then \([a, z] \in H \) for any \( a \in S \) and any \( z \in Z(H) \).

We have \([S, Z(H)] = \langle [a, z] \mid a \in S, z \in Z(H) \rangle \subseteq H \), moreover this subgroup is normalized by \( S \). If then \( y \in H \) we have, with the above notations, \([a, z]^y = [a', z] \); therefore \([S, Z(H)] \) is a normal subgroup of \( G \) contained in \( H \).

If we consider \([a_1, z_1] \) and \([a_2, z_2] \) with \( a_1, a_2 \in S \) and \( z_1, z_2 \in Z(H) \) then \([a_1, z_1]^{a_2} \in H \) and therefore \([a_1, z_1]^{a_2}z_2 = [a_1, z_1]^{a_2}z_2 = [a_1, z_1] \).

We conclude that \([S, Z(H)] \) is abelian.

In \( \overline{G} = G/[S, Z(H)] \), we have that \( Z(H)[S, Z(H)]/[S, Z(H)] \) is central; therefore the subgroup \( H_1 = Z(H)[S, Z(H)] \) is normal in \( G \) and it is abelian since \([S, Z(H)] \subseteq H \).

Finally, \( \langle S, Z(H) \rangle = \langle S, [S, Z(H)], Z(H) \rangle = SH_1 \) is metabelian by Itô’s theorem. \( \square \)

3. Proof of the main result

**Proposition 1.** If \( H \) is hypercentral then \( G \) is hyper-abelian. If \( H \) is nilpotent (of class \( c \)) then \( G \) is soluble (of derived length at most \( 2c \)).

**Proof.** If \( N \) denotes the normal closure of \( Z(H) \) in \( G \) then \( 1 \neq N = Z(H)^G = Z(H)^{H_S} = Z(H)^S \subseteq \langle S, Z(H) \rangle \). By Lemma 3, \( \langle S, Z(H) \rangle \) is metabelian. \( \square \)

If \( H \) is nilpotent then \( G \) is soluble and (by Lemma 2) periodic and therefore locally finite. This property holds also under the hypothesis that \( H \) is hypercentral:

**Proposition 2.** If \( H \) is hypercentral then \( G = O_2^c(G) \), moreover \( G \) is locally finite.

**Proof.** To prove the first part of the statement, it is enough to show that if \( O_2^c(G) = \{1\} \) then \( S \) is normal in \( G \). Let then \( O_2^c(G) = \{1\} \) and \( F = F(G) \) be the Fitting subgroup of \( G \). By Lemma 2, \( F \) is factorized. Then there exists \( S_0 \leq S \) and \( H_0 \leq H \) with \( F = S_0H_0 \).

Since \( F \) is locally nilpotent and \( \pi(S_0) \cap \pi(H_0) = \emptyset \), we have \( F = S_0 \times H_0 \) and \( H_0 \) is normal in \( G \). Then \( H_0 = \{1\} \) and \( F \leq S \); since \( S \) is abelian, we also have \( S \leq C_G(F) \).

By Proposition 1, \( G \) is hyper-abelian and by [7, Lemma 2.17] \( C_G(F) \leq F \) which implies \( S = F \) normal in \( G \).

Since \( G \) is hypercentral and periodic, it is locally finite and therefore the normal series of \( \{1\} \leq O_2^c(G) \leq O_2^c(G) \leq O_2^c(G) \) has locally finite quotients: it follows that \( G \) is locally finite [7, Theorem 1.45]. \( \square \)

**Proposition 3.** If \( H \) is hypercentral then every finite subgroup of \( G \) is contained in a finite factorized subgroup of \( G \).

**Proof.** Let \( \langle g_1, g_2, \ldots, g_n \rangle \) be a subgroup of \( G \) (which is finite because \( G \) is locally finite) and \( g_i = a_i x_i \) for some \( a_i \in S, x_i \in H, i = 1, 2, \ldots, n \). If we put \( S_0 = \langle a_1, a_2, \ldots, a_n \rangle \leq S \) and \( H_0 = \langle x_1, x_2, \ldots, x_n \rangle \leq H \), it is enough to prove that there exist two finite subgroups \( S_1 \leq S \) and \( H_1 \leq H \) such that \( S_0 \leq S_1, H_0 \leq H_1 \) and \( S_1 H_1 = H_1 S_1 \).
By the preceding proposition, \(<S_0, H_0> \leq O_2^2(G)H_0\), and we can take \(G = O_2^2(G)H_0\).

