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A spin-foam vertex amplitude with the correct semiclassical limit

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ABSTRACT

Spin-foam models are hoped to provide a dynamics for loop quantum gravity. These start from the Plebanski formulation of gravity, in which gravity is obtained from a topological field theory, BF theory, through constraints, which, however, select more than one gravitational sector, as well as an unphysical degenerate sector. We show this is why terms beyond the needed Feynman-prescribed one appear in the semiclassical limit of the EPRL spin-foam amplitude. By quantum mechanically isolating a single gravitational sector, we modify this amplitude, yielding a spin-foam amplitude for loop quantum gravity with the correct semiclassical limit.

1. Introduction

Loop quantum gravity (LQG) [1–4] offers a compelling kinematical framework in which discreteness of geometry is *derived* from a quantization of general relativity (GR) rather than postulated. The discreteness has enabled well-defined proposals for the Hamiltonian constraint — defining the dynamics of the theory — in which one sees how diffeomorphism invariance eliminates normally problematic ultraviolet divergences. However, the lack of manifest space–time covariance, inherent in any canonical approach, is often suspected as a reason for the presence of ambiguities in the quantization of the Hamiltonian constraint. This has motivated the *spin-foam* program [1,5–7], which aims to provide a space–time covariant, path integral version of the dynamics of LQG a la Feynman. The histories summed over in the path integral arise from loop quantization methods, each representing a 'quantum space–time', and referred to as a *spin-foam*.

At the heart of the path integral approach is the prescription that the contribution to the transition amplitude by each history should be the exponential of i times the action. The use of such an expression has roots tracing back to Paul Dirac's *Principles of Quantum Mechanics* [8], and is central to the successful derivation of the classical limit of the path integral. In spin-foams, the 'quantum space–times' have a classical geometric interpretation only in the semiclassical limit $\hbar \to 0$. It is in this limit that one seeks a spin-foam amplitude equal to the exponential of i times the classical action. We call this the 'semiclassical limit' of a spin-foam amplitude, following [9]. As highlighted in these remarks, having such a correct semiclassical limit is key in recovering the correct *classical* limit of the theory in the standard way.

The *method* used for constructing the individual amplitudes in a spin-foam sum is to use the *Plebanski* formulation of gravity, or variations thereof. In this formulation of gravity, one takes advantage of the fact that GR can be formulated as a topological field theory whose spin-foam quantization is well-understood — BF theory [10] — supplemented by so-called *simplicity* constraints. Within the last several years, a spin-foam model of quantum gravity was, for the first time, introduced whose kinematics match those of LQG and therefore realize the original goal of the spin-foam program: to provide a path integral dynamics for LQG. This is known in the literature as EPRL [11–14]; when the Barbero–Immirzi parameter [15,16] γ, a certain quantization ambiguity, is less than 1, this model is identical to the Freidel–Krasnov model [17]. Despite its success, the EPRL amplitude still has difficulty in obtaining the correct semiclassical limit: (non-geometric) degenerate configurations are not suppressed, and even if one restricts to non-degenerate configurations, the semiclassical limit of the simplest component of the amplitude, the vertex amplitude, has four terms instead of the desired one term of the form exponential of i times the action [18]. Both of these problems cause unphysical configurations to dominate in the semiclassical limit, as we will show. (See also additional arguments [19–22] on the importance of having only the one exponential term, reviewed in the final discussion.) Furthermore, we will show that both of these problems are directly due to a deficiency in the way gravity is recovered from BF theory: When one imposes the simplicity constraints, one isolates not just a single gravitational sector, but multiple sectors, not all physical. The other 4-d spin-foam models of gravity have similar problems with a similar source [9,23,24].

In the present work, we show how, by formulating the restriction to what we call the Einstein–Hilbert sector classically first, quantizing it, and incorporating it into the EPRL vertex definition, one can define a modified vertex for which the extra terms in

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the semiclassical limit are eliminated, degenerate configurations are exponentially suppressed, and one achieves a vertex amplitude with *the correct semiclassical limit*. This new modified vertex, which we call the proper EPRL vertex, additionally continues to be compatible with loop quantum gravity, linear in the boundary state, and $SU(2)$ invariant. The key condition of linearity in the boundary state ensures that the final transition amplitude defined by the spin-foam model is linear in the initial state and anti-linear in the final state.

