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Note

A characterization of strongly chordal graphs

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Abstract

In this paper, we present a simple charactrization of strongly chordal graphs. A chordal graph is strongly chordal if and only if every cycle on six or more vertices has an induced triangle with exactly two edges of the triangle as the chords of the cycle.

Keywords: Chordal graph; Strongly chordal graph; Characterization

1. Introduction

A graph is said to be *chordal* if it contains no induced cycle of length 4 or greater. A chord of a cycle of a graph is said to be a *strong chord* if it joins two vertices of the cycle with an odd distance in the cycle. A chordal graph is said to be *strongly chordal* if every cycle on six or more vertices contains a strong chord. The class of strongly chordal graphs is an interesting subclass of chordal graphs since there are several combinatorial graph problems which are NP-hard in chordal graphs but polynomially solvable in strongly chordal graphs [1,3,4]

Just as (u,v) denotes an edge joining the vertices u and v, let (u,v,w) denote a triangle joining the vertices u, v and w. In cycle $\langle 1,2,3,4,5,6,7,8,1\rangle$ of Fig. 1, $(1,2)$ is a cycle edge and $(2,5)$ is a chord of the cycle $(2,3,4)$ is an induced triangle with exactly one edge of the triangle as the chord of the cycle and $(1,2,5)$ is an induced triangle with exactly two edges of the triangle as the chords of the cycle.

Farber [2], Lubiw [4] and Chang and Nemhauser [1] have given many interesting characterizations of strongly chordal graphs. In this paper, we exhibit a simple charactrization of strongly chordal graphs.

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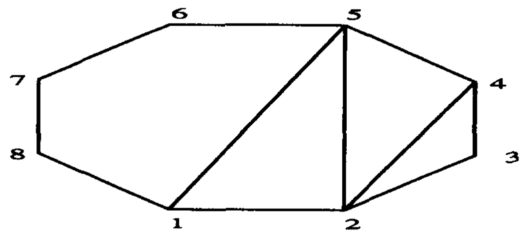

Fig. 1. Induced triangle in a cycle

An induced triangle $\triangle$ of a cycle is called *2-chord triangle* if two edges of the triangle $\triangle$ are chords of the cycle and the third edge is an edge of the cycle. In cycle $\langle 1,2,3,4,5,6,7,8,1\rangle$ of Fig. 1, $(1,2,5)$ is a 2-chord triangle of the cycle. An induced triangle $\triangle$ of a cycle is called *1-chord triangle* if one edge of the triangle $\triangle$ is a chord of the cycle and the other two edges of the triangle $\triangle$ are the consecutive edges of the cycle. In cycle $\langle 1,2,3,4,5,6,7,8,1\rangle$ of Fig. 1, $(2,3,4)$ is a 1-chord triangle of the cycle.

Our charactrization of strongly chordal graphs is similar to the following charactrization of chordal graphs which is due to Chang and Nemhauser [1].

Theorem 1.1 (Chang and Nemhauser [1]). *A graph is chordal if and only if every cycle of length greater than* 3 *has an induced* 1*-chord triangle.*

Our main result of this paper is the following characterzation of strongly chordal graphs:

Theorem 1.2. *A graph is strongly chordal if and only if it has no chordless cycle on four vertices and every cycle on at least five vertices has an induced* 2*-chord triangle.*

2. The main theorem

We need the following result [1] to prove Theorem 1.2.

Lemma 2.1 (Chang and Nemhauser [1]). *If B is a cycle of a chordal graph then for every edge (u,v) of B there is a vertex w of B that is adjacent to both u and v.*

Lemma 2.2. *Every cycle on* 6 *or more vertices of a strongly chordal graph G has an induced* 2*-chord triangle.*

Proof. Let $B=\langle v_1,v_2,\ldots,v_p,v_1\rangle$ be a cycle of G such that $p\geqslant 6$. Since G is strongly chordal, cycle B induces a strong chord, say (v_i,v_j). Without loss of generality, we assume that the cycle $\bar{B}=\langle v_i,v_{i+1},\ldots,v_j,v_i\rangle$ is of even length and hence $|j-i|$ is odd. Furthermore, we assume that (v_i,v_j) is a strong chord that minimizes the value of $|j-i|$.

Now, consider edge (v_i, v_j) and cycle $\bar{B}$. By Lemma 2.1, there exists a vertex v_k such that (v_i, v_j, v_k) forms an induced triangle of the cycle $\bar{B}$. Now (v_i, v_j, v_k) is either a 1-chord triangle or a 2-chord triangle of the cycle $\bar{B}$. If (v_i, v_j, v_k) is a 1-chord triangle of the cycle $\bar{B}$, then (v_i, v_j, v_k) is a 2-chord triangle of the cycle B and thus we are through. If (v_i, v_j, v_k) is a 2-chord triangle of the cycle $\bar{B}$, then either (v_i, v_k) or (v_j, v_k) is a strong chord of the cycle $\bar{B}$. Without loss of generality, we assume that (v_i, v_k) is a strong chord of the cycle $\bar{B}$ and the cycle $\bar{\bar{B}} = \langle v_i, v_{i+1}, \ldots, v_k, v_i \rangle$ is of even length. Notice that $|k - i|$ is odd and that the length of the cycle $\bar{\bar{B}}$ is strictly smaller than that of the cycle $\bar{B}$, i.e., $|k - i| < |j - i|$, a contradiction. □

Theorem 1.2 follows from Lemma 2.2 and from the fact that a 2-chord triangle of a cycle always induces a strong chord of the cycle (one of the two chords will be a strong chord of the cycle).

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