In \(\overline{G} = G/O_2^2(G)\), let \(\overline{S}_1 = \overline{S}_0^{H_0} = \overline{S}_0^{H_0}\). Since \(\overline{S}\) is elementary abelian and \(\overline{H}_0\) is finite, also \(\overline{S}_1\) is finite.

If \(O_2^2(G)S_1\) is the preimage of \(\overline{S}_1\) in \(G\) then \(<S_0, H_0> \leq O_2^2(G)S_1H_0\) and we can suppose \(G = O_2^2(G)S_1H_0\) that is \(S = S_1\).

Then \(<S_1, H_0>\) is finite and factorized (since it contains \(S\)), so it is \(S_1H_1\) where \(H_1 = H \cap <S_1, H_0>\).

Using the preceding proposition and (essentially) Theorem A of Hall and Higman [4] (see also [5]), we get:

**Corollary 1.** If \(H\) is nilpotent of class \(c\), then \(G\) is soluble of derived length at most \(c + 1\).

**Proof.** It is enough to apply [4, Theorem 1.2.4]. □

**Corollary 2.** If \(H\) is hypercentral and soluble with derived length \(d\) then \(G\) is soluble with derived length at most \(2d\).

**Proof.** By Proposition 3, we can suppose \(G\) finite. By induction on the order of \(G\), we can reduce to the case in which \(H\) is a \(p\)-group \((p \neq 2)\). From [4] we have \(H^{(d-1)} \leq O_p(G)\) and therefore in \(\overline{G} = G/O_p(G)\) the subgroup \(\overline{H}\) has derived length at most \(d - 1\). By Proposition 2, \(\overline{S}\) is a normal subgroup of \(\overline{G}\) and therefore \(\overline{G}\) has derived length at most \(d\). We conclude recalling that \(O_p(G) \leq H\) has derived length at most \(d\). □

The bounds in the preceding corollaries are not, in general, the best possible. In fact, by [2, Theorem 1] we get that if \(H\) has derived length 2, then \(G\) has derived length at most 3.

### 4. Sylow’s theory

We recall that a Sylow \(p\)-subgroup of a group \(X\) is a maximal \(p\)-subgroup of \(X\) (see [3, Chapter 2]).

The structure of \(G\) allows us to develop a Sylow’s theory for 2-subgroups also in the case in which \(H\) is not hyper-abelian.

We begin with some easy results.

**Lemma 4.** Every involution in \(G\) is conjugate to an element of \(S\) by an element of \(H\).

**Proof.** If \(g \in G\) is an involution then \(g = ax\) for some \(a \in S\) and \(x \in H\); then \(ag = x \in H\) has odd order \(\delta(x)\) and \((a, g)\) is isomorphic to the dihedral group of order \(2\delta(x)\), in which all the involutions are conjugate. □

**Lemma 5.** There does not exist any element of order 4 in \(G\).
Proof. Suppose, by contradiction, that \( g \in G \), \( g^2 \neq 1 = g^4 \) and \( g = ax \) for some \( a \in S \) and \( x \in H \). By the preceding lemma, we can suppose \( g^2 \in S \); then \( g^2 = axax \) and \( xax = (axax)^a = (g^2)^a = g^2 \). It follows that \( 1 = g^4 = axaxaxax = xax^2ax \) and \( (x^2)^a = x^{-2} \), but \( x \) has odd order and then \( x^a = x^{-1} \); that is, \( g^2 = axax = x^ax = 1 \); a contradiction. \( \square \)

Lemma 6. \( S \) is a Sylow \( 2 \)-subgroup of \( G \).

Proof. If \( \Sigma \) is a \( 2 \)-subgroup of \( G \) containing \( S \), by Lemma 4, \( \Sigma \) is elementary abelian. If \( g \in \Sigma \), then \( g = ax \) for some \( a \in S \) and \( x \in H \). Then \( x = ag \in \Sigma \) should be a \( 2 \)-element, while \( H \) does not contain involutions. Hence \( x = 1 \) and \( g = a \in S \). \( \square \)

Lemma 7. Let \( T \) be a \( 2 \)-subgroup of \( G \). Then:

(a) \( T \) is isomorphic to a subgroup of \( S \);
(b) if \( T \) is finite, there exists \( x \in H \) with \( T^x \subseteq S \).