To begin, we review the classical discrete framework, review the EPRL vertex, point out its problems, and then derive the solution, leading to the definition of the proper EPRL vertex. In the final discussion, we note how the proper vertex may also solve other problems in the literature, in addition to the above one originally motivating this work. This Letter provides a summary of the work, with emphasis on motivation and broader consequences, detailed proofs being left to the two longer articles [25,26].

2. A review of EPRL from a new perspective

The quantum histories used in spin-foam sums are usually based on a triangulation of space–time into 4-simplices. The probability amplitude for a given spin-foam history breaks up into a product of amplitudes associated to each component of the triangulation [1,7]. The most important of these amplitudes is the *vertex amplitude*, which provides the probability amplitude for data associated to a single 4-simplex. In the following, as we are concerned specifically with the vertex amplitude, for conceptual clarity, we focus on a single 4-simplex σ. (Though the EPRL vertex has been generalized to arbitrary cells [27], we restrict ourselves to the simplicial case, as certain key elements will depend on the combinatorics of this case. See final discussion.) Let triangles and tetrahedra of σ be denoted respectively by f and t and decorations thereof. Fix a transverse orientation of each f within the boundary of σ. Furthermore, fix an affine structure, which is equivalent to fixing a flat connection ∂_a, on σ; this is a pure gauge choice [26]. The basic variables for the single 4-simplex σ consist in 5 group elements $(G_t \in Spin(4))_{t\in\sigma}$, and 20 algebra elements $(B^{IJ}_{tf} \in \mathfrak{so}(4))_{f\in t\in\sigma}$, $I, J = 0, 1, 2, 3$. These are subject to constraints: (1.) 'orientation', $G_t \triangleright B_{ft} = -G_{t'} \triangleright B_{ft'}$, where $\triangleright$ denotes adjoint action, (2.) 'closure', $\sum_{f\in t} B_{ft} = 0$, and (3.) 'linear simplicity', $(B_{ft})^{ij} = 0$, $i, j = 1, 2, 3$. Each of these three constraints either restrict the allowed histories in the spin-foam sum or are imposed in the sense that violations are exponentially suppressed. Constraints (1.) and (2.) imply [25,28] that there exists a *unique two-form* $B^{IJ}_{\mu\nu}$, constant with respect to ∂_a, such that, for all t, f with $f \in t$,

$$G_t \triangleright B^{IJ}_{ft} = \int_f B^{IJ}. \tag{1}$$

In this Letter, $\mu, \nu, \ldots$ denote tensor indices over σ as a manifold. When the constraint (3.), linear simplicity, is additionally imposed, $B^{IJ}_{\mu\nu}$ takes one of the three forms [25]

$$\text{(II}\pm\text{)}\ B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL} e^K \wedge e^L \quad \text{for some const. } e^I_\mu,$$

$$\text{(deg)}\ \epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = 0 \quad \text{(degenerate } B\text{)}, \tag{2}$$

where ϵ_{IJKL} is the Levi-Civita array, and the names for these sectors have been taken from [25,29]. In sectors (II+) and (II−), e^I_μ has the interpretation of a *co-tetrad*, determining the space–time metric via $g_{\mu\nu} := \eta_{IJ} e^I_\mu e^J_\nu$, where $\eta_{IJ} := \mathrm{diag}(-1, 1, 1, 1)$. Note that, despite the spatial indices ij appearing in the constraint (3.), the e^I_μ arising in this way has full $SO(4)$ freedom intact: For all $H \in Spin(4)$, under $G_t \mapsto HG_t$, we have $e^I_\mu \mapsto H^I{}_J e^J_\mu$ where $H^I{}_J$ is the $SO(4)$ matrix canonically associated to H (see, e.g., [18,26]).

If $B^{IJ}_{\mu\nu}$ is non-degenerate, it additionally defines a dynamically determined orientation of σ, which we represent by its sign relative to the fixed orientation $\mathring{\epsilon}$ of σ:

$$\omega := \mathrm{sgn}\big(\mathring{\epsilon}^{\mu\nu\rho\sigma} \epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma}\big).$$

For convenience, define $\omega = 0$ when $B^{IJ}_{\mu\nu}$ is degenerate. Additionally, let $\nu := \pm 1, 0$ according to whether $B^{IJ}_{\mu\nu}$ is in (II±) or (deg). If $\nu \neq 0$, the BF Lagrangian is related to the Einstein–Hilbert Lagrangian by

$$\mathcal{L}_{BF} = \omega\nu\mathcal{L}_{EH}.$$

When $\omega\nu = +1$, $\mathcal{L}_{BF} = \mathcal{L}_{EH}$ and we say that $B^{IJ}_{\mu\nu}$, and the data (B^{IJ}_{ft}, G_t) determining $B^{IJ}_{\mu\nu}$, are in the *Einstein–Hilbert sector*.

What we have described until now are the discrete *space–time* variables of the model. These determine the *phase space* variables (G_f, J^{IJ}_{ft}) on the boundary via

$$G_f := G_{t'_f t_f} := G^{-1}_{t'_f} G_{t_f} \in Spin(4),$$

$$J^{IJ}_{ft} := \frac{1}{8\pi G}\left(B^{IJ}_{ft} + \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL} B^{IJ}_{ft}\right).$$

Here t_f, t'_f are respectively the tetrahedron 'above' and 'below' f within the boundary $\partial\sigma$ of σ. The J^{IJ}_{ft} are conjugate to the G_f in the sense that they generate left or right translations on G_f depending on whether $t = t_f$ or $t = t'_f$. The generators of (internal) spatial rotations in terms of these are then $L^i_{ft} := \frac{1}{2}\epsilon^i{}_{jk} J^{jk}_{ft}$.

In quantum theory, the simplicity constraint reduces the boundary Hilbert space of the quantum BF theory to that of LQG, yielding an embedding of LQG boundary states into $Spin(4)$ BF theory boundary states [14]. Let us recall this embedding both because it is at the heart of the EPRL vertex amplitude, and because it will be key in the modification we propose.

The LQG Hilbert space associated to $\partial\sigma$ is $L^2(\times_f SU(2))$. A (generalized) *spin-network* $\Psi_{(k_f,\psi_{ft})}$ in this space is labeled by one spin k_f and two states $\psi_{ft'_f} \in V^*_{k_f}$, $\psi_{ft_f} \in V_{k_f}$ per triangle f, where V_k denotes the spin-k representation of $SU(2)$. $\Psi_{(k_f,\psi_{ft})} \in L^2(\times_f SU(2))$ is given explicitly by

$$\Psi_{(k_f,\psi_{ft})}((g_f)) := \prod_f \langle \psi_{ft'_f} | \rho(g_f) | \psi_{ft_f} \rangle, \tag{3}$$

where $\rho(g)$ denotes the action of $g \in SU(2)$. The embedding ι from LQG states to $Spin(4)$ BF theory boundary states is defined in terms of the basis (3) by

$$(\iota\Psi_{(k_f,\psi_{ft})})((G_f)) := \prod_f \langle \psi_{ft'_f} | \iota^{k_f} \rho(G_f) \iota_{k_f} | \psi_{ft_f} \rangle,$$

where here and throughout this Letter we set $s^\pm := \frac{1}{2}|1 \pm \gamma|k$, $\iota_k : V_k \to V_{s^-} \otimes V_{s^+}$ denotes the intertwiner among the indicated $SU(2)$ representations, scaled such that it is isometric in the Hilbert space inner products, $\iota^k : V_{s^-} \otimes V_{s^+} \to V_k$ denotes its Hermitian conjugate, and $\rho(G)$ denotes the action of $G \in Spin(4)$ in the appropriate representation. Note that in order to ensure that $s^\pm_f$ are half integers, the values of k_f must be restricted; the resulting spectra of geometric operators then become continuous in the semiclassical limit if and only if γ is rational, so that γ must be rational in order for the theory to be viable [14,30]. (This is an artifact of the Euclidean EPRL theory and does not persist in the Lorentzian theory [14].) Let $\mathcal{K}_\gamma$ denote the set of allowed values for each k_f, and let $\mathcal{H}^\gamma_{\partial\sigma}$ be the span of $SU(2)$ spin-networks (3) with $\{k_f\} \subset \mathcal{K}_\gamma$, so that $\iota : \mathcal{H}^\gamma_{\partial\sigma} \to L^2(\times_f Spin(4))$.