Proof. (a) By Lemma 4 there are not elements of order 4 in \( G \) and therefore \( T \) is elementary abelian. Then to show the statement, it is enough to show that \( |T| \leq |S| \). For any \( y \in T \) there exists unique \( a \in S \) and \( x \in H \) with \( t = ax \). We conclude if we prove that the map \( T \to S \) defined by \( t = ax \mapsto a \) is injective. In fact, if \( t_1, t_2 \in T \), \( t_1 = ax \) and \( t_2 = ay \) (with \( a \in S \) and \( x, y \in H \)), recalling that \( t_1 \) is an involution, we get \( x^a = x^{-1} \) and \( t_1t_2 = axay = x^{-1}y \). Since \( H \) does not contain involutions, we must have \( x = y \) and \( t_1 = t_2 \).

(b) By induction on the order of \( T \): if \( |T| = 2 \) we conclude by Lemma 6. We then suppose that \( |T| > 2 \) and let \( t \) be a nontrivial element of \( T \). By Lemma 5, there exists \( x \in H \) with \( t^x \in S \). Let \( G_0 = C_G(t^x) \); then \( T^x \leq G_0 \), \( S \leq G_0 \) and \( G_0 \) is factorized: \( G_0 = SH_0 \). In \( G_0 = G_0/(t^x) \) we have \( |T^x| < |T| \) and therefore by induction hypothesis there exists \( \overline{y} \in \overline{H} \) such that \( (T^x)^{\overline{y}} \leq \overline{S} \). If \( y \) is the preimage of \( \overline{y} \) in \( H_0 \), we get \( T^{xy} \leq S \). \( \square \)

The following example is inspired by \( [6] \) (see \( [3, \text{Example 2.2.14}] \)).

Example 1. Let \( A \) be a countable elementary abelian \( 2 \)-group:

\[ A = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle \times \cdots \]

and \( B \) an elementary abelian \( 2 \)-group with \( |B| > \aleph_0 \). Let \( S \) be \( A \times B \) and \( X = \langle x \rangle \) be a group of prime order \( p \neq 2 \). Let \( H \) be the set of maps \( \rho : S \to X \) such that for every \( b \in B \) we have \( \rho((a, b)) \neq 1 \) only for a finite number of \( a \in A \).

For any \( \rho, \sigma \in H \) and any \( s \in S \) we put \( (\rho \sigma)(s) = \rho(s)\sigma(s) \) and \( (\rho^{-1})(s) = \rho(s)^{-1} \). Then \( H \) is an elementary abelian \( p \)-group and \( S \) acts over \( H \) in a natural way, putting \( (\rho^k)(s) = \rho(st) \).

If \( G = HS \) is the semidirect product of \( H \) by \( S \), it can be easily verified that \( G \) is a group satisfying \( (\ast) \).
We denote by \( a \) the element \((a, 1) \in S\) and \( b \) the element \((1, b) \in S\). If \( n \geq 1 \), we put \( A_n = \langle a_1, a_2, \ldots, a_n \rangle \) and define \( \rho_n \in H \) by

\[
\rho_n((a, b)) = \begin{cases} x, & \text{if } a \in A_n \text{ and } b \in B \setminus \{1\}, \\ 1, & \text{otherwise.} \end{cases}
\]

For any \( i < n \) and \( s \in S \) we have \( \rho(s) = \rho(a_is) = \rho^{a_i}(s) \) and therefore the elements of \( A_{n-1} \) commute with \( \rho_n \). We denote by \( \sigma_n = \rho_1 \rho_2 \ldots \rho_n, t_n = a_n^{\sigma_n}, T = \langle t_1, t_2, \ldots, t_n, \ldots \rangle \) and \( G_0 = HA = HT \). We prove that \( T \) is a Sylow 2-subgroup of \( G \) and moreover \( CG(T) = T \). In fact, if \( g \not\in T \) is an element of \( G \) such that \( T = \langle T, g \rangle \) is a 2-group then we have \( G_0 < H \overline{T} = HS_1 \) with \( A < S_1 < S \). There exist \( b \in B \) and \( \rho \in H \) with \( \rho b \in C_G(T) \).