The *EPRL vertex amplitude* $A_\sigma : \mathcal{H}^\gamma_{\partial\sigma} \to \mathbb{C}$, in terms of the above is

$$A_\sigma(\Psi_{(k_f,\psi_{ft})}) := \int_{Spin(4)^5} \times_t dG_t (\iota\Psi_{(k_f,\psi_{ft})})((G_f)) = \int_{Spin(4)^5} \times_t dG_t \prod_f \langle \psi_{ft'_f} | \iota^{k_f} \rho(G_f) \iota_{k_f} | \psi_{ft_f} \rangle. \quad (4)$$

3. EPRL asymptotics and their problem

The vertex amplitude (4) can be specialized to *Livine–Speziale* coherent states [31]. Each such state $\Psi_{(k_f,n_{ft})}$ is labeled by one spin k_f per f, and one unit 3-vector n_{ft} per f,t with $f \in t$, and are obtained from the states (3) by setting $\psi_{ft'_f} := \langle -n_{ft'_f}|$ and $\psi_{ft_f} := |n_{ft_f}\rangle$, where $|n_{ft}\rangle \in V_{k_f}$ denotes the $SU(2)$ *Perelomov* coherent state [32] peaked on the $SU(2)$ generators $\hat{L}^i$ via $\langle n_{ft}|\hat{L}^i|n_{ft}\rangle = k_f n^i_{ft}$. One then has

$$A_\sigma(\Psi_{(k_f,n_{ft})}) = \int_{Spin(4)^5} \times_t dG_t \prod_f \langle -n_{ft'_f} | \iota^{k_f} \rho(G_f) \iota_{k_f} | n_{ft_f} \rangle. \quad (5)$$

The $Spin(4)$ boundary state $\iota\Psi_{(k_f,n_{ft})}$ is a coherent state peaked on the classical configuration of B^{IJ}_{ft}'s taking the values

$$B^{IJ}_{ft} = 16\pi G k_f \delta^{[I}_0 n^{J]}_{ft}, \quad (6)$$

where $n^0_{ft} := 0$. B^{IJ}_{ft} values of this form are the most general satisfying linear simplicity — constraint (3.) enumerated in the last section. Furthermore, the integral over the G_t's in (5) is in a precise sense [26] a *path integral* over possible G_t's, which one identifies with the five group elements in the discrete classical framework reviewed at the start of Section 2 above.

If the data (k_f, n_{ft}) are such that, for each t, the span of $\{n_{ft}\}_{f\in t}$ is three dimensional, and is such that there exist group elements G_t allowing all the constraints (1.), (2.), and (3.) to be satisfied, then a unique *Regge geometry* [33] of the 4-simplex is determined and the data are called *Regge-like*. In this case the overall phase of the coherent state $\Psi_{(k_f,n_{ft})}$ can be fixed uniquely, giving rise to what is called the *Regge state* $\Psi^{\mathrm{R}}_{(k_f,n_{ft})}$ [18]. For such states, the asymptotics of the EPRL vertex are

$$A_\sigma(\Psi^{\mathrm{R}}_{(\lambda k_f,n_{ft})}) \sim N_1 e^{iS_{\mathrm{R}}} + N_1 e^{-iS_{\mathrm{R}}} + N_2 e^{\frac{i}{\gamma}S_{\mathrm{R}}} + N_3 e^{-\frac{i}{\gamma}S_{\mathrm{R}}} \quad (7)$$

for large $\lambda \in \mathbb{R}^+$, where the error term is bounded by a constant times λ^{-12}, S_{R} denotes the Regge action determined by the data $(\lambda k_f, n_{ft})$, and N_i are real functions of (k_f, n_{ft}). If the span of $\{n_{ft}\}_{f\in t}$ is not three dimensional for some t, but there still exist group elements G_t such that constraints (1.), (2.), and (3.) are satisfied, the asymptotics of the vertex are

$$A_\sigma(\Psi^{\mathrm{R}}_{(\lambda k_f,n_{ft})}) \sim N, \quad (8)$$

where again the error term is bounded by a constant times λ^{-12}.

The presence of the four distinct terms in (7) is enough to spoil the semiclassical limit of the model when multiple 4-simplices are involved. To see this, consider a spin-foam on a triangulation Δ whose data we assume, for simplicity, is Regge-like at each 4-simplex. The full amplitude then takes the form

$$A(\Delta) = \prod_f A_f \prod_t A_t \prod_\sigma A_\sigma, \quad (9)$$

where A_f and A_t are the factors associated to each triangle f and tetrahedron t in Δ [14]. Let $S_{f,t}$ denote the phase angle of the product $\prod_f A_f \prod_t A_t$ (defined modulo 2π). As A_f and A_t are always real, $S_{f,t}$ is 0 or π. The asymptotics of *each* factor A_σ in (9) now has four terms as in Eq. (7). On multiplying out these terms, the semiclassical limit of the full amplitude takes the form

$$A(\Delta) \sim \sum_{(\lambda_\sigma \in \{\pm 1, \pm 1/\gamma\})} N_{(\lambda_\sigma)} e^{iS_{(\lambda_\sigma)}},$$

where $S_{(\lambda_\sigma)} := S_{f,t} + \sum_\sigma \lambda_\sigma S_R(\sigma)$, and the sum is over all possible assignments of a coefficient $\lambda_\sigma \in \{\pm 1, \pm 1/\gamma\}$ to each 4-simplex σ. $S_{(\lambda_\sigma)}$ is the Regge action for Δ (modulo 2π) if and only if all the λ_σ are 1. Because, however, the λ_σ can vary from 4-simplex to 4-simplex, $S_{(\lambda_\sigma)}$ is in general *not equal to the Regge action, even upto to rescaling by a constant*, and its stationary points do not in general solve the Regge equations of motion. One thus has sectors in the semiclassical limit of the model which do not represent general relativity. These are in addition to the spin-foams which persist in the semiclassical limit whose data are degenerate and do not even represent a space–time geometry.

The most obvious way to correct the problem with the semiclassical limit of the amplitude is to somehow alter the vertex amplitude such that all but the first term in (7) is eliminated. How might one do this? Each term in the asymptotics (7) corresponds to a critical point of the integral (5), and hence to a particular value of the variables G_t, which, together with the boundary data (6), via Eq. (7) determine a continuum two-form $B^{IJ}_{\mu\nu}$ which is in one of the three Plebanski sectors labeled by $\nu = 0, \pm 1$, and which determines an orientation labeled by $\omega = 0, \pm 1$. The values of ν and ω corresponding to each of the four terms in (7) satisfy $\omega\nu = +1, -1, 0, 0$, respectively.[1] Therefore, to isolate the first term, one must impose $\omega\nu = +1$ — that is, one must restrict to

[1] On looking at the asymptotics in Eq. (7), it may be confusing that the last two terms correspond to the degenerate sector ($\omega\nu = 0$). If $B^{IJ}_{\mu\nu}$ is degenerate in this sector, what is the space–time metric corresponding to the non-degenerate geometry determining the Regge action appearing in these terms in the asymptotics? Furthermore, it would seem that if one isolates one of these last two terms (instead of the Einstein–Hilbert term), one would also have the correct action up to rescaling by $1/\gamma$ in the asymptotics, a rescaling which does not change the equations of motion but rather the value of the effective Newton constant. That is: it appears that the sectors corresponding to these last two terms are also gravitational sectors.

This suspicion turns out to be correct, and, as far as the author knows, has heretofore not been pointed out in the literature. It is certainly true that in these sectors $\epsilon_{IJKL} B^{IJ} \wedge B^{KL} = 0$, so that in this sense B is degenerate. Furthermore, in these sectors there does *not* exist a tetrad such that B is of the form (2). Nevertheless, in these sectors, B still determines a unique, non-degenerate space–time geometry. Specifically, the self-dual and anti-self-dual parts of B each determines an Urbantke metric [29,34] on space–time, and these two metrics are equal up to an overall sign because of linear simplicity. Furthermore, when the equation of motion for the connection is satisfied (see [35]), the BF action reduces to a constant times the Einstein–Hilbert action. Thus, these are indeed gravitational sectors.