Then for any \( n \) we have:

\[
a_n^{\rho \sigma_n} = a_n^{\rho}
\]

and since \( b \) centralizes \( A \), the map \( \tau_n = \sigma_n \rho(\sigma^{-1})b \in H \) centralizes \( a_n \). Then

\[
\tau_n(1) = \sigma_n(1)\rho(1)\sigma_n(b)^{-1} = \rho(1)x^{-n}
\]

and

\[
\tau_n(1) = \tau_n^{\rho}(1) = \sigma_n(a_n)\rho(a_n)\sigma_n(ba_n)^{-1} = \rho(a_n)x^{-1}
\]

and therefore

\[
\rho(a_n)x^{n-1} = \rho(1) \quad \text{for all } n \in \mathbb{N}.
\]

Then for an infinite number of \( a \in A \) we have \( \rho(a) \neq 1 \), and this contradicts the definition of \( H \). Therefore \( C_G(T) = T \). Then \( |T| = |A| < |S| \) and \( T \) is not isomorphic to \( S \).

However, we can prove

**Proposition 4.** If \( S \) is countable (or finite) all the Sylow 2-subgroups of \( G \) are isomorphic.

**Proof.** If \( S \) is finite, the statement follows from Lemma 7 (in this case all the Sylow 2-subgroups of \( G \) are conjugate). Let \( S \) be countable and \( T \) be a Sylow 2-subgroup of \( G \). By Lemma 7, we have \( |T| \leq |S| \). Moreover, \( T \) is not finite since in this case by Lemma 7 we can find \( x \in H \) with \( T^x \leq S \) and then \( S^{x^{-1}} \) is 2-subgroup of \( G \), containing \( T \) properly, against the hypothesis. Then \( |T| = |S| \) and therefore, recalling that \( S \) and \( T \) are elementary abelian, we get \( T \cong S \). \( \square \)

The following is a classical example (see [3, Example 2.6.6]) and shows that also under the hypothesis of the preceding proposition, it is not true that the Sylow 2-subgroups of \( G \) are conjugate.
Example 2. Let \((d_1, d_2, \ldots, d_n, \ldots)\) be a sequence of natural odd numbers strictly greater than 1 and let \(D_i = \langle h_i, s_i \mid h_i^{d_i} = s_i^2, \ h_i^{-1} = h_i^{-1}\rangle\) be the dihedral group with \(2d_i\) elements. Let \(S\) be the direct sum of \(\langle s_i \rangle\), \(H\) the direct sum of \(\langle h_i \rangle\) and \(G\) the direct sum of \(D_i\). Then \(G = HS\) is a countable group satisfying (\(\ast\)); but the Sylow 2-subgroups of \(G\) are \(2^{\aleph_0}\) and therefore they cannot be conjugated by elements of \(S\).

We need an additional hypothesis on \(H\) to conclude that the Sylow 2-subgroups of \(G\) are conjugate.

Lemma 8. If \(T\) is a Sylow 2-subgroup of \(G\) not contained in \(S\), then it has a conjugate \(T^*\) in \(\langle S, T \rangle\) such that \(T\) does not lie in \(\langle S, T^* \rangle\).

Proof. Note that \(S \cap T\) is central in \(\langle S, T \rangle\). Like any subgroup of \(G\) containing \(S\), \(\langle S, T \rangle\) is factorized; moreover, (\(\ast\)) also holds with \(G\) and \(S\) replaced by \(\langle S, T \rangle\), \(S\) and \(\langle S, T \rangle \cap H\).

Since \(T/(S \cap T)\) is a nontrivial Sylow 2-subgroup of \(\langle S, T \rangle/(S \cap T)\) intersecting \(S/(S \cap T)\) trivially, by Lemma 4 it has a conjugate \(T^*/(S \cap T)\) which intersects \(S/(S \cap T)\) nontrivially, that is, \(S \cap T^* > S \cap T\). If we had \(T \lec (S, T^*)\), then \(T\) would centralize \(S \cap T^*\) and so \(S \cap T^*\) would be a 2-group. But \(T\) is a Sylow subgroup and \(S \cap T^*\) does not lie in it, so this cannot happen.

Proposition 5. If \(H\) satisfies \(\text{min}\) on subgroups then all the Sylow 2-subgroups of \(G\) are conjugate.

Proof. Suppose that \(T\) is a Sylow 2-subgroup of \(G\) such that no conjugate of \(T\) lies in \(S\). Set \(T_0 = T\) and, if \(k > 0\), \(T_k = T_{k-1}^*\) as in the proof of the previous lemma. Then \(G_k = \langle S, T_k \rangle = SH_k\) form a strictly descending infinite chain of factorized (since \(S \lec G_k\)) subgroups. Then also \(H_k\) is a strictly descending infinite chain of subgroups of \(H\); this contradicts the assumption that \(H\) satisfies \(\text{min}\) on subgroups.

References