Thus, in the space of solutions to the linear simplicity constraint, one finds *four* sectors of general relativity, each corresponding to a different rescaling of the gravitational action, and two of which are found in what is usually called the degenerate sector because B is degenerate there. In addition to these four gravitational sectors, there still remains a "truly" degenerate sector, in which not only B, but also the two Urbantke metrics determined by B, are degenerate. This truly degenerate sector is the one giving rise to the asymptotics in Eq. (8), and also corresponds to $\omega\nu = 0$.

In principle, one could restrict to any one of the four gravitational sectors, thereby isolating any one of the four terms in (7), and obtain a spin-foam model with the correct semiclassical limit — this is an ambiguity in the definition of the model which we have chosen to resolve in what might be called an "obvious" way. What is important, however, is that only *one* of these sectors are included. For, if more than one is included, because the action is rescaled differently in the different sectors, the semiclassical limit would be ruined by the cross-terms in the asymptotics, as discussed above.

the *Einstein–Hilbert sector* as we have defined it. This, at the same time, will eliminate the degenerate sector represented in Eq. (8), in which one also has $\omega\nu = 0$.

4. A condition selecting the Einstein–Hilbert sector and its quantization

Our strategy is first to find a classical condition on the basic variables that selects the Einstein–Hilbert sector, quantize this condition, and then use it to modify the EPRL vertex (4, 5). For each face f define $\beta_f((G_{\tilde{t}}))$ by

$$\beta_f((G_{\tilde{t}})) := -\operatorname{sgn}\big[\epsilon_{ijk}(G_{t_f t_1})^i{}_0 (G_{t_f t_2})^j{}_0 (G_{t_f t_3})^k{}_0 \cdot \epsilon_{lmn}(G_{t'_f t_1})^l{}_0 (G_{t'_f t_2})^m{}_0 (G_{t'_f t_3})^n{}_0\big],$$

where $G^I{}_J$ denotes the $SO(4)$ matrix canonically associated to a given *Spin*(4) element G, t_1, t_2, t_3 are the tetrahedra in σ *not* containing f, in any order, and sgn is defined to be zero when its argument is zero. The constant 2-form $B^{IJ}_{\mu\nu}$ determined by (1) is then in the Einstein–Hilbert sector iff

$$\beta_f((G_{\tilde{t}})) \cdot (G_t)^i{}_0 \cdot (L_{ft})_i > 0 \tag{10}$$

for any f, t, with $f \in t$ [26]. This is the condition which we seek to quantize and use to modify the vertex integral (4), (5). Normally this would be done by inserting into the path integral (5) the Heaviside function

$$\Theta\big(\beta_f((G_{\tilde{t}})) \cdot (G_t)^i{}_0 \cdot (L_{ft})_i\big), \tag{11}$$

where $\Theta(\cdot)$ is zero when its argument is zero. However, if one inserts this into (5), one obtains a vertex amplitude which is *non-linear in the boundary state*, spoiling the property of the final spin-foam sum that it be linear in the initial state and anti-linear in the final state, a property necessary for it to be interpreted as a transition amplitude. Instead, we *partially quantize* the expression (11) before inserting it into (4), (5), by replacing L^i_{ft} with $SU(2)$ generators $\hat{L}^i$ acting on the coherent state $|n_{ft}\rangle$. Because the generators $\hat{L}^i$ are peaked on $L^i_{ft} = j_f n^i_{ft}$ when acting on $|n_{ft}\rangle$, such insertions will still impose the desired condition (10) in the semiclassical limit, and so remove the unwanted sectors, while at the same time preserving the necessary linearity in the boundary state. We emphasize that we quantize only the dependence on $(L_{ft})^i$ in (11), and not the dependence on the G_t variables, as only this is necessary to preserve linearity in the boundary state, and one wants to stay as close as possible to the usual path integral strategy, that is, to keep as much as possible of the insertion a classical discrete quantity, rather than an operator. This decision to quantize only the $(L_{ft})^i$ dependence will also lead to a simpler modified vertex. Thus we insert the following group-variable dependent operator on V_{k_f}:

$$\hat{P}_{ft}((G_{\tilde{t}})) := P_{(0,\infty)}\big(\beta_f((G_{\tilde{t}})) \cdot (G_t)^i{}_0 \cdot \hat{L}_i\big),$$

where $P_{\mathcal{S}}(\hat{O})$ denotes the spectral projector for the operator $\hat{O}$ onto the portion $\mathcal{S}$ of its spectrum. Inserting this into the vertex path integral (4), one obtains what we call the *proper EPRL vertex amplitude*:

$$A^{(+)}_\sigma(\Psi_{\{k_f,\psi_{ft}\}}) = \int_{Spin(4)^5} \bigtimes_t dG_t \prod_f \langle \psi_{ft'_f} | \iota^{k_f} \rho(G_f) \iota_{k_f} \hat{P}_{ft_f}((G_{\tilde{t}})) | \psi_{ft_f} \rangle. \tag{12}$$

One can equivalently write this expression with the projector on the left side of each integrand factor [26]. This vertex amplitude is manifestly linear in the boundary state (3), and one can furthermore show that it is $SU(2)$ invariant [26]. Furthermore, despite the use of internal spatial indices $i, j, k, \ldots$ at certain points, this vertex amplitude does not break *Spin*(4) symmetry [26]. For the coherent state $\Psi_{(\lambda k_f, n_{ft})}$, for large λ, the proper EPRL vertex is exponentially suppressed unless (k_f, n_{ft}) describes a non-degenerate Regge geometry, in which case it furthermore now has *precisely the required asymptotics* [26]

$$A^{(+)}_\sigma\big(\Psi^{\mathrm{R}}_{(\lambda k_f, n_{ft})}\big) \sim N_1 e^{iS_{\mathrm{R}}}.$$

5. Discussion

By implementing quantum mechanically a restriction to the Einstein–Hilbert sector, the EPRL vertex amplitude has been modified, yielding what we call the proper EPRL vertex. The resulting vertex is linear in the boundary state, $SU(2)$ invariant, and leads to a correct semiclassical limit.

Let us remark first on the non-triviality of the removal of the degenerate sector that has been achieved. In the work [9], the degenerate sector of the Freidel–Krasnov model (equal to EPRL for $\gamma < 1$) is removed by using a path integral representation based on coherent states, similar to the path integral representation of the EPRL vertex given in (5) above. However, in the conclusion of the work [9], the authors mention that they do not know how to rewrite the resulting restricted path integral as a spin-foam sum — that is, as a sum over histories of spin-foams labeled with spins and intertwiners, similar to (4). The reason for this difficulty arguably can be traced to the same reason for our rejection of the "naive" prescription of inserting the non-quantized Heaviside function (11) into (5): because the resulting transition amplitude is non-linear in the boundary state. Thus, as far as removal of the degenerate sector is concerned, the new element of the present work is precisely the fact that the removal is achieved in such a way that *linearity in the boundary state is preserved*, so that the vertex amplitude can continue to be used to define transition amplitudes between canonical states in the usual sense.

Beyond the removal of the degenerate sector, the proper vertex furthermore achieves isolation of the Einstein–Hilbert sector, in which the sign of the Lagrangian relative to the Einstein–Hilbert Lagrangian is restricted to be consistently positive, ensuring the correct equations of motion in the semiclassical limit. In doing this, linearity in the boundary state is again preserved. This contrasts with the modification proposed in [21], in which the undesired term in the asymptotics is removed by direct means without understanding first its deeper meaning in terms of Plebanski sectors and orientations, and without removing the degenerate sector.

In addition to ensuring the correct equations of motion, the fact that the proper vertex asymptotically has only a single term with a single sign in front of the action may solve other problems as well. In particular, such asymptotics seem necessary in order for spin-foams to be consistent with the positive frequency condition in loop quantum cosmology [36,37]. They have also been advocated by Oriti [19] as a way of implementing causality in the sense introduced by Teitelboim [38]. Finally, from studies of 3-d gravity, there are indications that such asymptotics may completely eliminate a certain divergence in spin-foam sums present until now [22].

The expression (12) can be easily generalized to the Lorentzian signature [26]. One open issue is to provide a derivation of this generalization as well as to verify that, like the Euclidean proper vertex above, it has the desired asymptotics. A second open issue is to generalize this work to an arbitrary cell [27]. This second generalization will likely require an entirely new perspective, as the combinatorics of the 4-simplex are presently used in a key way not only in the derivation of the proper vertex, but in its very definition.

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